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A Three-Parameter Exponential Frechet Distribution: Theory and Application

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ABSTRACT

A novel three-parameter distribution of the Exponential G-family distribution called the Exponential Frechet Distribution has been proposed. Some properties of the model like moment generating function, characteristic function, quantile function, survival function and hazard function were derived. Bayesian and MLE approach was used to estimate the parameter. The Exponential Frechet Distribution flexibility was demonstrated by both simulation and application to remission time of bladder cancer patients. The results revealed that the Exponential Frechet Distribution yielded a better fit in comparison with other extended distribution.

1. Introduction

The Exponential distribution is used in Poisson processes and gives a description of the time between events and mostly applied in life testing experiments. It has memoryless property with a constant failure rate which makes the distribution unsuitable for real life problems and hence creating a vital problem in statistical modeling and applications. Besides the applications of Exponential distribution and its attractive properties, its usage has been limited in modeling real life situations due to the fact that it has a constant failure rate (Lemonte, 2013). Numerous standard distributions have been extensively used over the past decades for modeling data in several fields such as Engineering, Economics, Finance, Biological, Environmental and Medical Sciences etc. However, generalizing these standard distributions has produced several compound distributions that are more flexible compared to the baseline distributions. For this reason, several methods for generating new families of distributions have been studied. Eraikhuemen *et al.* (2020), propose a three-parameter Weibull-Lindley. Ahmed *et al.* (2023) introduce the modified Frechet-Exponential (MFE) distribution, which is an extension of the Frechet model. Bourguignon *et al.*, (2014) developed a new wider family of distributions called the Weibull generalized (Weibull-G) family of distributions. This family can reduce to the Exponential-G family when one of the shape parameters is equal to one. Cordeiro *et al.* (2014) proposed a new class of distributions called the Lomax generator with two extra positive parameters to generalize any continuous baseline distribution. Oguntunde *et al.* (2015) also introduced the Weibull-Exponential distribution (*WED*) using the generator by Bourguignon *et al.*, (2014). El-Bassiouny *et al.* (2015) introduced a generalization of Lomax distribution called exponential Lomax distribution. Some statistical properties of this distribution were derived and discussed; the maximum likelihood estimators of the parameters were derived. Ieren *et al.* (2018) proposed a new distribution called Weibull-Lindley distribution using the new Weibull-G family by Tahir *et al.* (2016).

2 Methods

2.1 The Exponential-Frechet Distribution.

According to Bouguignon *et al.,* (2014), the cumulative distribution function (cdf) and the probability density function (pdf) (for $x > 0$) of the Exponential-G family of distributions with an additional shape parameter ($\alpha > 0$) are defined by:

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$$F(x) = 1 - \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{1}$$

and

$$f(x) = \frac{\alpha g(x)}{\left[1-G(x)\right]^2}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{2}$$

where $g(x)$ and $G(x)$ represent the pdf and the cdf of the continuous distribution to be modified, which in this case stands for the pdf and cdf of the Frechet distribution respectively.
A random variable x is said to follow a Fréchet distribution with parameters θ *and* β if its probability density function (*pdf*) is given by

$$g(x) = \theta\beta^{\theta}x^{-\theta-1}\mathrm{e}^{-\left(\frac{\beta}{x}\right)^{\theta}} \tag{3}$$

and the corresponding cumulative distribution function (*cdf*) is given as

$$G(x) = \mathrm{e}^{-\left(\frac{\beta}{x}\right)^{\theta}} \tag{4}$$

For $x > 0, \theta > 0, \beta > 0$ where θ *and* β are the shape and scale parameters of the Fréchet distribution respectively. Using equation (3) and (4) in (1) and (2) and simplifying, we obtain the cdf and pdf of the Exponential-Frechet distribution (EFrD) as follows:

$$F(x) = 1 - \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\}$$

$$F(x) = 1 - \exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-\mathrm{e}^{-\left(\frac{\beta}{x}\right)^{\theta}}}\right]\right\}$$

$$F(x) = 1 - \exp\left\{-\alpha\left(e^{-\left(\frac{\beta}{x}\right)^{\theta}} - 1\right)^{-1}\right\} \tag{5}$$

And

$$f(x) = \frac{\alpha g(x)}{\left[1-G(x)\right]^2}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\}$$

$$f(x) = \frac{\alpha\theta\beta^{\theta}x^{-\theta-1}\mathrm{e}^{-\left(\frac{\beta}{x}\right)^{\theta}}}{\left[1-\mathrm{e}^{-\left(\frac{\beta}{x}\right)^{\theta}}\right]^2}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\left(\frac{\beta}{x}\right)^{\theta}}}{1-\mathrm{e}^{-\left(\frac{\beta}{x}\right)^{\theta}}}\right]\right\}$$

$$f(x)=\frac{\alpha\theta\beta^{\theta}x^{-\theta-1}e^{\left(\frac{\beta}{x}\right)^{\theta}}}{\left(e^{-\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{2}}\exp\left\{-\alpha\left(e^{-\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}\right\} \qquad (6)$$

For $x>0$, $\alpha,\beta,\theta>0$ where $\alpha>0$ and $\beta>0$ are the shape parameters while $\theta>0$ is a scale parameter. Hence equation (5) and (6) are the cdf and pdf of the Exponential-Frechet distribution (EFrD).

2.1.1 Graphical representation of the PDF

Given some values for the parameters α and β and θ, we provide some possible shapes for the *pdf* and the *cdf* of the Exponential-Frechet distribution (EFrD) as shown in figure 1 and figure 2 below:

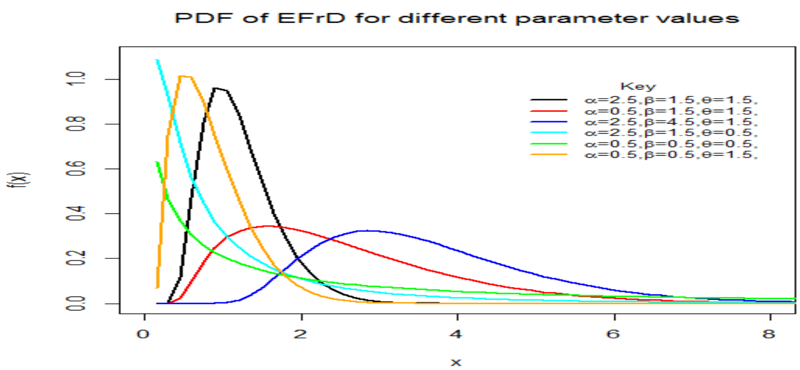

Figure 1: PDF of the EFrD for different parameter values

Figure **1** indicates that the Exponential-Frechet distribution (EFrD) distribution has various shapes such as right-skewed and symmetry shapes depending on the parameter values. This means that the distribution can be very useful for datasets with different shapes.

2.1.2 Graphical representation of the CDF

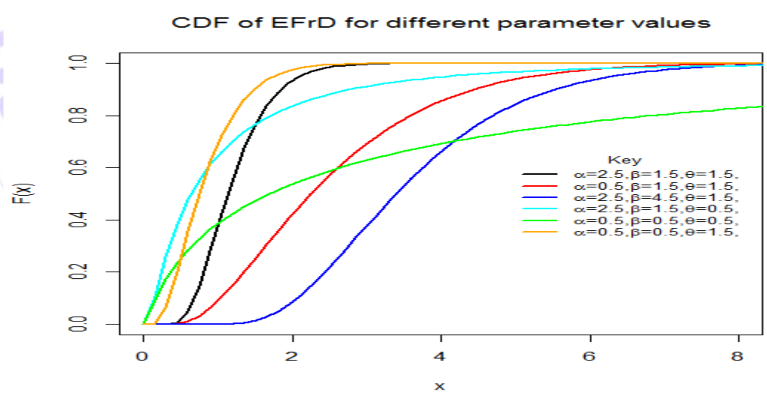

Figure. 2: CDF of the EFrD for different parameter values

Figure 2 shows the *cdf* plot, the *cdf* increases when X increases, and approaches 1 when X becomes large, as expected.

2 .2 Moment Generating Function

The moment generating function of a random variable X can be obtained as

$$M_x(t)=E\left[e^{tx}\right]=\int_{-\infty}^{\infty}e^{tx}f(x)dx \qquad (7)$$

Recall that by power series expansion,

$$e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r \tag{8}$$

Using the result in equation (8) and simplifying the integral in (7) therefore we have:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{-k-2}{l} \frac{(-1)^{k+l+1} \alpha^{k+1} \beta^r \Gamma\left(1-\frac{r}{\theta}\right)}{(k+l+1)^{1-\frac{r}{\theta}} k!} \right] \tag{9}$$

2.3 Characteristics Function

A representation for the characteristics function is given by

$$\phi_x(t) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x) dx \tag{10}$$

Recall that by power series expansion,

$$\mathrm{e}^{itx} = \sum_{r=0}^{\infty} \frac{(itx)^r}{r!} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} x^r \tag{11}$$

Hence, simple algebra and use of (10) above produces the following results:

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r f(x) dx = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E\left(X^r\right) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\mu_r^{'}\right]$$

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{-k-2}{l} \frac{(-1)^{k+l+1} \alpha^{k+1} \beta^r \Gamma\left(1-\frac{r}{\theta}\right)}{(k+l+1)^{1-\frac{r}{\theta}} k!} \right] \tag{12}$$

2.4 Quantile Function

Let $Q(u) = F^{-1}(u)$ be the quantile function (qf) of $F(x)$ for $0 < u < 1$.

Taking $F(x)$ to be the *cdf* of the Exponential-Frechet distribution (EFrD) and inverting it as above will give us the quantile function as follows.

$$F(x) = 1 - \exp\left\{ -\alpha \left(e^{\left(\frac{\beta}{x}\right)^{\theta}} - 1 \right)^{-1} \right\}$$

Inverting $F(x) = u$

$$F(x) = 1 - \exp\left\{ -\alpha \left(e^{\left(\frac{\beta}{x}\right)^{\theta}} - 1 \right)^{-1} \right\} = u \tag{13}$$

Simplifying equation (13) above, we obtain:

$$1-u=\exp\left\{-\alpha\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}\right\}$$

$$\ln(1-u)=-\alpha\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}$$

$$-\frac{\ln(1-u)}{\alpha}=\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}$$

$$1+\left(-\frac{\ln(1-u)}{\alpha}\right)^{-1}=\mathrm{e}^{\left(\frac{\beta}{x}\right)^{\theta}}$$

$$x=\beta\left\{\ln\left[1+\left(-\frac{\ln(1-u)}{\alpha}\right)^{-1}\right]\right\}^{-\frac{1}{\theta}}$$

$$Q(u)=X_q=\beta\left\{\ln\left[1+\left(-\frac{\ln(1-u)}{\alpha}\right)^{-1}\right]\right\}^{-\frac{1}{\theta}} \tag{14}$$

2.5 Reliability Analysis

2.5.1 Survival Function

Survival function is the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1-F(x) \tag{15}$$

Now, taking $F(x)$ to be the *cdf* of the proposed EFrD and substituting, we have;

$$F(x)=1-\exp\left\{-\alpha\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}\right\}$$

$$S(x)=1-\left\{1-\exp\left\{-\alpha\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}\right\}\right\}$$

$$S(x)=\exp\left\{-\alpha\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}\right\} \tag{16}$$

2.5.2 Hazard Function

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)} \tag{17}$$

Taking the proposed Exponential-Frechet distribution (EFrD) *PDF* and *CDF* as *f(x)* and *F(x)* respectively:

$$f(x)=\frac{\alpha\theta\beta^{\theta}x^{-\theta-1}e^{\left(\frac{\beta}{x}\right)^{\theta}}}{\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{2}}\exp\left\{-\alpha\frac{1}{\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)}\right\}$$

$$F(x)=1-\exp\left\{-\alpha\frac{1}{\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)}\right\}$$

Substitue for *f(x)* and *F(x)* and simplify gives

$$h(x)=\frac{\dfrac{\alpha\theta\beta^{\theta}x^{-\theta-1}e^{\left(\frac{\beta}{x}\right)^{\theta}}}{\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{2}}\exp\left\{-\alpha\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}\right\}}{1-\left\{1-\exp\left\{-\alpha\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{-1}\right\}\right\}}$$

$$h(x)=\frac{\alpha\theta\beta^{\theta}x^{-\theta-1}e^{\left(\frac{\beta}{x}\right)^{\theta}}}{\left(e^{\left(\frac{\beta}{x}\right)^{\theta}}-1\right)^{2}} \tag{18}$$

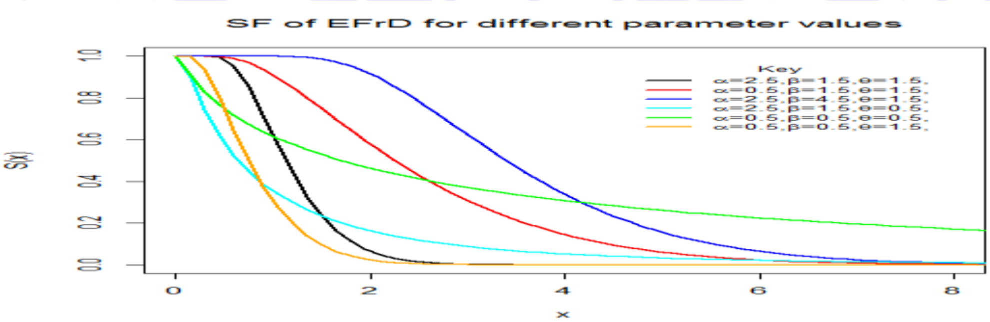

Figure 3: Survival Function

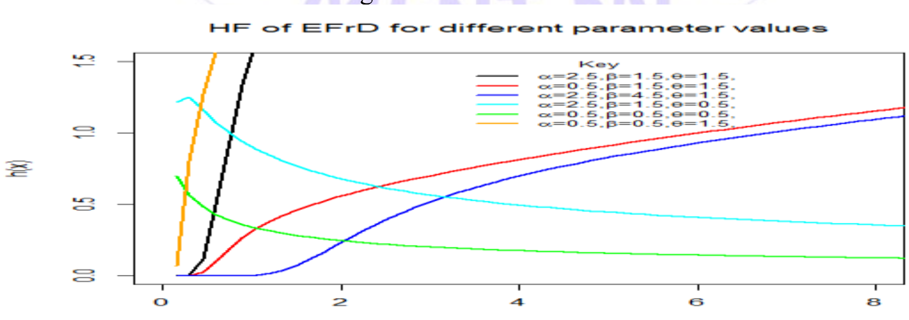

Figure 4: Hazard Function

3 Results

3.1 Simulation

Table 1: Mean Squared Errors and Average Estimates, why varying sample sizes under three loss functions and three priors for the form parameter ($\hat{\alpha}$) with $b = 1.0$ $a = 1.0$, $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$

n	Compute	MLE	Gamma Prior			Uniform Prior			Jeffrey's Prior		
			PLF	QLF	SELF	PLF	QLF	SELF	PLF	QLF	SELF
10	Estimate	0.5561	0.6015	0.4712	0.5759	0.6389	0.6117	0.6389	0.5832	0.4449	0.5561
	MSE	0.0422	0.0496	0.025	0.0418	0.0708	0.0597	0.0708	0.0499	0.028	0.0422
25	Estimate	0.5212	0.5406	0.4897	0.5305	0.5523	0.542	0.5523	0.5315	0.4795	0.5212
	MSE	0.012	0.0135	0.0099	0.0124	0.0157	0.0143	0.0157	0.013	0.0102	0.012
77	Estimate	0.5078	0.5143	0.4979	0.511	0.5177	0.5144	0.5177	0.5111	0.4946	0.5078
	MSE	0.0035	0.0037	0.0033	0.0036	0.0039	0.0038	0.0039	0.0036	0.0033	0.0035
130	Estimate	0.5044	0.5082	0.4986	0.5063	0.5102	0.5083	0.5102	0.5064	0.4967	0.5044
	MSE	0.002	0.0021	0.0019	0.002	0.0021	0.0021	0.0021	0.002	0.0019	0.002
200	Estimate	0.5028	0.5053	0.4991	0.5041	0.5066	0.5053	0.5066	0.5041	0.4978	0.5028
	MSE	0.0013	0.0013	0.0012	0.0013	0.0013	0.0013	0.0013	0.0013	0.0012	0.0013

Table 2: Mean Squared Errors and Average Estimates, why varying sample sizes under three loss functions and three priors for the form parameter ($\hat{\alpha}$) for $b = 1.0$, $\alpha = 3.5, \beta = 0.5, a = 1.0$ and $\theta = 0.5$

n	Compute	MLE	Gamma Prior			Uniform Prior		
			PLF	QLF	SELF	PLF	QLF	SELF
10	Estimate	3.8927	3.1457	2.4642	3.0118	4.4723	3.5034	4.2819
	MSE	2.0667	0.6852	1.4163	0.7514	3.4699	1.5491	2.9255
25	Estimate	3.6481	3.3583	3.042	3.2955	3.8663	3.5021	3.794
	MSE	0.5887	0.3749	0.5009	0.3835	0.7707	0.5223	0.6994
77	Estimate	3.5547	3.462	3.3518	3.44	3.6239	3.5085	3.6009
	MSE	0.1723	0.1476	0.159	0.1479	0.1913	0.165	0.1839
130	Estimate	3.5309	3.4765	3.4104	3.4633	3.5716	3.5037	3.5581
	MSE	0.0984	0.0899	0.094	0.0901	0.1048	0.0959	0.1023
200	Estimate	3.5198	3.4846	3.4414	3.4759	3.5462	3.5022	3.5374
	MSE	0.0616	0.0582	0.06	0.0582	0.0643	0.0606	0.0633

As shown in Table 1, for sample sizes n=10,25,77,130,200, the MSE values associated with Bayesian estimation using Jeffrey's prior, Gamma prior, and Uniform prior were consistently lower than those obtained through MLE. Specifically, the Bayesian estimation under the quadratic loss function (QLF) yielded MSE values of *0.028, 0.0102, 0.0033, 0.0019*, and *0.0012*, respectively, under Jeffrey's prior. Similarly, the MSE values under Gamma prior were *0.025, 0.009, 0.0033, 0.0019*, and *0.0012*, while those under the Uniform prior were *0.0316, 0.0107, 0.0034, 0.002*, and *0.0012*. These findings indicate that, regardless of the sample size, the Bayesian estimation approach—particularly when applying the QLF—provides a better fit for the dataset compared to the MLE method. The consistency in lower MSE values across different sample sizes suggests that the Bayesian approach is more robust and reliable, with QLF outperforming other loss functions and priors in estimating the parameters of the Exponential-Frechet distribution.

In addition to examining the performance of different estimation methods across various sample sizes, the study also analyzed the impact of changing the shape parameter, θ from 0.5 to 3.5. Table 2 presents the MSE values under these conditions, revealing a pattern similar to that observed in Table 1. The QLF under all priors (Gamma, Uniform, and Jeffrey's) continued to produce lower MSE values than MLE. Specifically, the MSE values under Gamma prior were *1.4163*, *0.5009, 0.1590, 0.0940*, and *0.0600*, respectively, while those under Uniform prior were *1.5491*, *0.5223, 0.1650, 0.0959*, and *0.0606*. Under Jeffrey's prior, the MSE values were *1.3729, 0.5004*, *0.1620, 0.0950*, and *0.0603*, respectively. The fact that QLF consistently yields lower MSE values, even as the shape parameter is varied, suggests that this estimation method is relatively insensitive to changes in the distribution's shape parameter. This insensitivity indicates that the QLF method provides stable and reliable estimates across different parameter values, further establishing its superiority over MLE and other loss functions.

3.2 Applications of the proposed Model

This session presents descriptive statistics of the dataset and application to some selected generalized probability distributions. It has compared the adequacy of the EFrD to that of five other extended models. The models are: Exponential-Lindley distribution (ExLinD) by Ieren and Balogun (2021), extended power Lindley distribution (EPLD) by Alkani (2015), exponentiated power Lindley distribution (ExPLD) by Ashour and Eltehiwy (2015), Lomax inverse exponential distribution (LomINExD) by Abdulkadir *et al*. (2020) as well as the conventional inverse exponential distribution (INExD) by Keller and Kamath (1982).

The model's performance mentioned above was evaluated using minimum information criteria: CAIC (Consistent Akaike Information Criterion), AIC (Akaike Information Criterion), HQIC (Hannan Quin information criterion) and BIC (Bayesian Information Criterion). The following are the formulas for these statistics:

$$AIC = -2ll + 2k$$

$$BIC = -2ll + k\log(n),$$

$$CAIC = -2ll + \frac{2kn}{(n-k-1)}$$

$$HQIC = -2ll + 2k\log\left[\log(n)\right]$$

Where k is the number of model parameters, n is the sample size, ll denotes the log-likelihood function evaluated at the *MLEs*.

Decision bench mark: The model with the lowest values of these statistics would be chosen as the best model to fit the data.

Table 3: The information shows the number of months that a random sample of 128 people with bladder cancer experienced remission.

Parameters	n	Min	Q_1	Median	Q_1	Mean	Max	Variance	Skewness	Kurtosis
Values	128	0.0800	3.348	6.395	11.840	9.366	79.05	110.425	3.3257	19.158

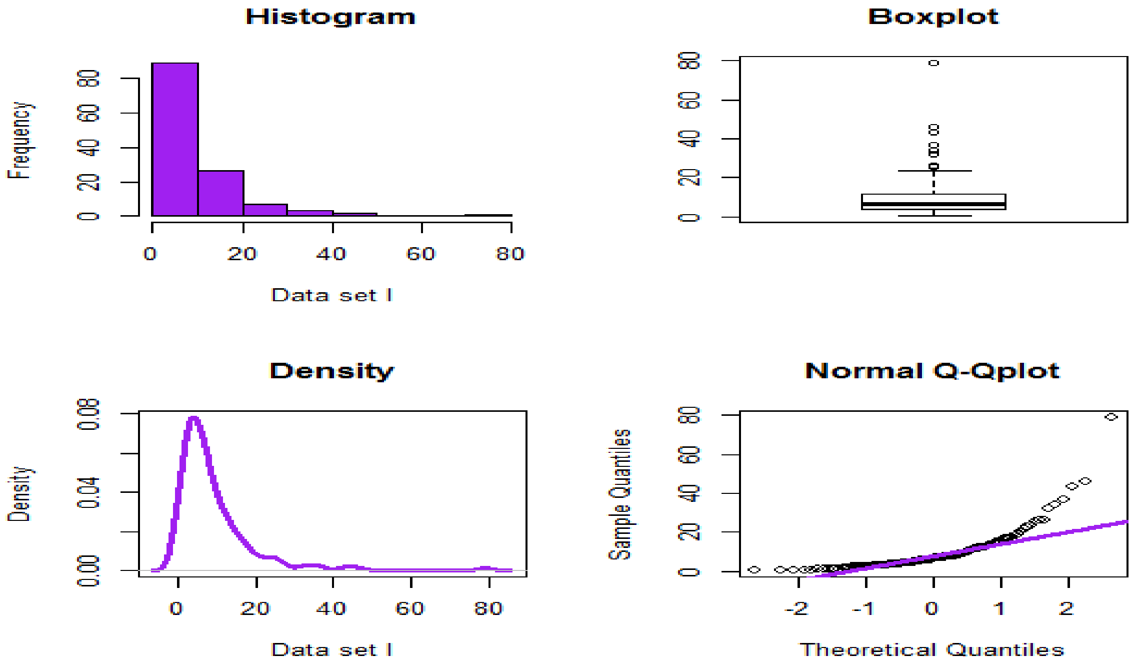

Figure 5: A graphical summary of dataset

Table 4: Estimates of Maximum Likelihood (Dataset)

Distribution	Parameter Estimates			
EFrD	$\hat{\theta}$=4.7162	$\hat{\beta}$=0.0001	$\hat{\alpha}$=3.9325	-
ExLinD	$\hat{\theta}$=9.9639	-	$\hat{\alpha}$=9.9596	-
EPLD	$\hat{\theta}$=0.6624	$\hat{\alpha}$=0.5338	$\hat{\beta}$=8.4146	-
ExPLinD	$\hat{\theta}$=0.8172	$\hat{\alpha}$=2.5439	$\hat{\beta}$=0.6889	-
LomINExD	$\hat{\theta}$= 2.2357	$\hat{\alpha}$= 8.3074	$\hat{\beta}$= 8.5288	-
INExD	$\hat{\theta}$= 2.4847	-	-	-

Table 5: The AIC, statistics ℓ, BIC, HQIC and CAIC for dataset

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	**Ranks**
EFrD	5.39881	16.79762	16.99117	25.35371	20.274	1st
ExLinD	9.414409	22.82882	22.92482	28.53288	25.14641	2rd
ExPLinD	210.4056	426.8111	427.0047	435.3672	430.2875	3th
EPLD	423.6476	853.2953	853.4888	861.8514	856.7717	4th
INExD	460.3823	922.7646	922.7963	925.6166	923.9234	5th
LomINExD	463.9814	933.9629	934.1564	942.519	937.4393	6th

Table 3 presents descriptive of Remission times for bladder cancer patients with mean, variance, skewness, kurtosis (9.3660, 110.4250, 3.3257, 19.1537) signifying the presence of extreme values. Table 4 provide the parameter estimates for various distributions fitted to the dataset. Table 5 present the consistent superiority of the Exponential-Frechet distribution (EFrD) from the dataset underscores the robustness and versatility of these models. The smaller values of CAIC, AIC, HQIC, and BIC for the EFrD indicate that this distributions provide the best balance between goodness of fit and model complexity, making it the most suitable for modeling the remission times of bladder cancer patients. These findings are particularly significant given the diverse nature of the datasets. The ability of the EFrD to consistently outperform other distributions across such varied datasets suggests that these models are not only flexible but also generalizable across different types of data. The results of this study align with theoretical expectations derived from the properties of the Exponential-G family, which is part of the broader Weibull-G family of distributions. The Exponential-G family is known for its ability to generate distributions with enhanced flexibility, particularly in capturing skewness and heavy tails, which are common in real-world data.

4. Conclusion

This study had proposed Exponential-Frechet Distribution, derive its properties and estimate the new shape parameter using both Bayesian and Maximum Likelihood Estimator. It has compared the adequacy of EFrD to that of five other extended models. The AIC, CAIC, BIC AND HQIC revealing that the EFrD exhibits the smallest values. In comparison with Exponential-Lindley distribution (ExLinD), Extended power Lindley distribution (EPLD), exponentiated power Lindley distribution (ExPLD), Inverse exponential distribution (INExD) and Lomax inverse exponential distribution (LomINExD) imply that AIC for Exponential Frechet Distribution is lower than others indicating the best fit. The study has made momentous contributions to the field of Mathematical statistics particularly within the realm of extreme value theory. This distribution can be used in extreme value theory to model the tails of distributions, particularly in fields such as finance, environmental science, and hydrology, also can help to understand the behavior of extreme events (e.g extreme weather, financial crashes) can aid in risk management and mitigation strategies.

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