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Mathematical Model on the Dynamics of Crimes of Murder Incorporating Amnesty and Rehabilitation as Control Strategies

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ABSTRACT

In this paper, a mathematical model for the dynamics of murderous crimes is developed, considering rehabilitation and amnesty as control strategies. The basic features of the model, such as the positivity and boundedness of the model solution, were explored. Three positive equilibrium points were computed: a murder-free equilibrium, an amnesty and rehabilitation-free equilibrium, and a murder-present equilibrium with amnesty and rehabilitation. Using a next-generation matrix approach, a murder threshold quantity, represented by R_0, was computed. This quantity measures the average number of non-murderers that a single murderer will persuade to commit murder. A local stability analysis of the murder-free equilibrium point was conducted, revealing that, under certain conditions, the equilibrium point is locally asymptotically stable. Furthermore, a global stability analysis of the murder-free equilibrium point was carried out, and the result showed that the equilibrium point is globally asymptotically stable, provided that ex-convicts who received amnesty do not commit murder again. Moreover, computational results of the model equations demonstrated that the murder conversion rate, police arrest rate, and amnesty grant rate play important roles in a community's ability to control murder cases.

1. Introduction

Murder can be described as the wilful killing of another person without cause or legitimate explanation, particularly when it involves the unlawful killing of a human being (Section 316 of the Criminal Code Act Cap C38 Laws of the Federation of Nigeria 2004). Murder is defined as the deliberate, knowing, careless, or reckless taking of another person's life. It also implies manslaughter under the criminal code, murder, or culpable homicide that is either not punishable by death or punishable by life under the penal code (Onamade, 2009). The act of a murderer can be used to identify them, such as infanticide, slaying, mass murder, parricide, and gun for hire, cutthroat, or fratricide. According to Kate and Jane (2018), professional killers, like cutthroats, are accountable for a range of fatalities, including slaying and killing. Obiorah and Atanda's (2013) study found that firearms were the most often used weapon in 1,004 homicide incidents in Port Harcourt and 113 in Kano. The availability of weapons, gender roles, religion, socio-cultural disparities, the rule of law, and drug use were all significant variables.

Lisa (2015) found that every prisoner reacts to homicides due to trauma, based on her research on the causes of murder in high-security facilities. The majority of murderers result from trauma, and environmental factors interact with genes. Despite a hereditary propensity for aggression, epigenetic research reveals that experiences impact gene expression. In support of the crime-wealth hypothesis, Adenuga and Nors (2016) study in Nigeria discovered a short-term relationship between crime and poverty. The report recommends that governments adopt policies related to economic growth as a means of reducing both crime and poverty. Ethelbert (2015) analysis of Nigerian homicide rates identified nine subtopics. His recommendations included reducing jail overcrowding, enhancing rehabilitation efforts, and bolstering police resources. Except in some cases, he categorized homicide into legal and illegal categories. The

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ultimate goal is to achieve the lowest feasible level of homicide. Pelumi *et al*. (2018) study on crime patterns in Nigeria highlighted the severe effects of crime on Nigerian society, finding a negative association between murder, armed robbery, assault, manslaughter, bribery, and corruption. Due largely to the government's failure to address the causes, armed robberies have emerged as Nigeria's most common crime, accounting for 50% of fatalities between 2006 and 2015 (Ukoji and Okolie-Osemene, 2016). Emmanuel (2017) study on Nigeria's assembly of the condemned found that, despite the abolition of the death penalty in other countries, it does not deter criminal activity. Instead, the paper suggests addressing societal inequalities and providing rehabilitation for offenders. Odulaja (2013) study on the efficacy of the death penalty in Nigeria found that, although violent crime rates have declined, criminals are still deterred by the death sentence. This suggests that Nigerian governments should prioritize job creation and community policing. According to Barisiagbon and Aderinto (2018) study, there is a significant level of corruption, unfavourable working conditions, and inadequate court facilities in Nigeria. They suggest that the government take measures to combat corruption, such as evaluating the working conditions of officers and remodelling courts. Jashandeep (2019) suggests reducing India's crime rate by using artificial intelligence, management information systems, smart eye glasses, and sharing criminal databases with other jurisdictions. A South African study reveals that courts now consider social psychological factors in murder trials, despite sentencing others to death, highlighting the growing influence of psychology in international legal cases (Andrew, 2011).

To better understand social issues like crime and support political and economic decision-making, mathematical modelling is essential. Compartmental models connect mathematics to modern problems such as diseases, unemployment, crime, and climate change. Using the "broken windows effect" and "prison as a crime school," Jongo and Pilwon (2019) study investigates the dynamics of crime within and outside of prisons, demonstrating the complex relationship between serious and minor crimes. They offer a mathematical model to explain how minor offenders become major offenders, implying that the frequency of significant crimes can be increased by properly monitoring inmates' contacts. Crime models are a recent field of study that combines social studies and mathematics. Following the creation of a statistical model for burglary by Short et al. (2008), both linear and nonlinear models were constructed. Soemarsono *et al*., (2021) added a crime compartment to represent the impact of unemployment on a mathematical model developed by Munoli and Gani (2016). Vocational training and employment solutions are more effective in reducing crime, according to Mataru *et al.,* (2023) study, which employed a deterministic mathematical model to investigate unemployment-crime dynamics. Murder cases have surged in Nigeria and beyond, with an estimated 464,000 victims in 2017. El Salvador has the highest rate of murder at 84.25%, while Nigeria has a rate of 63.86%.

Numerous studies have explored homicide patterns in Nigeria, political assassination, and armed robbery. However, none have utilized mathematical models to study murder dynamics. Therefore, this paper aims to develop a mathematical model to study the dynamics of murder and its possible control strategies. The model incorporates police and judicial actions, as well as the effects of correctional centres. This paper is organized as follows: the model is formulated in Section 2, and analysed in Section 3. The results of the numerical simulations and discussion are presented in Sections 4 and 5, respectively. Finally, the conclusion is provided in Section 6.

2. Model Formulation

2.1 The Main Assumptions of the Model

The following are some of the main assumptions of the model:

(i) The model assumes that suspected murderers may commit suicide.

(ii) Suspected murderers found guilty will face one of the following outcomes: condemnation, life imprisonment, regaining freedom through amnesty, returning to the murderer class, undergoing further investigation, or being converted to lifers through amnesty.

(iii) Only lifers receive training in murder cases while in custodial centers.

2.2 Description of the Model

The total population at time t, denoted by $T(t)$, is divided into ten mutually exclusive classes namely: the non-Murderers, $N(t)$, the suspected Murderers before arrest, $M_s(t)$, suspected Murderers in police custody, $P_c(t)$, the arrested murder suspect in court for trial, $C(t)$, , suspected Murderers in custodial centre awaiting trial, $A_T(t)$, condemned Murderer waiting for execution, $C_c(t)$, Lifers, $L(t)$, released Lifers, $R(t)$, unskilled ex-convict, $U_x(t)$ and skilled ex-convict, $S_x(t)$. Therefore, the total population is given by

$$T(t)=N(t)+M_s(t)+P_c(t)+C(t)+A_T(t)+C_c(t)+L(t)+R(t)+L(t)+U_x(t)+S_x(t) \qquad (1)$$

The non-Murderers' class is generated by a constant recruitment rate μ and by those who were found innocent (discharge and acquitted by the court) at the rate φ. The non-Murderers sub-population is reduced due to natural death at the rate θ which is assumed the same for all the classes, it further reduces by those who resort to crime of murder after getting contact with a Murderer at the rate α. The Murderer sub-population is increased by those who become Murderer at the rate α and the unskilled and skilled ex-convict at rates ε_1 and ε_2 respectively, and reduces by those who were arrested by police at the rate η. The Police custody sub population increases as a result of police arrest and those who were brought from court for further investigation at the rate η and τ respectively, it reduces by those taken to court at the rate β. The Murderers in court for trials sub population increases by those who were brought from police custody for trial at the rate β and the sub-population decreases by those kept as awaiting trial and by those who were taken back to police for further investigation at the rate γ and τ respectively. The awaiting trial sub-population is generated by those who were brought from court as awaiting trial and it reduces by those that were discharged and acquitted at the rate φ, it further reduces by those that were sentenced to death at the rate π as well as those that were sentenced to life imprisonment at the rate δ. Furthermore, the condemned criminal sub population increases by those who were sentenced to death at the rate π and it reduces by those who were executed at the rate ρ and by those who were granted amnesty at the rate ϕ. The lifer sub-population increases by those who were sentenced to life imprisonment at the rate δ and also increases by the condemned Murderers who were granted amnesty at the rate ϕ and further reduces by those who were released by amnesty at the rate ξ. Similarly, the released sub-population increases by the lifers who were granted amnesty at the rate ξ and reduces by the unskilled and skilled ex-convict at the rate $q(1-\psi_s)$ and $q\psi_s$ respectively. Also, the unskilled ex-convict sub-population increases by the unskilled beneficiaries of amnesty at the rate $q(1-\psi_s)$ and reduces by those who commit murder again at the rate ε_1. Finally, the skilled ex- convict sub-population increases by the skilled beneficiaries of amnesty at the rate $q\psi_s$ and reduces by those who commit murder again at the rate ε_2.

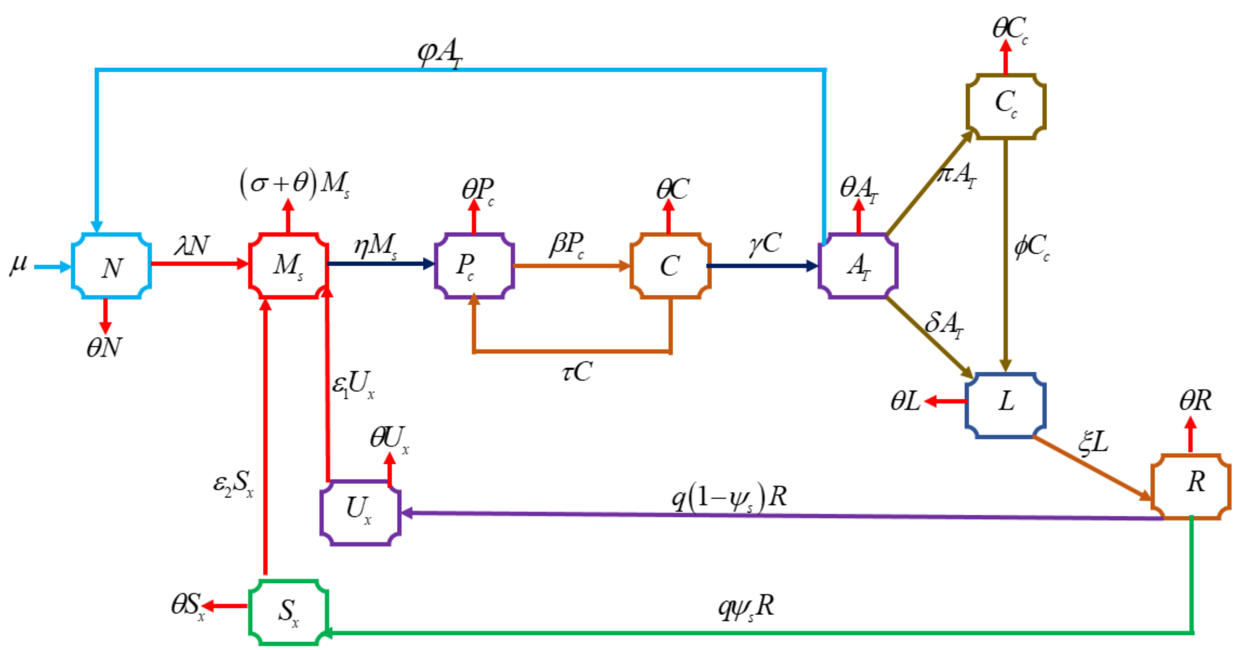

Figure 1: Schematic diagram of the model

2.3 The Model Equations

Based on the model assumptions, descriptions of the model state variables and parameters provided in Tables 1 and 2, and the model description and schematic diagram, we obtain the following system of first-order nonlinear ordinary differential equations:

$$\frac{dN}{dt} = \mu + \varphi A_T - (\theta + \lambda) N, \tag{2}$$

$$\frac{dM_s}{dt} = \lambda N + \varepsilon_1 U_x + \varepsilon_2 S_x - (\eta + \theta + \sigma) M_s, \tag{3}$$

$$\frac{dP_c}{dt} = \eta M_s + \tau C - (\theta + \beta) P_c, \tag{4}$$

$$\frac{dC}{dt} = \beta P_c - (\theta + \gamma + \tau) C, \tag{5}$$

$$\frac{dA_T}{dt} = \gamma C - (\theta + \varphi + \pi + \delta) A_T, \tag{6}$$

$$\frac{dC_c}{dt} = \pi A_T - (\theta + \rho + \phi) C_c, \tag{7}$$

$$\frac{dL}{dt} = \delta A_T + \phi C_c - (\theta + \xi) L, \tag{8}$$

$$\frac{dR}{dt} = \xi L - (q + \theta) R, \tag{9}$$

$$\frac{dU_x}{dt} = q(1 - \psi_s) R - (\theta + \varepsilon_1) U_x, \tag{10}$$

$$\frac{dS_x}{dt} = q \psi_s R - (\theta + \varepsilon_2) S_x, \tag{11}$$

$$T = N + M_s + P_c + C + A_T + C_c + L + R + U_x + S_x, \tag{12}$$

$$\lambda = \frac{\alpha M}{T},$$

with the following initial conditions:

$$N(0) > 0, M_s(0) \geq 0, P_c(0) \geq 0, C(0) \geq 0, A_T(0) \geq 0, C_c(0) \geq 0, L(0) \geq 0, R(0) \geq 0, U_x(0) \geq 0, S_x(0) \geq 0. \tag{13}$$

Table 1: Description of state variables of the model

Symbol	Description
$N(t)$	Number of non-Murderers at time (t)
$M_s(t)$	Number of suspected Murderers before arrest at time (t)
$P_c(t)$	Number of suspected Murderers in police custody at time (t)
$C(t)$	Number of arrested Murder suspect in court for trial at time (t)
A_T(t)	Number of suspected Murderers in custodial centre as awaiting trial at time (t)
$C_c(t)$	Number of condemned Murderer waiting for execution at time (t)
L(t)	Number of Murderer sentenced for life imprisonment (lifer) at time (t)
R(t)	Number of released lifers at time (t)
U_x(t)	Number of unskilled ex-convict at time (t)
S_x(t)	Number of skilled ex-convict at time (t)

Table 2: Description of parameters of the model

Parameter	Description
μ	Recruitment rate into the non-Murderers compartment
θ	Natural mortality rate
α	Effective murder conversion rate
η	The rate at which suspected Murderers are arrested by the police
σ	Rate at which Murderers commit suicide before being arrested
β	Rate at which murder suspect in police custody are taking to trial in court
τ	Rate at which Murderers return to police custody for further investigation
γ	Rate at which Murderers are taken to custodial centre
φ	Rate at which suspected Murderers are released from custody
π	Rate at which Murderers are condemned (sentenced to death)
δ	Rate at which Murderers are sentenced to life imprisonment
ρ	Rate at which the Murderers are executed
ϕ	Rate at which a condemned Murderer is converted to lifer
ξ	Rate at which a lifer is pardoned by amnesty
q	Transition rate from the release class to either skilled or unskilled
ψ_S	Proportion of release Murderer who acquired skilled
ε_1	The rate of recidivism for unskilled ex-convict
ε_2	The rate of recidivism for skilled ex-convict

3. Model Analysis

3. 1 *Positivity of the Model Solution*

The Positivity of the model solution which is one of the basic properties of the model will be explored using the following theorem:

Theorem 1

Solution of the model equations given by system [(2) – (12)] with positive initial data will remain positive for all time $t>0$.

Proof:

Let $t^* = \{t>0:\ N>0, M_s>0, P_c>0, C>0, A_T>0, C_c>0, L>0, R>0, U_x>0, S_x>0\} \in [0,t]$.

Therefore, $t^*>0$. Considering the first equation of the model i.e. equation (2) we have

$$\frac{dN}{dt} = \mu + \varphi A_T - (\theta+\lambda)N.$$

This can be rewritten as

$$\frac{dN}{dt} \geq -(\theta+\lambda)N.$$

It follows that

$$\frac{N(t^*)}{N(0)} \geq e^{-\left[\theta t^* + \int_0^{t^*} \lambda(u)du\right]} \text{ or } N(t^*) \geq N(0)e^{-\left[\theta t^* + \int_0^{t^*} \lambda(u)du\right]} > 0.$$ Thus, $N(t)>0$ for all $t>0$.

Using similar approach, it has been shown that $P_c(t)>0,\ A_T(t)>0, C_c(t)>0, L(t)>0,\ R(t)>0, U_x(t)>0,$ $S_x(t)>0$, for $t>0$. Hence, the proof is complete.

3.2 Invariant Region of the Model

The invariant region of the model will be established using the following theorem:

Theorem 2

The closed set $\Omega = \left\{ \left(N, M_s, P_c, C, A_T, C_c, L, R, U_x, S_x \right) \in R_+^{10} : T \le \frac{\mu}{\theta} \right\}$ is positively invariant.

Proof:
Adding equations [(2) - (12)] we obtain

$$\frac{dT}{dt} = \mu - \theta N - \theta M_s - \sigma M_s - \theta P_c - \theta C - \theta A_T - \theta C_c - \theta L - \theta R - \theta U_x - \theta S_x - \rho C_c ,$$

$$\frac{dT}{dt} \le \mu - \theta T .$$

Using integrating factor method and further simplification yield,

$$T(t) \le \frac{\mu}{\theta} + T(0) e^{-\theta t} - \frac{\mu e^{-\theta t}}{\theta},$$

$$\le T(0) \mathrm{e}^{-\theta t} + \frac{\mu}{\theta}\left[1 - e^{-\theta t} \right].$$

Therefore, $\lim_{t \to \infty} \sup T(t) \le \frac{\mu}{\theta}$. Hence, the closed set Ω is positively invariant, hence, it is satisfactory to study the dynamics of the model in this region.

3.3 Equilibrium points of the model

The model has three positive equilibrium points which were obtained by first setting the right hand sides of the model equations given by system [(2)-(12)] to zero and solve for the state variables.

3.3.1 Murder-free equilibrium point of the model

In the absence of murder we set $M_s = P_c = C = A_T = C_c = L = R = U_x = S_x = 0$.

Thus, equations (2) – (12) reduced to

$$\mu - \theta N = 0, \Rightarrow N = \frac{\mu}{\theta}.$$

Let E_0 denotes the Murder-free Equilibrium point (MFE) so that,

$$E_0 = \left(N^0, M_s^0, \mathrm{P}_c^0, \mathrm{C}^0, \mathrm{A}_T^0, C_c^0, L^0, R^0, U_x^0, S_x^0 \right) = \left(\frac{\mu}{\theta}, 0,0,0,0,0,0,0,0,0 \right).$$

3.3.2 Amnesty and rehabilitation-free equilibrium point of the model

In the absence of amnesty and rehabilitation we set $R^* = U_x^* = S_x^* = 0$, in the model equations we obtain the amnesty and rehabilitation-free equilibrium point denoted by E_1 and is presented as follows:

$$E_1 = \left(N^*, M_s^*, P_c^*, \mathrm{C}^*, A_T^*, C_c^*, L^*, 0,0,0 \right),$$

where

$N^* = \frac{T^*(\eta + \theta + \sigma)}{\alpha}$, $M_s^* = \frac{\left(\mu + \varphi A_T^* - \theta N^* \right) T^*}{\alpha N^*}$ with $\mu + \varphi A_T^* > \theta N^*$, $P_c^* = \frac{\eta M_s^* + \tau C^*}{\theta + \beta}$,

$C^* = \frac{\beta P_c^*}{\theta + \gamma + \tau}$, $\mathrm{A}_T^* = \frac{\gamma C^*}{\theta + \varphi + \pi + \delta}$, $C_c^* = \frac{\pi A_T^*}{\theta + \rho + \phi}$ and $\mathrm{L}^* = \frac{\delta A_T^* + \phi C_c^*}{\theta + \xi}$.

3.3.3 Amnesty and rehabilitation present equilibrium point of the model

The amnesty and rehabilitation present equilibrium point denoted by E_2 is given by

$$E_2 = \left(N^{**}, M_s^{**}, P_c^{**}, C^{**}, A_T^{**}, C_c^{*}, L^{**}, R^{**}, U_x^{**}, S_x^{**}\right),$$

where

$$N^{**} = \left(\frac{\mu + \varphi A_T^{**}}{\theta T^{**} + \alpha M_s^{**}}\right) T^{**},\ M_s^{**} = \frac{T^{**}\left(\varepsilon_1 U_x^{**} + \varepsilon_2 S_x^{**}\right)}{T^{**}\left(\eta + \theta + \sigma\right) - \alpha N^{**}},\ \ T^{**}\left(\eta + \theta + \sigma\right) > \alpha N^{**},$$

$$P_c^{**} = \frac{\eta M_s^{**} + \tau C^{**}}{\theta + \beta},\ C^{**} = \frac{\beta P_c^{**}}{\theta + \gamma + \tau},\ A_T^{**} = \frac{\gamma C^{**}}{\theta + \varphi + \pi + \delta},\ C_c^{**} = \frac{\pi A_T^{**}}{\theta + \rho + \phi},\ L^{**} = \frac{\delta A_T^{**} + \phi C_c^{**}}{\theta + \xi},$$

$$R^{**} = \frac{\xi L^{**}}{q + \theta},\ U_x^{**} = \frac{q(1 - \psi_s)\mathrm{R}^{**}}{\left(\theta + \varepsilon_1\right)},\ \text{and } S_x^{**} = \frac{q\psi_s \mathrm{R}^{**}}{\left(\theta + \varepsilon_2\right)}.$$

3.4 Basic Reproduction Number (Murder Threshold Quantity)

To determine the Murder threshold quantity, we consider the murder-related equations [(3)-(8)]. Let F_i denotes the rate of appearance of new Murderer in the compartment i and V_i denotes the rate of transfer of individuals out of compartment i, given the murder-free equilibrium point, then R_0 is defined as the spectral radius (largest eigenvalue) of the next generation matrix denoted by $R_0 = FV^{-1}$ or $R_0 = \rho\left(FV^{-1}\right)$,

where

$$F = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{\Theta_1} & 0 & 0 & 0 & 0 & 0 \\ \frac{\eta\Theta_3}{\beta\tau\Theta_1 + \Theta_1\Theta_2\Theta_3} & \frac{\Theta_3}{\beta\tau + \Theta_2\Theta_3} & \frac{\tau}{\beta\tau + \Theta_2\Theta_3} & 0 & 0 & 0 \\ \frac{\beta\eta}{\beta\tau\Theta_1 + \Theta_1\Theta_2\Theta_3} & \frac{\beta}{\beta\tau + \Theta_2\Theta_3} & \frac{\Theta_2}{\beta\tau + \Theta_2\Theta_3} & 0 & 0 & 0 \\ \frac{\beta\gamma\eta}{\Theta_1\Theta_2\Theta_3 + \beta\tau\Theta_1\Theta_4} & \frac{\beta\gamma}{\beta\tau\Theta_4 + \Theta_2\Theta_3\Theta_4} & \frac{\gamma\Theta_2}{\beta\tau\Theta_4 + \Theta_2\Theta_3\Theta_4} & \frac{1}{\Theta_4} & 0 & 0 \\ \frac{\pi\beta\gamma\eta}{\beta\tau\Theta_1\Theta_4\Theta_5 + \Theta_1\Theta_2\Theta_3\Theta_5} & \frac{\pi\beta\gamma}{\Theta_2\Theta_3\Theta_4\Theta_5 + \beta\tau\Theta_4\Theta_5} & \frac{\pi\gamma\Theta_2}{\Theta_2\Theta_3\Theta_4\Theta_5 + \beta\tau\Theta_4\Theta_5} & \frac{\pi}{\Theta_4\Theta_5} & \frac{1}{\Theta_5} & 0 \\ \frac{\beta\gamma\delta\eta\Theta_5 - \pi\beta\gamma\phi\eta}{\Theta_6(\beta\tau\Theta_1\Theta_4\Theta_5 + \Theta_1\Theta_2\Theta_3\Theta_4\Theta_5)} & \frac{\pi\beta\gamma\phi - \beta\gamma\delta\Theta_5}{\Theta_6(\Theta_2\Theta_3\Theta_4\Theta_5 + \beta\tau\Theta_4\Theta_5)} & \frac{\pi\gamma\phi\Theta_2 - \gamma\delta\Theta_2\Theta_5}{\Theta_6(\Theta_2\Theta_3\Theta_4\Theta_5 + \beta\tau\Theta_4\Theta_5)} & \frac{(\pi\phi - \delta\Theta_5)}{\Theta_4\Theta_6} & \frac{\phi}{\Theta_5\Theta_6} & \frac{1}{\Theta_6} \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} \frac{\alpha}{(\eta + \theta + \sigma)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $R_0 = \dfrac{\alpha}{(\eta + \theta + \sigma)}$.

Remarks:

In epidemiology, the basic reproduction number represents the number of secondary cases that a single infected person will generate in a totally susceptible population. In this work, we refer to the basic reproduction number as a "murder threshold quantity," which measures the number of non-murderers that a single murderer will persuade to commit murder in a total non-murderer population.

3.5 Local Stability of the Murder-free Equilibrium Point

The local stability of the murder-free equilibrium point of model equations given by system [(2) – (12)] can be established by showing that all the eigenvalues of the Jacobian matrix evaluated at the murder-free equilibrium point, E_0 , have negative real parts. The local stability of the murder-free equilibrium point will be achieved through the following theorem:

Theorem 3

The murder-free equilibrium point given by $\left(E_0\right)$, is locally asymptotically stable provided that the following inequalities hold:

(i) $R_0 < 1$.

(ii) $(\theta+\beta)^2 > 2(\theta+\beta)(\theta+\gamma+\tau)+(\theta+\gamma+\tau)^2+4\beta\tau$.

(iii) $(\theta+\beta)+(\theta+\gamma+\tau) > \sqrt{(\theta+\beta)^2-2(\theta+\beta)(\theta+\gamma+\tau)+(\theta+\gamma+\tau)^2+4\beta\tau}$.

Proof:

Evaluating the Jacobean matrix of the model equations at the murder-free equilibrium, we obtain

$$J(E_0)=\begin{bmatrix} -\theta & -\alpha & 0 & 0 & \varphi & 0 & 0 & 0 & 0 & 0 \\ 0 & \Theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_1 & \varepsilon_2 \\ 0 & \eta & \Theta_2 & \tau & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & \Theta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & \Theta_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi & \Theta_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & \phi & \Theta_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi & \Theta_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Theta_8 & -\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q\psi_s & 0 & -\theta \end{bmatrix},$$

where $\Theta_1=\alpha-(\eta+\theta+\sigma)$, $\Theta_2=-(\theta+\beta)$, $\Theta_3=-(\theta+\gamma+\tau)$, $\Theta_4=-(\theta+\varphi+\pi+\delta)$, $\Theta_5=-(\theta+\rho+\phi)$, $\Theta_6=-(\theta+\xi)$, $\Theta_7=-(q+\theta)$ *and* $\Theta_8=q(1-\psi_s)$

The characteristic polynomial is given by

$$\left|J\left(E_0\right)-\lambda I\right|=0 \tag{14}$$

Thus, solving the characteristic equations given by equation (14) we obtained the following eigenvalues:

$$\begin{gathered} \lambda_1=\lambda_2=\lambda_3=-\theta, \lambda_4=\Theta_5, \\ \lambda_5=\Theta_4, \lambda_6=\Theta_7, \lambda_7=\Theta_6, \lambda_8=\Theta_1 \\ \lambda_9=\frac{1}{2}\left(\Theta_2+\Theta_3+\sqrt{\Theta_2^2-2\Theta_2\Theta_3+\Theta_3^2+4\beta\tau}\right) \\ \lambda_{10}=\frac{1}{2}\left(\Theta_2+\Theta_3-\sqrt{\Theta_2^2-2\Theta_2\Theta_3+\Theta_3^2+4\beta\tau}\right) \end{gathered} \tag{15}$$

From equation (15) it clear that the eigenvalues $\lambda_i < 0, \ (\mathrm{i}=1, 2, \ldots, 7)$ since, all the model parameters are positive. For $\lambda_8 < 0$, we must have $\alpha < (\eta+\theta+\sigma)$ which implies $R_0 < 1$. Furthermore, for $\lambda_i \left(i = 9,10\right)$ to satisfy the negativity requirements for local stability, the following conditions must hold:

(i) $(\theta+\beta)^2 > 2(\theta+\beta)(\theta+\gamma+\tau)+(\theta+\gamma+\tau)^2+4\beta\tau$,

(ii) $(\theta+\beta)+(\theta+\gamma+\tau) > \sqrt{(\theta+\beta)^2-2(\theta+\beta)(\theta+\gamma+\tau)+(\theta+\gamma+\tau)^2+4\beta\tau}$.

Theorem 4

The murder-free equilibrium point (E_0) of the model is globally asymptotically stable (GAS) provided that $R_0 < 1$ and $\varepsilon_1 = \varepsilon_2 = 0$, otherwise is unstable.

Proof:

Following Castillo-Chavez *et al.,* (2002), the model equations given by [(2) - (12)] is rewritten as

$$\frac{dX}{dt} = H(X, \mathrm{Z}),$$

$$\frac{dZ}{dt} = G(X, Z), G(X, 0) = 0,$$

with the element of $X = (\mathrm{N}, R, \mathrm{U}_x, S_x) \in \mathbb{R}^4$ denotes the population of non-Murderers related components and $Z = (\mathrm{M}_s, \mathrm{P}_c, \mathrm{C}, \mathrm{A}_T, \mathrm{C}_c, \mathrm{L}) \in \mathbb{R}^6$ denotes the population of Murderers related components. The murder-free equilibrium point is now denoted by $\mathrm{E}_0 = (X_0, 0)$, where $X_0 = \left(\frac{\mu}{\theta}, 0, 0, 0\right)$. We have to prove that the following two conditions are satisfied:

(i) For $\frac{dX}{dt} = H(\mathrm{X}, 0), \mathrm{X}_0$ is globally asymptotically stable,

(ii) $G(\mathrm{X}, \mathrm{Z}) = PZ - \hat{G}(\mathrm{X}, \mathrm{Z}), \quad \hat{G}(\mathrm{X}, \mathrm{Z}) \geq 0$, for $(\mathrm{X}, \mathrm{Z}) \in \Omega$.

Where $P = \Delta_z G(\mathrm{X}_0, 0)$ is an M-matrix (the off-diagonal elements of P are non-negative) and Ω is the region where the model makes biological sense. Let

$$\frac{dX}{dt} = H(X, Z) = \begin{bmatrix} \mu + \varphi A_T - \left(\theta + \frac{\alpha M_s}{T}\right) N \\ \xi L - (q + \theta)) R \\ q(1 - \psi_s) R - \left(\theta + \frac{\varepsilon \alpha N M_s}{T}\right) U_x \\ q(\psi)_s R - \left(\theta + \frac{\in \alpha N M_s}{T}\right) S_x \end{bmatrix},$$

So that

$$H(\mathrm{X}, 0) = \begin{pmatrix} \mu - \theta N \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (16)$$

Considering the first component of system (16) we have

$$\frac{dN}{dt} = \mu - \theta N. \tag{17}$$

Rewriting equation (15) we have

$$\frac{dN}{dt} + \theta N = \mu \tag{18}$$

Applying integrating factor method we obtain

$$N(t) = \frac{\mu}{\theta} + \left(N(0) - \frac{\mu}{\theta} \right) e^{-\theta t}.$$

As $t \to \infty$ $N(t) \to \frac{\mu}{\theta}$, similarly, it can be verified that $R(t) \to 0, U_x(t) \to 0, S_x(t) \to 0$, as $t \to \infty$, irrespective of the initial values of the model equations. Hence, $X_0 = \left(\frac{\mu}{\theta}, 0, 0, 0 \right)$ is globally asymptotically stable in the region Ω. Thus, the first condition is satisfied. For the second condition we have $G(X,Z) = PZ - \hat{G}(X,Z)$, we wish to show that $\hat{G}(X,Z) \geq 0$. Let $P = J[G(X,Z)]$, where *J* is the Jacobian matrix of $G(X,Z)$ evaluated at the Murder-free equilibrium point. Thus, we have

$$P = \begin{bmatrix} \frac{\alpha N}{T} - (\eta+\theta+\sigma) & 0 & 0 & 0 & 0 & 0 \\ \eta & -(\theta+\beta) & \tau & 0 & 0 & 0 \\ 0 & \beta & -(\theta+\gamma+\tau) & 0 & 0 & 0 \\ 0 & 0 & \gamma & -(\theta+\varphi+\pi+\delta) & 0 & 0 \\ 0 & 0 & 0 & \pi & -(\theta+\rho+\phi) & 0 \\ 0 & 0 & 0 & \delta & \phi & -(\theta+\xi) \end{bmatrix}, PZ = \begin{bmatrix} \frac{\alpha N M_s}{T} - (\eta+\theta+\sigma) M_s \\ \eta M_s - (\theta+\beta) P_s + \tau C \\ \beta P_C - (\theta+\gamma+\tau) C \\ \gamma C - (\theta+\varphi+\pi+\delta) A_T \\ \pi A_T - (\theta+\rho+\phi) C_c \\ \delta A_T + \phi C_C - (\theta+\xi) L \end{bmatrix}$$

$$G(X,Z) = PZ - G(X,Z). \tag{19}$$

Now, evaluating equation (19), it follows that

$$G(X,Z)=\begin{bmatrix} -(\varepsilon_1 U_x+\varepsilon_2 S_x) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{20}$$

From equation (20), we have $G(X,Z)\geq 0$ if and only if $\varepsilon_1=\varepsilon_2=0$. Hence, conditions (ii) is also satisfied. Therefore, we conclude that murder-free equilibrium point is globally asymptotically stable provided that $\varepsilon_1=\varepsilon_2=0$

Remark:

Theorem 4 is physically interpreted to mean that a community can be free from murder crime provided that the ex-convicts will not commit murder again, (i.e. if $\varepsilon_1=\varepsilon_2=0$).

4. Results of the Numerical Simulations

To investigate the behaviour of the system and the effects of the effective murder conversion rate, arrest rate, and amnesty grant rate on the murder population, we conducted various numerical experiments using MATLAB. The baseline values of the state variables and model parameters, as given in Tables 3 and 4, were used in the numerical simulations. The results are presented in Figures 2, 3, and 4.

Table 3: Baseline Values for the State Variable of the Model

Variables	Values	Reference
$N(t)$	2,365,040	CIA World fact book (2018)
$M_s(t)$	3000	Assumed
$P_c(t)$	825	Assumed
$C(t)$	800	Assumed
$A_T(t)$	745	(NCoS, 2019)
$C_c(t)$	301	(NCoS, 2019)
$L(t)$	21	(NCoS, 2019)
$R(t)$	2	(NCoS, 2019)
$U_x(t)$	0	(NCoS, 2019)
$S_x(t)$	2	(NCoS, 2019)

Table 4: Baseline Values for the Model Parameters

Parameters	Values	Reference
μ	25000	Assumed
θ	0.0125	(CIA, 2018)
α	0.2	(NCoS, 2019)
σ	0.0004	(NCoS, 2019)
η	0.2	(NCoS, 2019)
β	0.97	(NCoS, 2019)
τ	0.931	(NCoS, 2019)
γ	0.06	(NCoS, 2019)
π	0.404	(NCoS, 2019)
δ	0.282	(NCoS, 2019)
ϕ	0.07	(NCoS, 2019)
ρ	0.007	(NCoS, 2019)
ζ	0.095	(NCoS, 2019)
ψ_s	0.92	(NCoS, 2019)
φ	0.05	(NCoS, 2019)
q	0.0062	(NCoS, 2019)
ε_1	0.08	Assumed
ε_2	0.04	Assumed

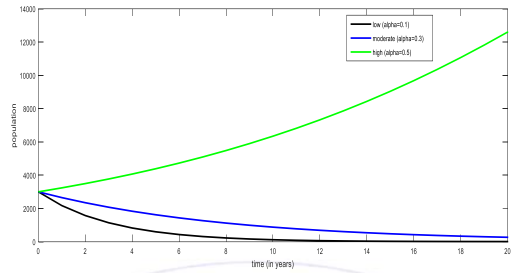

Figure 2: Simulations result showing the effect of effective murder conversion rate $(\alpha = 0.1, 0.3, 0.5)$ on the population Murderers.

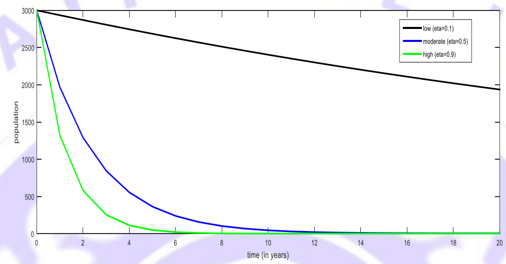

Figure 3: Simulations result showing the effect of the rate at which suspected Murderers are arrested $(\eta = 0.1,\ 0.5, 0.9)$ on the population Murderers.

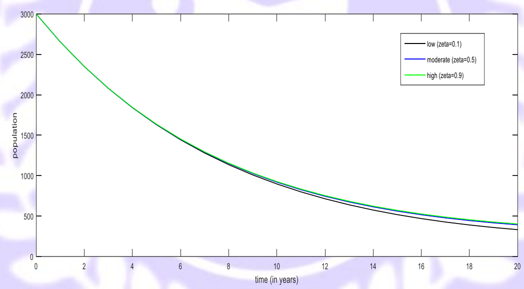

Figure 4: Simulation result showing the effect of the rate at which amnesty are granted to the lifers $(\zeta = 0.1, 0.5, 0.9)$ on the population Murderers.

5. Discussion

Figure 2 illustrates how the murderer population is affected when the effective murder conversion rate is varied. The graph clearly shows that an increase in the effective murder conversion rate leads to an increase in the population of murderers. Therefore, any control strategy that reduces the effective murder conversion rate will be effective in combating murder crime. As shown in Figure 2, reducing the effective murder conversion rate to 10% results in the population of murderers approaching zero after 10 years. This finding suggests that achieving a community free of murder depends on lowering the murder conversion rate. Figure 3 depicts the impact of the arrest rate of suspected murderers on the population of murderers. The findings indicate that increasing the arrest rate of suspected murderers leads to a decrease in the number of murderers. This result highlights the critical role that arrests play in combating homicide.

According to the simulation results, a 10% arrest rate leads to a decrease in the murder population from 3000 to 1900 over 20 years. A 50% arrest rate results in the population approaching zero in 12 years. Therefore, a society free of murder is attainable with a high arrest rate of suspected murderers. Figure 4 shows the impact of the amnesty grant rate on the population of murderers. The results indicate that there is no discernible difference in the number of murderers among those who have received amnesty. According to available data, amnesty is extremely rare, with only two murderers receiving it between 1998 and 2018. Although the effects of amnesty on the murderers' population are negligible, it may encourage recidivism.

6. Conclusion

This paper presents a mathematical model for the dynamics of murderers, incorporating rehabilitation and amnesty as control strategies. The linearization approach was employed to conduct a local stability analysis of the murder-free equilibrium point of the model. The results of the local stability analysis of the murder-free equilibrium indicate that the equilibrium point is locally asymptotically stable under certain conditions. Furthermore, a global stability analysis of the murder-free equilibrium was conducted using the Castillo-Chavez (2002) approach, revealing that the equilibrium point is globally asymptotically stable in the absence of recidivism. Numerical simulations demonstrated that reducing and eradicating murder in a given community critically depend on the following factors: decreasing the effective murder conversion rate, increasing the rate of arrest for murderers, and decreasing the rate of amnesty grants.

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