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Formulation of Variable Step Block Backward Differentiation Formula for the Computation of Nonlinear Fuzzy Differential Equations

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ABSTRACT

The research paper formulates a variable step block backward differentiation formula (VSBBDF) for solving nonlinear fuzzy differential equations (FDEs). Developed to address uncertainties within differential equations by using fuzzy environments, VSBBDF offers a flexible approach to solve equations with triangular fuzzy numbers. This method incorporates a dynamic step-size selection, allowing it to adapt to changes and reduce computational costs while maintaining accuracy. The paper further discusses the stability and convergence properties of VSBBDF, demonstrating that it is both zero-stable and of high accuracy. The results obtained highlight the method's computational efficiency and reliability, particularly when compared with other existing methods for solving nonlinear FDEs.

1. Introduction

FDEs, a result of integrating fuzzy set theory into differential equations, are essential for handling uncertainties inherent in real-world applications. Variables, parameters, and initial conditions in differential equations often involve uncertainties due to measurement errors or natural variability. Fuzzy set theory, pioneered by Zadeh in 1965, enables the inclusion of such uncertainties, allowing more flexible modeling through FDEs. This approach is especially useful in fields where parameters are not strictly deterministic, such as biological systems, engineering, and finance. Nonlinear FDEs are solved numerically due to the complexity of obtaining analytical solutions. Several methods exist for solving FDEs, including hybrid block methods, extended Runge-Kutta methods, and predictor-corrector formulas.

The variables, parameters and initial conditions within a model in most cases are assumed to be exact. However, in reality, these initial conditions may contain errors like observation error, experiment error or measurement error. According to Mallak *et al*. (2022), this usually results in incorrect, incomplete or even fuzzy information. To overrule or circumvent such ambiguity or uncertainties, fuzzy environments can be used in the variables, parameters and initial conditions instead of fixed (exact) ones by simply transforming the differential equation into FDEs.

This research focuses on the formulation of VSBBDF for the computation of nonlinear FDEs of the form,

$$y'(t) = f\left(t, y(t)\right),\ y(t_0) = y_0,\ t \in [t_0, T], \tag{1}$$

where $f : I \times E \to E$ is a continuous fuzzy mapping (fuzzy function of the crisp variable t), y is fuzzy variable, y' is fuzzy derivative and y_0 is initial fuzzy number (interval). Note that E is a set of fuzzy numbers and I is an interval defined as $I = \left[0, T\right]$ for $T > 0$ or $I = \left[0, \infty\right)$.

FDEs find applications in various fields of human endeavours, such as the study of barometric pressure models, decay models, population dynamics models, hyperchaotic systems, friction models, control design, cooling problems, Amarti *et al*. (2018). Researchers have also formulated so many methods for the solution of FDEs.

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These methods include Runge-Kutta methods (Ramli et al., 2016); Adomian decomposition method (Biswan et al., 2016); homotopy perturbation method (Sekar and Sakthieval, 2015); Simpson rule (Devi and Ganesa, 2016); cubic spline method (Karpagappriya et al., 2021); extension principle method (Dogan et al., 2023), among others. The following authors also formulated block methods for the direct solution of systems of FDEs directly, Wu et al. (2021), Shams et al. (2022), Sunday (2018), Isa et al. (2018), Onyekonwu et al. (2024), Hussain et al. (2023), among others. One of the advantages of block methods is that they simultaneously generate approximate solutions at several points within the interval of integration. The hybrid block method has an additional advantage of increasing the order of accuracy of the method while maintaining a small step number (Sunday et al. 2015; Soomro et al. 2022; Kamoh, Kumleng and Sunday, 2021). The proposed VSBBDF which is in predictor-corrector mode, is in the form of hybrid block method.

2. Preliminaries

2.1 Basic Concepts

In this section, some basic concepts and notations that are relevant to this study will be defined.

Definition 2.1.1 (Fuzzy Number): Let $u:\Re\rightarrow[0,1]$ be a mapping of fuzzy number, where $\Re$ is a set of real numbers. A fuzzy set u is called a fuzzy number if it possesses the following properties:

(i) u is normal; that is, there exists $x_0\in\Re$ such that $u(x_0)=1$,

(ii) u is convex; that is, for all $x,y\in\Re$ and $\lambda\in[0,1]$, the condition $u(\lambda x+(1-\lambda)y)\geq\min(u(x),u(y))$ holds,

(iii) u is upper semi-continuous; that is, for $x_0\in\Re$, the condition $u(x_0)\geq\lim\limits_{x\to x_0^{\pm}}u(x)$ holds, and

(iv) the support of u must be bounded, that is, $\operatorname{supp}u=\{x\in\Re|u(x)>0\}$. Also, its closure $cl(\operatorname{supp}u)$ is compact, (Isa et al., 2018).

In clear terms, a fuzzy number is an extension of a regular number in the sense that it does not refer to one single value but rather a connected set of possible values where each possible value has its own weight between 0 and 1. This weight is called the membership function.

Definition 2.1.2 (α-cut): Let E be the set of all fuzzy numbers on $\Re$. The α-level (or α-cut) set of a fuzzy number $u\in E$, $\alpha\in[0,1]$ denoted by $[u]_\alpha$ is a crisp set that contains all elements of E that have membership values in u greater than or equal to α. That is,

$$[u]_\alpha=\begin{cases}\{x\in\Re|u(x)>\alpha\}, & \alpha\in[0,1]\\ cl(\operatorname{supp}u), & \alpha=0\end{cases}$$

with its closed, bounded interval $[\underline{u}(x),\overline{u}(x)]$. This is also called the interval of confidence at level α or α-cut. Note that $\underline{u}(x)$ and $\overline{u}(x)$ denotes lower bound (left-hand endpoint) and upper bound (right-hand endpoint) of $[u]_\alpha$ respectively, (Hussain et al., 2023).

Definition 2.1.3 (Parametric Representation of Fuzzy Number): A fuzzy number is represented by an ordered pair of functions $(\underline{u}(\alpha),\overline{u}(\alpha))$, $\alpha\in[0,1]$ that satisfy the following conditions:

(i) $\underline{u}(\alpha)$ is a bounded left continuous increasing (non-decreasing) function for any $\alpha\in[0,1]$,

(ii) $\bar{u}(\alpha)$ is a bounded right continuous decreasing (non-increasing) function for any $\alpha \in [0,1]$, and

(iii) $\underline{u}(\alpha) \leq \bar{u}(\alpha)$ for any $\alpha \in [0,1]$, (Fook et al., 2016).

Definition 2.1.4 (Triangular Fuzzy Number): A fuzzy number u is called a triangular fuzzy number if defined by three numbers a, b and c, $a < b < c$ and its membership function $u(x;a,b,c)$ is of the form:

$$u(x;a,b,c) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x > c \end{cases}. \tag{2}$$

and its α -levels are defined by

$$[u]_\alpha = [a + \alpha(b-a), c - \alpha(c-b)], \ \alpha \in [0,1] \tag{3}$$

The base of the triangle $[a, c]$ is called the support (or length of support). The height of a fuzzy set E is the largest membership grade obtained by any element in that set. That is, $h(E) = \sup u_E(x)$.

Definition 2.1.5 (Symmetric Triangular Fuzzy Number): A symmetric triangular fuzzy number is a triangular fuzzy number that has the same left and right width of membership function from the centre, that is, $b-a = c-b$. It is non-symmetric type 1 if $b-a > c-b$ and non-symmetric type 2 if $b-a < c-b$, (Isa et al., 2018).

Definition 2.1.6 (Fuzzy Ordinary Differential Equation): An ordinary differential equation

$$y'(t) = f\left(k, y(t)\right),$$

with initial condition

$$y(t_0) = y_0.$$

is called fuzzy ordinary differential equation if it satisfies any of the following conditions:

(i) only the initial condition, y_0, is a fuzzy number (Type-I),

(ii) only the coefficient, k, is a fuzzy number (Type-II), or

(iii) both the initial condition and coefficients, y_0 and k, are fuzzy numbers (Type-III or fully FDEs), (Mondal et al., 2016).

3. Formulation of the VSBBDF

In this section, VSBBDF will be formulated for computing nonlinear FDEs of the form (1). The formulation is carried out in the interval $[t_{n-1}, t_{n+2}]$ with stepsize rh of the previous block and stepsize $h/2$ of the computed block, where r is the stepsize ratio. The stepsize ratio is selected at three values, $r=1$, $r=2$ and $r=1/2$ corresponding to stepsize being maintained, halved and doubled respectively. Let the function $f\left(t, y(t)\right)$ in equation (1) be approximated by the Lagrange interpolation polynomial

$$P_w(t) = L_{w,1/2}(t) f\left(t_{n+1/2}\right) + L_{w,3/2}(t) f\left(t_{n+3/2}\right) + \sum_{j=-1}^{2} L_{w,1-j}(t) f\left(t_{n+1-j}\right), \tag{4}$$

where

$$L_{w,1/2}(t)=\left(\prod_{i=-1}^{2}\frac{t-t_{n+1-i}}{t_{n+1/2}-t_{n+1-i}}\right)\left(\frac{t-t_{n+3/2}}{t_{n+1/2}-t_{n+3/2}}\right),$$

$$L_{w,3/2}(t)=\left(\prod_{i=-1}^{2}\frac{t-t_{n+1-i}}{t_{n+3/2}-t_{n+1-i}}\right)\left(\frac{t-t_{n+1/2}}{t_{n+3/2}-t_{n+1/2}}\right),$$

$$L_{w,1-j}(t)=\left(\prod_{\substack{i=-1\\ i\neq j}}^{2}\frac{t-t_{n+1-i}}{t_{n+1-j}-t_{n+1-i}}\right)\left(\frac{t-t_{n+1/2}}{t_{n+1-j}-t_{n+1/2}}\right)\left(\frac{t-t_{n+3/2}}{t_{n+1-j}-t_{n+3/2}}\right),\ j=-1,0,1,2.$$

The polynomial in (4) is assumed to interpolate the approximate solution to (1) at the points (t_{n-1},y_{n-1}), (t_n,y_n), $(t_{n+1/2},y_{n+1/2})$, (t_{n+1},y_{n+1}), $(t_{n+3/2},y_{n+3/2})$ and (t_{n+2},y_{n+2}). Let $s=(t-t_{n+2})/h$, so that $t=t_{n+2}+sh$ is substituted in the interpolating polynomial (4). This gives,

$$\begin{aligned}P(t)=P(t_{n+2}+sh)=&\left(\frac{(s+1)(s+2)(2s+1)(2s+3)}{6}\right)\left(\frac{r+s+2}{r+2}\right)y_{n+2}-\left(\frac{8s(s+1)(s+2)(2s+3)}{3}\right)\left(\frac{r+s+2}{2r+3}\right)y_{n+\frac{3}{2}}\\&+s(s+2)(2s+1)(2s+3)\left(\frac{r+s+2}{r+1}\right)y_{n+1}-\left(\frac{8s(s+1)(s+2)(2s+1)}{3}\right)\left(\frac{r+s+2}{2r+1}\right)y_{n+\frac{1}{2}}\\&+\left(\frac{s(s+1)(2s+1)(2s+3)}{6}\right)\left(\frac{r+s+2}{r}\right)y_n-s(s+1)(s+2)(2s+1)\left(\frac{2s+3}{r(r+1)(r+2)(2r+1)(2r+3)}\right)y_{n-1}.\end{aligned} \tag{5}$$

Equation (5) is differentiated with respect to s. This gives

$$\begin{aligned}hP'(t_{n+2}+sh)=&\left(\frac{25r+190s+60rs^2+16rs^3+70rs+225s^2+112s^3+20s^4+56}{6(r+2)}\right)y_{n+2}-\left(\frac{8\left(6r+64s+27rs^2+8rs^3+26rs+93s^2+52s^3+10s^4+12\right)}{3(2r+3)}\right)y_{n+\frac{3}{2}}\\&+\left(\frac{6r+88s+48rs^2+16rs^3+38rs+153s^2+96s^3+20s^4+12}{(r+1)}\right)y_{n+1}-\left(\frac{8\left(2r+32s+21rs^2+8rs^3+14rs+63s^2+44s^3+10s^4+4\right)}{3(2r+1)}\right)y_{n+\frac{1}{2}}\\&+\left(\frac{3r+50s+36rs^2+16rs^3+22rs+105s^2+80s^3+20s^4+6}{2r}\right)y_n-\left(\frac{\left(20s^4+80s^3+105s^2+50s+6\right)}{r\left(4r^4+20r^3+35r^2+25r+6\right)}\right)y_{n-1}.\end{aligned} \tag{6}$$

Substituting $s=-3/2,-1,-1/2$ and $s=0$ into equation (6) respectively gives,

$$\left.\begin{aligned}hf_{n+\frac{1}{2}}&=\frac{1}{6}\frac{y_{n+2}}{r+2}\left(r+\frac{1}{2}\right)-\frac{8}{3}\frac{y_{n+3/2}}{2r+3}\left(\frac{3}{4}r+\frac{3}{8}\right)+\frac{y_{n+1}}{r+1}\left(3r+\frac{3}{2}\right)-\frac{8}{3}\frac{y_{n+1/2}}{2r+1}\left(\frac{5}{4}r-\frac{1}{8}\right)-\frac{y_n}{6r}\left(3r+\frac{3}{2}\right)+\frac{3}{2r}\left(\frac{y_{n-1}}{4r^4+20r^3+35r^2+25r+6}\right),\\hf_{n+1}&=-\frac{y_{n+2}}{6}\frac{r+1}{r+2}+\frac{8}{3}\frac{y_{n+3/2}}{2r+3}(r+1)+\frac{y_{n+1}}{r+1}-\frac{8}{3}\frac{y_{n+1/2}}{2r+1}(r+1)+\frac{y_n}{6r}(r+1)-\frac{1}{r}\left(\frac{y_{n-1}}{4r^4+20r^3+35r^2+25r+6}\right),\\hf_{n+\frac{3}{2}}&=\frac{1}{6}\frac{y_{n+2}}{r+2}\left(3r+\frac{9}{2}\right)+\frac{8}{3}\frac{y_{n+3/2}}{2r+3}\left(\frac{5}{4}r+\frac{21}{8}\right)-\frac{y_{n+1}}{r+1}\left(3r+\frac{9}{2}\right)+\frac{8}{3}\frac{y_{n+1/2}}{2r+1}\left(\frac{3}{4}r+\frac{9}{8}\right)-\frac{y_n}{6r}\left(r+\frac{3}{2}\right)+\frac{3}{2r}\left(\frac{y_{n-1}}{4r^4+20r^3+35r^2+25r+6}\right),\\hf_{n+2}&=\frac{1}{6}\frac{y_{n+2}}{r+2}(25r+56)-\frac{8}{3}\frac{y_{n+3/2}}{2r+3}(6r+12)+y_{n+1}\left(\frac{6r+12}{r+1}\right)-\frac{8}{3}\frac{y_{n+1/2}}{2r+1}(2r+4)+\frac{y_n}{6r}(3r+6)-\frac{3}{r}\left(\frac{y_{n-1}}{4r^4+20r^3+35r^2+25r+6}\right).\end{aligned}\right\} \tag{7}$$

Equation (7) is solved for $y_{n+1/2},y_{n+1}$, $y_{n+3/2}$ and y_{n+2} respectively to obtain,

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= \frac{\left(\frac{1}{6}\frac{y_{n+2}}{r+2}\left(r+\frac{1}{2}\right)-\frac{8}{3}\frac{y_{n+3/2}}{2r+3}\left(\frac{3}{4}r+\frac{3}{8}\right)+\frac{y_{n+1}}{r+1}\left(3r+\frac{3}{2}\right)-\frac{y_n}{6r}\left(3r+\frac{3}{2}\right)+\frac{3}{2r}\left(\frac{y_{n-1}}{4r^4+20r^3+35r^2+25r+6}\right)-hf_{n+\frac{1}{2}}\right)}{\left(\frac{8}{3(2r+1)}\right)\left(\frac{5}{4}r-\frac{1}{8}\right)}, \\
y_{n+1} &= \frac{\left(\frac{y_{n+2}}{6}\frac{r+1}{r+2}-\frac{8}{3}\frac{y_{n+3/2}}{2r+3}(r+1)+\frac{8}{3}\frac{y_{n+1/2}}{2r+1}(r+1)-\frac{y_n}{6r}(r+1)+\frac{1}{r}\left(\frac{y_{n-1}}{4r^4+20r^3+35r^2+25r+6}\right)+hf_{n+1}\right)}{\left(\frac{1}{r+1}\right)}, \\
y_{n+\frac{3}{2}} &= \frac{\left(-\frac{1}{6}\frac{y_{n+2}}{r+2}\left(3r+\frac{9}{2}\right)+\frac{y_{n+1}}{r+1}\left(3r+\frac{9}{2}\right)-\frac{8}{3}\frac{y_{n+1/2}}{2r+1}\left(\frac{3}{4}r+\frac{9}{8}\right)+\frac{y_n}{6r}\left(r+\frac{3}{2}\right)-\frac{3}{2r}\left(\frac{y_{n-1}}{4r^4+20r^3+35r^2+25r+6}\right)+hf_{n+\frac{3}{2}}\right)}{\left(\frac{8}{3(2r+3)}\right)\left(\frac{5}{4}r+\frac{21}{8}\right)}, \\
y_{n+2} &= \frac{\left(\frac{8}{3}\frac{y_{n+3/2}}{2r+3}(6r+12)-y_{n+1}\left(\frac{6r+12}{r+1}\right)+\frac{8}{3}\frac{y_{n+1/2}}{2r+1}(2r+4)-\frac{y_n}{6r}(3r+6)+\frac{3}{r}\left(\frac{y_{n-1}}{4r^4+20r^3+35r^2+25r+6}\right)+hf_{n+2}\right)}{\left(\frac{1}{6(r+2)}\right)(25r+56)}.
\end{aligned}\right\} \tag{8}$$

At $r=1$, the equation (8) is given by

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= \frac{1}{60}y_{n-1}-\frac{3}{4}y_n+\frac{9}{4}y_{n+1}-\frac{3}{5}y_{n+\frac{3}{2}}+\frac{1}{12}y_{n+2}-hf_{n+\frac{1}{2}}, \\
y_{n+1} &= \frac{1}{45}y_{n-1}-\frac{2}{3}y_n+\frac{32}{9}y_{n+\frac{1}{2}}-\frac{32}{15}y_{n+\frac{3}{2}}+\frac{2}{9}y_{n+2}+2hf_{n+1}, \\
y_{n+\frac{3}{2}} &= -\frac{1}{124}y_{n-1}+\frac{25}{124}y_n-\frac{25}{31}y_{n+\frac{1}{2}}+\frac{225}{124}y_{n+1}-\frac{25}{124}y_{n+2}+\frac{15}{31}hf_{n+\frac{3}{2}}, \\
y_{n+2} &= \frac{2}{135}y_{n-1}-\frac{1}{3}y_n+\frac{32}{27}y_{n+\frac{1}{2}}-2y_{n+1}+\frac{32}{15}y_{n+\frac{3}{2}}-\frac{2}{9}hf_{n+2}.
\end{aligned}\right\} \tag{9}$$

At $r=2$, the equation (8) is given by

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= \frac{3}{2128}y_{n-1}-\frac{75}{152}y_n+\frac{75}{38}y_{n+1}-\frac{75}{133}y_{n+\frac{3}{2}}+\frac{25}{304}y_{n+2}-\frac{15}{19}hf_{n+\frac{1}{2}}, \\
y_{n+1} &= \frac{1}{280}y_{n-1}-\frac{3}{4}y_n+\frac{24}{5}y_{n+\frac{1}{2}}-\frac{24}{7}y_{n+\frac{3}{2}}+\frac{3}{8}y_{n+2}+3hf_{n+1}, \\
y_{n+\frac{3}{2}} &= -\frac{3}{3280}y_{n-1}+\frac{49}{328}y_n-\frac{147}{205}y_{n+\frac{1}{2}}+\frac{147}{82}y_{n+1}-\frac{147}{656}y_{n+2}+\frac{21}{41}hf_{n+\frac{3}{2}}, \\
y_{n+2} &= \frac{3}{1855}y_{n-1}-\frac{12}{53}y_n+\frac{256}{265}y_{n+\frac{1}{2}}-\frac{96}{53}y_{n+1}+\frac{768}{371}y_{n+\frac{3}{2}}+\frac{12}{53}hf_{n+2}.
\end{aligned}\right\} \tag{10}$$

At $r=1/2$, the equation (8) becomes

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= \frac{3}{20}y_{n-1} - \frac{3}{2}y_n + 3y_{n+1} - \frac{3}{4}y_{n+\frac{3}{2}} + \frac{1}{10}y_{n+2} - \frac{3}{2}hf_{n+\frac{1}{2}},\\
y_{n+1} &= \frac{1}{10}y_{n-1} - \frac{3}{4}y_n + 3y_{n+\frac{1}{2}} - \frac{3}{2}y_{n+\frac{3}{2}} + \frac{3}{20}y_{n+2} + \frac{3}{2}hf_{n+1},\\
y_{n+\frac{3}{2}} &= -\frac{3}{65}y_{n-1} + \frac{4}{13}y_n - \frac{12}{13}y_{n+\frac{1}{2}} + \frac{24}{13}y_{n+1} - \frac{12}{65}y_{n+2} + \frac{6}{13}hf_{n+\frac{3}{2}},\\
y_{n+2} &= \frac{12}{137}y_{n-1} - \frac{75}{137}y_n + \frac{200}{137}y_{n+\frac{1}{2}} - \frac{300}{137}y_{n+1} + \frac{300}{137}y_{n+\frac{3}{2}} + \frac{30}{137}hf_{n+2}.
\end{aligned}\right\} \tag{11}$$

Equations (9)-(11) collectively form the corrector-VSBBDF. Applying the same procedure, the predictor-VSBBDF for $y_{n+1/2}, y_{n+1}$, $y_{n+3/2}$ and y_{n+2} at $r=1$, $r=2$ and $r=1/2$ using the back (previous) values t_{n-1} and t_n as interpolating points are obtained as follows,

$$\left.\begin{aligned}
&\text{At } r=1, && y_{n+\frac{1}{2}} = y_{n+1} = y_{n+\frac{3}{2}} = y_{n+2} = -y_{n-1} + y_n\\
&\text{At } r=2, && y_{n+\frac{1}{2}} = y_{n+1} = y_{n+\frac{3}{2}} = y_{n+2} = -\frac{1}{2}y_{n-1} + \frac{1}{2}y_n\\
&\text{At } r=1/2, && y_{n+\frac{1}{2}} = y_{n+1} = y_{n+\frac{3}{2}} = y_{n+2} = -2y_{n-1} + 2y_n
\end{aligned}\right\} \tag{12}$$

The predictor-VSBBDF in equation (12) and the corrector-VSBBDF in equations (9)-(11) collectively form the VSBBDF.

4. Convergence and Stability Analyses of the VSBBDF

The convergence and stability properties of the VSBBDF will be investigated in this section.

4.1 Order and error constant

Definition 4.1.1: The general k-step linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^{\mu} \sum_{j=0}^{k} \beta_j f_{n+j} \tag{13}$$

and its associated linear difference operator L defined by

$$L\{y(t);h\} = \sum_{j=0}^{k}\left[\alpha_j y(t+jh) - h\beta_j y'(t+jh)\right] \tag{14}$$

are said to be of order p if $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \ldots = \bar{c}_p = 0,\ \bar{c}_{p+1} \neq 0$, (Fatunla, 1980).

The term $\bar{c}_{p+1} \neq 0$ is the error constant of the method. The general form for the constant c_p is defined as

$$\left.\begin{aligned} c_0 &= \sum_{j=0}^{k} \alpha_j \\ c_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ c_p &= \sum_{j=0}^{k} \left[\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!} j^{p-1} \beta_j \right], p = 2,3,\ldots \end{aligned}\right\} \tag{15}$$

Proposition 4.1.1: The order of accuracy of the generalized VSBBDF is five.

Proof

The generalized form of the proposed VSBBDF in equations (9)-(11) can be written as,

$$\left.\begin{aligned} &\varphi_{-1,\frac{1}{2}} y_{n-1} + \varphi_{0,\frac{1}{2}} y_n + \varphi_{\frac{1}{2},\frac{1}{2}} y_{n+\frac{1}{2}} + \varphi_{1,\frac{1}{2}} y_{n+1} + \varphi_{\frac{3}{2},\frac{1}{2}} y_{n+\frac{3}{2}} + \varphi_{2,\frac{1}{2}} y_{n+2} = \gamma_{\frac{1}{2},\frac{1}{2}} h f_{n+\frac{1}{2}}, \\ &\varphi_{-1,1} y_{n-1} + \varphi_{0,1} y_n + \varphi_{\frac{1}{2},1} y_{n+\frac{1}{2}} + \varphi_{1,1} y_{n+1} + \varphi_{\frac{3}{2},1} y_{n+\frac{3}{2}} + \varphi_{2,1} y_{n+2} = \gamma_{1,1} h f_{n+1}, \\ &\varphi_{-1,\frac{3}{2}} y_{n-1} + \varphi_{0,\frac{3}{2}} y_n + \varphi_{\frac{1}{2},\frac{3}{2}} y_{n+\frac{1}{2}} + \varphi_{1,\frac{3}{2}} y_{n+1} + \varphi_{\frac{3}{2},\frac{3}{2}} y_{n+\frac{3}{2}} + \varphi_{2,\frac{3}{2}} y_{n+2} = \gamma_{\frac{3}{2},\frac{3}{2}} h f_{n+\frac{3}{2}}, \\ &\varphi_{-1,2} y_{n-1} + \varphi_{0,2} y_n + \varphi_{\frac{1}{2},2} y_{n+\frac{1}{2}} + \varphi_{1,2} y_{n+1} + \varphi_{\frac{3}{2},2} y_{n+\frac{3}{2}} + \varphi_{2,2} y_{n+2} = \gamma_{2,2} h f_{n+1}. \end{aligned}\right\} \tag{16}$$

Writing equation (16) in matrix form gives

$$\begin{bmatrix} 0 & 0 & \varphi_{-1,\frac{1}{2}} & \varphi_{0,\frac{1}{2}} \\ 0 & 0 & \varphi_{-1,1} & \varphi_{0,1} \\ 0 & 0 & \varphi_{-1,\frac{3}{2}} & \varphi_{0,\frac{3}{2}} \\ 0 & 0 & \varphi_{-1,2} & \varphi_{0,2} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} \varphi_{\frac{1}{2},\frac{1}{2}} & \varphi_{1,\frac{1}{2}} & \varphi_{\frac{3}{2},\frac{1}{2}} & \varphi_{2,\frac{1}{2}} \\ \varphi_{\frac{1}{2},1} & \varphi_{1,1} & \varphi_{\frac{3}{2},1} & \varphi_{2,1} \\ \varphi_{\frac{1}{2},\frac{3}{2}} & \varphi_{1,\frac{3}{2}} & \varphi_{\frac{3}{2},\frac{3}{2}} & \varphi_{2,\frac{3}{2}} \\ \varphi_{\frac{1}{2},2} & \varphi_{1,2} & \varphi_{\frac{3}{2},2} & \varphi_{2,2} \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = h \begin{bmatrix} \gamma_{\frac{1}{2},\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \gamma_{1,1} & 0 & 0 \\ 0 & 0 & \gamma_{\frac{3}{2},\frac{3}{2}} & 0 \\ 0 & 0 & 0 & \gamma_{2,2} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}. \tag{17}$$

Let

$$\alpha_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\alpha_1=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ \alpha_2=\begin{bmatrix}\varphi_{-1,\frac{1}{2}}\\ \varphi_{-1,1}\\ \varphi_{-1,\frac{3}{2}}\\ \varphi_{-1,2}\end{bmatrix},\alpha_3=\begin{bmatrix}\varphi_{0,\frac{1}{2}}\\ \varphi_{0,1}\\ \varphi_{0,\frac{3}{2}}\\ \varphi_{0,2}\end{bmatrix},\ \alpha_4=\begin{bmatrix}\varphi_{\frac{1}{2},\frac{1}{2}}\\ \varphi_{\frac{1}{2},1}\\ \varphi_{\frac{1}{2},\frac{3}{2}}\\ \varphi_{\frac{1}{2},2}\end{bmatrix},\alpha_5=\begin{bmatrix}\varphi_{1,\frac{1}{2}}\\ \varphi_{1,1}\\ \varphi_{1,\frac{3}{2}}\\ \varphi_{1,2}\end{bmatrix}$$

$$\alpha_6=\begin{bmatrix}\varphi_{\frac{3}{2},\frac{1}{2}}\\ \varphi_{\frac{3}{2},1}\\ \varphi_{\frac{3}{2},\frac{3}{2}}\\ \varphi_{\frac{3}{2},2}\end{bmatrix},\alpha_7=\begin{bmatrix}\varphi_{2,\frac{1}{2}}\\ \varphi_{2,1}\\ \varphi_{2,\frac{3}{2}}\\ \varphi_{2,2}\end{bmatrix},\ \beta_4=\begin{bmatrix}\gamma_{\frac{1}{2},\frac{1}{2}}\\0\\0\\0\end{bmatrix},\beta_5=\begin{bmatrix}0\\ \gamma_{1,1}\\0\\0\end{bmatrix},\ \beta_6=\begin{bmatrix}0\\0\\ \gamma_{\frac{3}{2},\frac{3}{2}}\\0\end{bmatrix},\beta_7=\begin{bmatrix}0\\0\\0\\ \gamma_{2,2}\end{bmatrix}$$

The linear difference operator (14) and the associated VSBBDF (16) clearly satisfy Definition 4.1.1 as follows

$$\left.\begin{aligned}
&c_0=\sum_{j=0}^{7}\alpha_j=\begin{bmatrix}0&0&0&0\end{bmatrix}^T, c_1=\sum_{j=0}^{7}\left(j\alpha_j-\beta_j\right)=\begin{bmatrix}0&0&0&0\end{bmatrix}^T,\\
&c_2=\sum_{j=0}^{7}\left(\frac{j^2\alpha_j}{2!}-j\beta_j\right)=\begin{bmatrix}0&0&0&0\end{bmatrix}^T,\ c_3=\sum_{j=0}^{7}\left(\frac{j^3\alpha_j}{3!}-\frac{j^2\beta_j}{2!}\right)=\begin{bmatrix}0&0&0&0\end{bmatrix}^T,\\
&c_4=\sum_{j=0}^{7}\left(\frac{j^4\alpha_j}{4!}-\frac{j^3\beta_j}{3!}\right)=\begin{bmatrix}0&0&0&0\end{bmatrix}^T, c_5=\sum_{j=0}^{7}\left(\frac{j^5\alpha_j}{5!}-\frac{j^4\beta_j}{4!}\right)=\begin{bmatrix}0&0&0&0\end{bmatrix}^T,\\
&c_6=\sum_{j=0}^{7}\left(\frac{j^6\alpha_j}{6!}-\frac{j^5\beta_j}{5!}\right)\neq\begin{bmatrix}0&0&0&0\end{bmatrix}^T.
\end{aligned}\right\}. \tag{18}$$

Therefore, the generalized VSBBDF is of order five. This can be verified considering the fact that the VSBBDF with stepsize ratio $r=1$ satisfies

$$\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=\begin{bmatrix}0&0&0&0\end{bmatrix}^T, \tag{19}$$

with error constant

$$\bar{c}_6=\begin{bmatrix}-7.8125\times10^{-4} & -1.3889\times10^{-3} & 6.3004\times10^{-4} & -1.3889\times10^{-3}\end{bmatrix}^T. \tag{20}$$

The same strategy can be applied in finding the orders of the VSBBDF at $r=2$ and $r=1/2$ in equations (10) and (11) respectively. □

4.2 Zero-stability

Definition 4.1.2: A method is zero-stable if no root of the first characteristic polynomial has a modulus greater than one and every root with modulus one is simple, (Dahlquist, 1963).

To determine the zero-stability of the VSBBDF, the linear scalar test equation

$$y'(t)=\lambda y(t), \tag{21}$$

is used. Take $\mathrm{Re}(\lambda)<0$, λ being a complex constant.

Proposition 4.1.2: The generalized VSBBDF is zero-stable.

Proof

To establish the zero-stability of the proposed VSBBDF, equation (21) is substituted into it to obtain

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= \varphi_{-1,\frac{1}{2}}y_{n-1}+\varphi_{0,\frac{1}{2}}y_n+\varphi_{1,\frac{1}{2}}y_{n+1}+\varphi_{\frac{3}{2},\frac{1}{2}}y_{n+\frac{3}{2}}+\varphi_{2,\frac{1}{2}}y_{n+2}+\gamma_{\frac{1}{2},\frac{1}{2}}h\lambda y_{n+\frac{1}{2}},\\ y_{n+1} &= \varphi_{-1,1}y_{n-1}+\varphi_{0,1}y_n+\varphi_{\frac{1}{2},1}y_{n+\frac{1}{2}}+\varphi_{\frac{3}{2},1}y_{n+\frac{3}{2}}+\varphi_{2,\frac{1}{2}}y_{n+2}+\gamma_{1,1}h\lambda y_{n+1},\\ y_{n+\frac{3}{2}} &= \varphi_{-1,\frac{3}{2}}y_{n-1}+\varphi_{0,\frac{3}{2}}y_n+\varphi_{\frac{1}{2},\frac{3}{2}}y_{n+\frac{1}{2}}+\varphi_{1,\frac{3}{2}}y_{n+1}+\varphi_{2,\frac{3}{2}}y_{n+2}+\gamma_{\frac{3}{2},\frac{3}{2}}h\lambda y_{n+\frac{3}{2}},\\ y_{n+2} &= \varphi_{-1,2}y_{n-1}+\varphi_{0,2}y_n+\varphi_{\frac{1}{2},2}y_{n+\frac{1}{2}}+\varphi_{1,2}y_{n+1}+\varphi_{\frac{3}{2},2}y_{n+\frac{3}{2}}+\gamma_{2,2}h\lambda y_{n+2}. \end{aligned}\right\} \tag{22}$$

Equation (22) is written in matrix form as

$$\begin{bmatrix} 1-\gamma_{\frac{1}{2},\frac{1}{2}}h\lambda & -\varphi_{1,\frac{1}{2}} & -\varphi_{\frac{3}{2},\frac{1}{2}} & -\varphi_{2,\frac{1}{2}}\\ -\varphi_{\frac{1}{2},1} & 1-\gamma_{1,1}h\lambda & -\varphi_{\frac{3}{2},1} & -\varphi_{2,1}\\ -\varphi_{\frac{1}{2},\frac{3}{2}} & -\varphi_{1,\frac{3}{2}} & 1-\gamma_{\frac{3}{2},\frac{3}{2}}h\lambda & -\varphi_{2,\frac{3}{2}}\\ -\varphi_{\frac{1}{2},2} & -\varphi_{1,2} & -\varphi_{\frac{3}{2},2} & 1-\gamma_{2,2}h\lambda \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\end{bmatrix}=\begin{bmatrix} 0 & 0 & \varphi_{-1,\frac{1}{2}} & \varphi_{0,\frac{1}{2}}\\ 0 & 0 & \varphi_{-1,1} & \varphi_{0,1}\\ 0 & 0 & \varphi_{-1,\frac{3}{2}} & \varphi_{0,\frac{3}{2}}\\ 0 & 0 & \varphi_{-1,2} & \varphi_{0,2}\end{bmatrix}\begin{bmatrix} y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_n\end{bmatrix}. \tag{23}$$

Equation (23) is further written as

$$AY_m = BY_{m-1}, \tag{24}$$

Where

$$A=\begin{bmatrix} 1-\gamma_{\frac{1}{2},\frac{1}{2}}h\lambda & -\varphi_{1,\frac{1}{2}} & -\varphi_{\frac{3}{2},\frac{1}{2}} & -\varphi_{2,\frac{1}{2}}\\ -\varphi_{\frac{1}{2},1} & 1-\gamma_{1,1}h\lambda & -\varphi_{\frac{3}{2},1} & -\varphi_{2,1}\\ -\varphi_{\frac{1}{2},\frac{3}{2}} & -\varphi_{1,\frac{3}{2}} & 1-\gamma_{\frac{3}{2},\frac{3}{2}}h\lambda & -\varphi_{2,\frac{3}{2}}\\ -\varphi_{\frac{1}{2},2} & -\varphi_{1,2} & -\varphi_{\frac{3}{2},2} & 1-\gamma_{2,2}h\lambda \end{bmatrix},\quad B=\begin{bmatrix} 0 & 0 & \varphi_{-1,\frac{1}{2}} & \varphi_{0,\frac{1}{2}}\\ 0 & 0 & \varphi_{-1,1} & \varphi_{0,1}\\ 0 & 0 & \varphi_{-1,\frac{3}{2}} & \varphi_{0,\frac{3}{2}}\\ 0 & 0 & \varphi_{-1,2} & \varphi_{0,2}\end{bmatrix},$$

$$Y_m=\begin{bmatrix} y_{n+\frac{1}{2}} & y_{n+1} & y_{n+\frac{3}{2}} & y_{n+2}\end{bmatrix}^T \text{ and } Y_{m-1}=\begin{bmatrix} y_{n-3} & y_{n-2} & y_{n-1} & y_n\end{bmatrix}^T.$$

The determinant of equation (24) is evaluated, that is $\det\left(At-B\right)$. This produces the stability polynomial of the VSBBDF. Substituting $\lambda h=0$ in the resultant stability polynomial and solving for t gives the roots. For the VSBBDF at $r=1$, equation (22) is given by

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= \frac{1}{60}y_{n-1} - \frac{3}{4}y_n + \frac{9}{4}y_{n+1} - \frac{3}{5}y_{n+\frac{3}{2}} + \frac{1}{12}y_{n+2} - h\lambda y_{n+\frac{1}{2}},\\
y_{n+1} &= \frac{1}{45}y_{n-1} - \frac{2}{3}y_n + \frac{32}{9}y_{n+\frac{1}{2}} - \frac{32}{15}y_{n+\frac{3}{2}} + \frac{2}{9}y_{n+2} + 2h\lambda y_{n+1},\\
y_{n+\frac{3}{2}} &= -\frac{1}{124}y_{n-1} + \frac{25}{124}y_n - \frac{25}{31}y_{n+\frac{1}{2}} + \frac{225}{124}y_{n+1} - \frac{25}{124}y_{n+2} + \frac{15}{31}h\lambda y_{n+\frac{3}{2}},\\
y_{n+2} &= \frac{2}{135}y_{n-1} - \frac{1}{3}y_n + \frac{32}{27}y_{n+\frac{1}{2}} - 2y_{n+1} + \frac{32}{15}y_{n+\frac{3}{2}} - \frac{2}{9}h\lambda y_{n+2}.
\end{aligned}\right\} \tag{25}$$

which is then written in the form of Equation (23) as,

$$\begin{bmatrix} 1+h\lambda & \frac{-9}{4} & \frac{3}{5} & \frac{-1}{12} \\ \frac{-32}{9} & 1-2h\lambda & \frac{32}{15} & \frac{-2}{9} \\ \frac{25}{31} & \frac{-225}{124} & 1-\frac{15}{31}h\lambda & \frac{25}{124} \\ \frac{-32}{27} & 2 & \frac{-32}{15} & 1+\frac{2}{9}h\lambda \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+2} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{60} & \frac{-3}{4} \\ 0 & 0 & \frac{1}{45} & \frac{-2}{3} \\ 0 & 0 & \frac{-1}{124} & \frac{25}{124} \\ 0 & 0 & \frac{2}{135} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}. \tag{26}$$

Equation (26) is then written compactly in the form of equation (24), where

$$A = \begin{bmatrix} 1+h\lambda & \frac{-9}{4} & \frac{3}{5} & \frac{-1}{12} \\ \frac{-32}{9} & 1-2h\lambda & \frac{32}{15} & \frac{-2}{9} \\ \frac{25}{31} & \frac{-225}{124} & 1-\frac{15}{31}h\lambda & \frac{25}{124} \\ \frac{-32}{27} & 2 & \frac{-32}{15} & 1+\frac{2}{9}h\lambda \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & \frac{1}{60} & \frac{-3}{4} \\ 0 & 0 & \frac{1}{45} & \frac{-2}{3} \\ 0 & 0 & \frac{-1}{124} & \frac{25}{124} \\ 0 & 0 & \frac{2}{135} & \frac{-1}{3} \end{bmatrix}$$

$$Y_m = \left[y_{n+\frac{1}{2}} \quad y_{n+1} \quad y_{n+\frac{3}{2}} \quad y_{n+2} \right]^T \text{ and } Y_{m-1} = \left[y_{n-3} \quad y_{n-2} \quad y_{n-1} \quad y_n \right]^T.$$

On the evaluation of $\det(At - B)$, the stability polynomial of the VSBBDF at $r = 1$ in equation (9), denoted by $R_1(t, H)$, where $H = h\lambda$, is given by

$$\begin{aligned} R_1(t,H) = &-\frac{20}{93}t^4H^4 + \frac{364}{279}t^4H^3 + \frac{65}{279}t^3H^3 - \frac{3593}{837}t^4H^2 + \frac{3719}{2790}t^3H^2 + \frac{1123}{8370}t^2H^2 \\ &+ \frac{20272}{2511}t^4H + \frac{30671}{5022}t^3H + \frac{3551}{5022}t^2H - \frac{17264}{2511}t^4 + \frac{4513}{810}t^3 + \frac{32737}{25110}t^2. \end{aligned} \tag{27}$$

Applying similar procedure on the VSBBDF at $r = 2$ and $r = 1/2$ given in equations (10) and (11) respectively, we obtain the respective stability polynomials

$$R_2(t,H)=-\frac{11340}{41287}t^4H^4+\frac{61641}{41287}t^4H^3+\frac{45441}{165148}t^3H^3-\frac{189387}{41287}t^4H^2+\frac{572589}{330296}t^3H^2+\frac{27}{330296}t^2H^2+\frac{340848}{41287}t^4H+\frac{442791}{82574}t^3H+\frac{63}{82574}t^2H-\frac{14832}{2173}t^4+\frac{29655}{4346}t^3+\frac{9}{4346}t^2. \quad (28)$$

and

$$R_{1/2}(t,H)=-\frac{405}{1781}t^4H^4+\frac{2727}{1781}t^4H^3+\frac{81}{137}t^3H^3-\frac{38709}{7124}t^4H^2+\frac{10341}{3562}t^3H^2+\frac{27}{7124}t^2H^2+\frac{19251}{1781}t^4H+\frac{15246}{1781}t^3H+\frac{63}{1781}t^2H-\frac{17109}{1781}t^4+\frac{16938}{1781}t^3+\frac{171}{1781}t^2. \quad (29)$$

The roots of the VSBBDF are determined by substituting $H=0$ into the stability polynomials (27)-(29). This gives

$$R_1(t,0)=-\frac{17264}{2511}t^4+\frac{4513}{810}t^3+\frac{32737}{25110}t^2. \quad (30)$$

$$R_2(t,0)=-\frac{14832}{2173}t^4+\frac{29655}{4346}t^3+\frac{9}{4346}t^2. \quad (31)$$

$$R_{1/2}(t,0)=\left(\frac{1}{1782}\right)\left(-17109t^4+16938t^3+171t^2\right). \quad (32)$$

The roots of the stability polynomials are obtained by equating equations (30)-(32) to zero and then solve for t. The resultant roots of the VSBBDF are displayed in Table 1.

Table 1. Roots of VSBBDF

Step-size ratio	**Roots of VSBBDF**
$r=1$	$t=-0.18963, 0, 0, 1$
$r=2$	$t=-3.0340\times10^{-4}, 0, 0, 1$
$r=1/2$	$t=-9.9947\times10^{-3}, 0, 0, 1$

Since all the roots in Table 1 satisfy $|t| \le 1$, the VSBBDF is said to be zero-stable. □

4.3 Consistency

Definition 4.1.3: A general k-step LMM (13) is consistent if its order $p \ge 1$, (Fatunla, 1980).

The VSBBDF is consistent since it is of order five.

4.4 Convergence

Theorem 4.1.1: For a method to be convergent, it is necessary and sufficient that it be consistent and zero-stable, (Dahlquist, 1963).

Since the VSBBDF is consistent and zero-stable, it is said to be convergent.

4.5 Stability regions

Definition 4.1.4: A method whose stability region contains the whole left half-plane $\mathrm{Re}(h\lambda) < 0$, is called A-stable, (Lambert, 1973).
Definition 4.1.5: A method is said to be $A(\alpha)$-stable if the stability region includes all complex numbers $z = \lambda h$ such that $-(\pi-\alpha) \le \arg(z) \le (\pi-\alpha)$, Lambert (1991).

The regions of absolute stability of the VSBBDF are displayed in Figure 1.

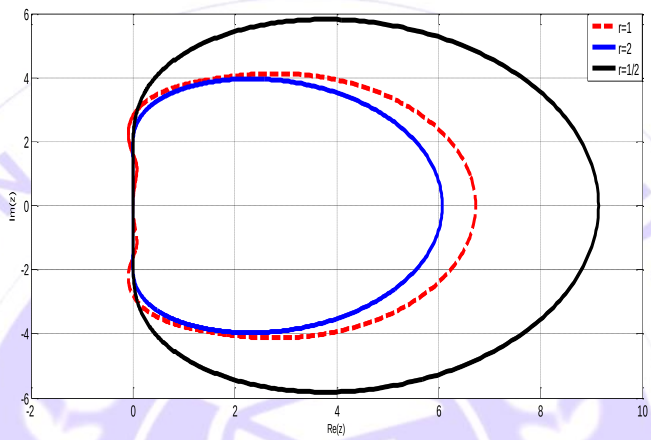

Figure 1. Regions of absolute stability of the VSBBDF

The regions of absolute stability of the VSBBDF are the exterior of the blue-, red-, and black-coloured contours for stepsize ratios $r = 2$, $r = 1$ and $r = 1/2$ respectively. This implies that the VSBBDF with $r = 2$ (corresponding to halved stepsize) has the largest stability region, then the VSBBDF with $r = 1$ (corresponding to constant stepsize) and then VSBBDF with $r = 1/2$ (corresponding to double stepsize) has the smallest region of absolute stability. Definition 4.1.4 clearly shows that the VSBBDF is A-stable at $r = 2$ and $r = 1/2$ and $A(\alpha)$-stable at $r = 1$. Table 2 shows the intervals of instability of the VSBBDF.

Table 2. Intervals of instability of the VSBBDF

Step-size Ratio	**Intervals of Instability**
$r = 1/2$	$(0, 9.139218)$
$r = 1$	$(0, 6.726702)$
$r = 2$	$(0, 6.070839)$

From Table 2, it is clear that the VSBBDF at $r = 2$ has the smallest interval of instability (implying the largest interval of stability) while the VSBBDF at $r = 1/2$ has the largest interval of instability (implying the smallest interval of stability).

5 Implementation of the VSBBDF

To implement the proposed VSBBDF, the Newton's iteration formula is employed in finding the approximate solutions $y_{n+1/2}$, y_{n+1}, $y_{n+3/2}$ and y_{n+2}. The generalized expression for the VSBBDF is given by

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= \varphi_{1,\frac{1}{2}} y_{n+1} + \varphi_{\frac{3}{2},\frac{1}{2}} y_{n+\frac{3}{2}} + \varphi_{2,\frac{1}{2}} y_{n+2} + \gamma_{\frac{1}{2},\frac{1}{2}} h f_{n+\frac{1}{2}} + \eta_{\frac{1}{2}},\\
y_{n+1} &= \varphi_{\frac{1}{2},1} y_{n+\frac{1}{2}} + \varphi_{\frac{3}{2},1} y_{n+\frac{3}{2}} + \varphi_{2,\frac{1}{2}} y_{n+2} + \gamma_{1,1} h f_{n+1} + \eta_{1},\\
y_{n+\frac{3}{2}} &= \varphi_{\frac{1}{2},\frac{3}{2}} y_{n+\frac{1}{2}} + \varphi_{1,\frac{3}{2}} y_{n+1} + \varphi_{2,\frac{3}{2}} y_{n+2} + \gamma_{\frac{3}{2},\frac{3}{2}} h f_{n+\frac{3}{2}} + \eta_{\frac{3}{2}},\\
y_{n+2} &= \varphi_{\frac{1}{2},2} y_{n+\frac{1}{2}} + \varphi_{1,2} y_{n+1} + \varphi_{\frac{3}{2},2} y_{n+\frac{3}{2}} + \gamma_{2,2} h f_{n+2} + \eta_{2}.
\end{aligned}\right\} \tag{33}$$

where $\eta_{\frac{1}{2}}$, η_1, $\eta_{\frac{3}{2}}$ and η_2 are back values. Equation (33) is written in matrix form as

$$(I-A)Y_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = hBF_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} + \zeta_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}, \tag{34}$$

where

$$I=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, Y_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}=\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}, A=\begin{bmatrix}0&\varphi_{1,\frac{1}{2}}&\varphi_{\frac{3}{2},\frac{1}{2}}&\varphi_{2,\frac{1}{2}}\\\varphi_{\frac{1}{2},1}&0&\varphi_{\frac{3}{2},1}&\varphi_{2,1}\\\varphi_{\frac{1}{2},\frac{3}{2}}&\varphi_{1,\frac{3}{2}}&0&\varphi_{2,\frac{3}{2}}\\\varphi_{\frac{1}{2},2}&\varphi_{1,2}&\varphi_{\frac{3}{2},2}&0\end{bmatrix},$$

$$B=\begin{bmatrix}\gamma_{\frac{1}{2},\frac{1}{2}}&0&0&0\\0&\gamma_{1,1}&0&0\\0&0&\gamma_{\frac{3}{2},\frac{3}{2}}&0\\0&0&0&\gamma_{2,2}\end{bmatrix}, F_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}=\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix}, \zeta_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}=\begin{bmatrix}\eta_{n+\frac{1}{2}}\\\eta_{n+1}\\\eta_{n+\frac{3}{2}}\\\eta_{n+2}\end{bmatrix}.$$

Suppose equation (34) is expressed as

$$\hat{F}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = (I-A)Y_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} - hBF_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} - \zeta_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = 0, \tag{35}$$

then the generalized Newton iteration formula for the VSBBDF is defined as

$$Y^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} - \frac{\hat{F}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}}{\hat{F}'_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}}. \tag{36}$$

Embedding equation (35) into (36) and solving the resultant equation gives the approximate solution,

$$Y^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} - Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = -\left[(I-A) - hB\frac{\partial F}{\partial Y}\left(Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right)\right]^{-1}\left[\begin{array}{l}(I-A)Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} \\ -hBF\left(Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right) - \zeta_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\end{array}\right] \quad (37)$$

where (i) and $(i+1)$ denote the previous and current iterations while $\frac{\partial F}{\partial Y}\left(Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{3},n+2}\right)$ is the Jacobian matrix of F for Y. Equation (37) is disintegrated into the following three matrices

$$E^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = Y^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}, \quad (38)$$

$$\hat{\mathrm{A}} = (I-A) - hB\frac{\partial F}{\partial Y}\left(Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right), \quad (39)$$

$$\hat{B} = -\left[(I-A)Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} - hBF\left(Y^{(i)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right) - \zeta_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}\right]. \quad (40)$$

Therefore, the approximate solution of nonlinear (1) is obtained by evaluating the linear system below at the lower bound of the solution component $\underline{Y}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}$ and the upper bound of the solution component $\overline{Y}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}$. That is,

$$\hat{\mathrm{A}}\, E^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2} = \hat{B}, \quad (41)$$

where $E^{(i+1)}_{n+\frac{1}{2},n+1,n+\frac{3}{2},n+2}$ is the increment value from i^{th} to $(i+1)^{th}$ iteration, and matrices $\hat{\mathrm{A}}$ and $\hat{B}$ are expressed as,

$$\hat{\mathrm{A}} = \begin{bmatrix} 1-\gamma_{\frac{1}{2},\frac{1}{2}}h\left(\frac{\partial f_{n+1/2}}{\partial y_{n+1/2}}\right) & \varphi_{1,\frac{1}{2}}-\gamma_{\frac{1}{2},\frac{1}{2}}h\left(\frac{\partial f_{n+1/2}}{\partial y_{n+1}}\right) & \varphi_{\frac{3}{2},\frac{1}{2}}-\gamma_{\frac{1}{2},\frac{1}{2}}h\left(\frac{\partial f_{n+1/2}}{\partial y_{n+3/2}}\right) & \varphi_{2,\frac{1}{2}}-\gamma_{\frac{1}{2},\frac{1}{2}}h\left(\frac{\partial f_{n+1/2}}{\partial y_{n+2}}\right) \\ \varphi_{\frac{1}{2},1}-\gamma_{1,1}h\left(\frac{\partial f_{n+1}}{\partial y_{n+1/2}}\right) & 1-\gamma_{1,1}h\left(\frac{\partial f_{n+1}}{\partial y_{n+1}}\right) & \varphi_{\frac{3}{2},1}-\gamma_{1,1}h\left(\frac{\partial f_{n+1}}{\partial y_{n+3/2}}\right) & \varphi_{2,1}-\gamma_{1,1}h\left(\frac{\partial f_{n+1}}{\partial y_{n+2}}\right) \\ \varphi_{\frac{1}{2},\frac{3}{2}}-\gamma_{\frac{3}{2},\frac{3}{2}}h\left(\frac{\partial f_{n+3/2}}{\partial y_{n+1/2}}\right) & \varphi_{1,\frac{3}{2}}-\gamma_{\frac{3}{2},\frac{3}{2}}h\left(\frac{\partial f_{n+3/2}}{\partial y_{n+1}}\right) & 1-\gamma_{\frac{3}{2},\frac{3}{2}}h\left(\frac{\partial f_{n+3/2}}{\partial y_{n+3/2}}\right) & \varphi_{2,\frac{3}{2}}-\gamma_{\frac{3}{2},\frac{3}{2}}h\left(\frac{\partial f_{n+3/2}}{\partial y_{n+2}}\right) \\ \varphi_{\frac{1}{2},2}-\gamma_{2,2}h\left(\frac{\partial f_{n+2}}{\partial y_{n+1/2}}\right) & \varphi_{1,2}-\gamma_{2,2}h\left(\frac{\partial f_{n+2}}{\partial y_{n+1}}\right) & \varphi_{\frac{3}{2},2}-\gamma_{2,2}h\left(\frac{\partial f_{n+2}}{\partial y_{n+3/2}}\right) & 1-\gamma_{2,2}h\left(\frac{\partial f_{n+2}}{\partial y_{n+2}}\right) \end{bmatrix}$$

and

$$\hat{B}=\begin{bmatrix} -y^{(i)}_{n+\frac{1}{2}}+\varphi_{1,\frac{1}{2}}y^{(i)}_{n+1}+\varphi_{\frac{3}{2},\frac{1}{2}}y^{(i)}_{n+\frac{3}{2}}+\varphi_{2,\frac{1}{2}}y^{(i)}_{n+2}+\gamma_{\frac{1}{2},\frac{1}{2}}hf^{(i)}_{n+\frac{1}{2}}+\eta_{\frac{1}{2}}, \\ \varphi_{\frac{1}{2},1}y^{(i)}_{n+\frac{1}{2}}-y^{(i)}_{n+1}+\varphi_{\frac{3}{2},1}y^{(i)}_{n+\frac{3}{2}}+\varphi_{2,\frac{1}{2}}y^{(i)}_{n+2}+\gamma_{1,1}hf^{(i)}_{n+1}+\eta_1, \\ \varphi_{\frac{1}{2},\frac{3}{2}}y^{(i)}_{n+\frac{1}{2}}+\varphi_{1,\frac{3}{2}}y^{(i)}_{n+1}-y^{(i)}_{n+\frac{3}{2}}+\varphi_{2,\frac{3}{2}}y^{(i)}_{n+2}+\gamma_{\frac{3}{2},\frac{3}{2}}hf^{(i)}_{n+\frac{3}{2}}+\eta_{\frac{3}{2}}, \\ \varphi_{\frac{1}{2},2}y^{(i)}_{n+\frac{1}{2}}+\varphi_{1,2}y^{(i)}_{n+1}+\varphi_{\frac{3}{2},2}y^{(i)}_{n+\frac{3}{2}}-y^{(i)}_{n+2}+\gamma_{2,2}hf^{(i)}_{n+2}+\eta_2. \end{bmatrix}.$$

6 Results and Discussions

The newly formulated VSBBDF will be used in computing nonlinear FDEs of the form (1). The abbreviations and notations below will be used in Figures 2-4 and Tables 3-5.

h: Step-size
r: Step-size ratio
α: α -cut fuzzy numbers
$\underline{Y}(t,\alpha),\overline{Y}(t,\alpha)$: Lower and upper bounded exact solution
$\underline{y}(t,\alpha),\overline{y}(t,\alpha)$: Lower and upper bounded approximate solution
NST: Total number of steps taken
NFE: Number of function evaluations
ET: Execution time in seconds
NR: Not recorded
HBM: One-step hybrid block method of order six developed by Sunday et al. (2020)
IBM: Fourth-order implicit block method developed by Isa et al. (2018)
RKM4: Fourth-order Runge-Kutta method
VSBBDF: Newly formulated variable step block backward differentiation formula

Let the exact solution and approximate solution at t_n be denoted by $Y_n(t,\alpha)=\left[\underline{Y}_n(t,\alpha),\overline{Y}_n(t,\alpha)\right]$ and $y_n(t,\alpha)=\left[\underline{y}_n(t,\alpha),\overline{y}_n(t,\alpha)\right]$ respectively, then the absolute error is defined as $\left|Y_n(t,\alpha)-y_n(t,\alpha)\right|$.

Problem 1

Consider the nonlinear FDE

$$y'(t)=e^{-y^2(t)},\ y(0)=\left[0.75+0.25\alpha,1.5-0.5\alpha\right], \tag{42}$$

defined for $t,\alpha\in\left[0,1\right]$. This problem does not have a closed form (exact) solution, Isa et al. (2018). Applying the VSBBDF on Problem 1 produces the numerical results in Table 3 and solution curves in Figure 2.

Table 3. Approximate solution of Problem 1 using VSBBDF and RKM4 at various α -cuts, $t=1$ and $h=0.1$

	$\underline{Y}_n(t,\alpha)$	
α -cuts	**VSBBDF**	**RKM4**
0.0	0.795175	0.795172
0.1	0.846307	0.846303
0.2	0.894281	0.894276
0.3	0.940199	0.940193
0.4	0.984974	0.984969
0.5	1.029403	1.029399
0.6	1.074204	1.074198
0.7	1.120034	1.120029
0.8	1.167524	1.167520
0.9	1.217278	1.217275
1.0	1.269873	1.269868
	$\overline{y}_n(t,\alpha)$	
α -cuts	**VSBBDF**	**RKM4**
0.0	2.017670	2.017674
0.1	1.925740	1.925744
0.2	1.836617	1.836622
0.3	1.750860	1.750863
0.4	1.668970	1.668972
0.5	1.591340	1.591344
0.6	1.518223	1.518228
0.7	1.449697	1.449698
0.8	1.385651	1.385652
0.9	1.325834	1.325836
1.0	1.269865	1.269868
NFE	451	451
NST	58	110
ET	0.284	0.468

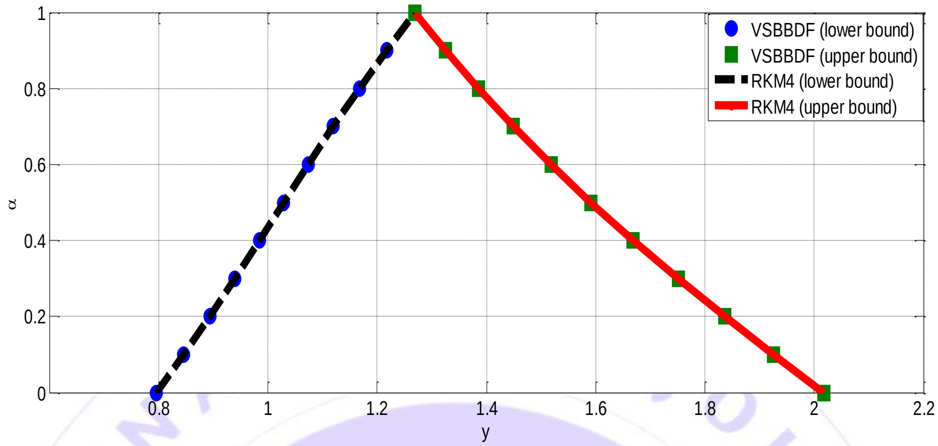

Figure 2. Solution curves for approximate solutions of Problem 1 at $t = 1$

Problem 2

Consider the nonlinear FDE,

$$y'(t) = 2y(t) + t^2 + 1,\ y(0) = [\alpha, 2-\alpha], \tag{43}$$

defined for $t, \alpha \in [0,1]$. The exact solution is given by

$$Y(t,\alpha) = \left[\left(\alpha + \frac{3}{4}\right)e^{2t} - \frac{1}{4}\left(2t^2 + 2t + 3\right), \left(\frac{11}{4} - \alpha\right)e^{2t} - \frac{1}{4}\left(2t^2 + 2t + 3\right)\right]. \tag{44}$$

Applying the VSBBDF on Problem 2 produces the numerical results in Table 4 and solution curves in Figure 3.

Table 4. Absolute errors of VSBBDF and IBM for Problem 2 at various α -cuts, $t=1$ and $h=0.1$

$\left|\underline{Y}_n(t,\alpha)-\underline{y}_n(t,\alpha)\right|$

α -cuts	VSBBDF	IBM
0.0	2.018123×10^{-09}	6.963978×10^{-05}
0.1	2.671872×10^{-09}	7.770009×10^{-05}
0.2	3.182229×10^{-09}	8.576039×10^{-05}
0.3	4.172342×10^{-09}	9.382069×10^{-05}
0.4	5.029249×10^{-09}	1.018810×10^{-04}
0.5	6.028211×10^{-09}	1.099413×10^{-04}
0.6	7.271168×10^{-09}	1.180016×10^{-04}
0.7	7.622781×10^{-09}	1.260619×10^{-04}
0.8	8.256271×10^{-09}	1.341222×10^{-04}
0.9	8.922567×10^{-09}	1.421825×10^{-04}
1.0	9.716928×10^{-09}	1.502428×10^{-04}

$\left|\overline{Y}_n(t,\alpha)-\overline{y}_n(t,\alpha)\right|$

α -cuts	VSBBDF	IBM
0.0	3.189002×10^{-06}	2.308459×10^{-04}
0.1	4.164572×10^{-06}	2.227856×10^{-04}
0.2	5.019200×10^{-06}	2.147253×10^{-04}
0.3	5.923231×10^{-06}	2.066649×10^{-04}
0.4	6.024271×10^{-06}	1.986046×10^{-04}
0.5	6.861898×10^{-06}	1.905443×10^{-04}
0.6	2.728161×10^{-07}	1.824840×10^{-04}
0.7	3.182229×10^{-07}	1.744237×10^{-04}
0.8	4.189822×10^{-07}	1.663634×10^{-04}
0.9	7.819291×10^{-07}	1.583031×10^{-04}
1.0	2.345162×10^{-08}	1.502428×10^{-04}
NFE	451	451
NST	56	66
ET	0.313	0.530

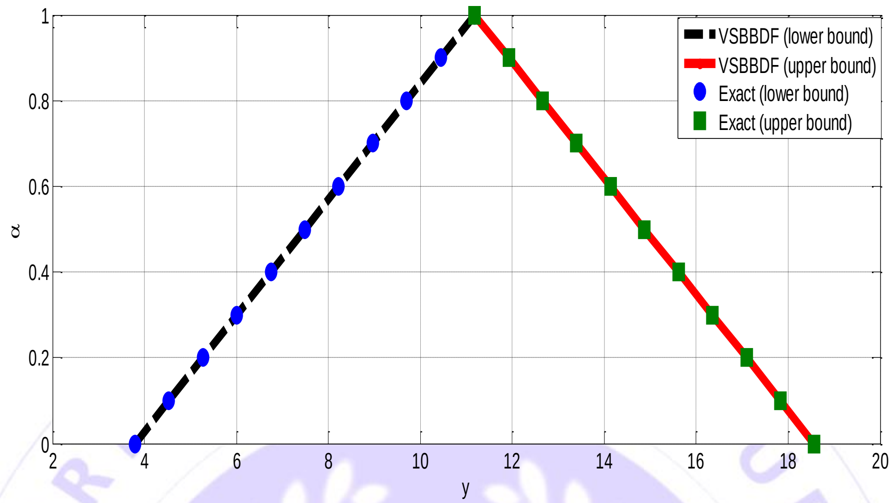

Figure 3. Solution curves for exact and approximate solutions of Problem 2 at $t = 1$

Problem 3

Consider the nonlinear FDE,

$$y'(t) = y(t)(1-2t),\ y(0) = \left[-\frac{\sqrt{1-\alpha}}{2},\ \frac{\sqrt{1-\alpha}}{2}\right], \tag{45}$$

defined for $t, \alpha \in [0,1]$. The exact solution is given by

$$Y(t,\alpha) = \left[-\frac{\sqrt{1-\alpha}}{2}e^{t-t^2},\ \frac{\sqrt{1-\alpha}}{2}e^{t-t^2}\right]. \tag{46}$$

Applying the VSBBDF on Problem 1 produces the numerical results in Table 5 and solution curves in Figure 4.

Table 5. Absolute errors of VSBBDF and HBM for Problem 3 at various α -cuts, $t=1$ and $h=0.1$

	$\left\lvert \underline{Y}_n(t,\alpha)-\underline{y}_n(t,\alpha)\right\rvert$	
α **-cuts**	**VSBBDF**	**HBM**
0.1	2.718×10^{-13}	4.535×10^{-12}
0.2	4.012×10^{-13}	4.652×10^{-12}
0.3	5.178×10^{-13}	4.789×10^{-12}
0.4	6.516×10^{-13}	5.108×10^{-12}
0.5	1.002×10^{-14}	5.234×10^{-13}
0.6	2.037×10^{-14}	6.873×10^{-13}
0.7	2.672×10^{-14}	6.986×10^{-13}
0.8	3.224×10^{-14}	7.256×10^{-13}
0.9	4.912×10^{-14}	7.456×10^{-13}
1.0	6.352×10^{-14}	7.845×10^{-13}
	$\left\lvert \overline{Y}_n(t,\alpha)-\overline{y}_n(t,\alpha)\right\rvert$	
α **-cuts**	**VSBBDF**	**HBM**
0.1	1.324×10^{-13}	4.535×10^{-12}
0.2	1.928×10^{-13}	4.652×10^{-12}
0.3	3.012×10^{-13}	4.789×10^{-12}
0.4	4.261×10^{-13}	5.108×10^{-12}
0.5	5.352×10^{-13}	5.234×10^{-13}
0.6	5.918×10^{-13}	6.873×10^{-13}
0.7	6.812×10^{-13}	6.986×10^{-13}
0.8	3.821×10^{-14}	7.256×10^{-13}
0.9	5.192×10^{-14}	7.456×10^{-13}
1.0	7.234×10^{-14}	7.845×10^{-13}
NFE	451	NR
NST	64	NR
ET	0.5210	0.7128

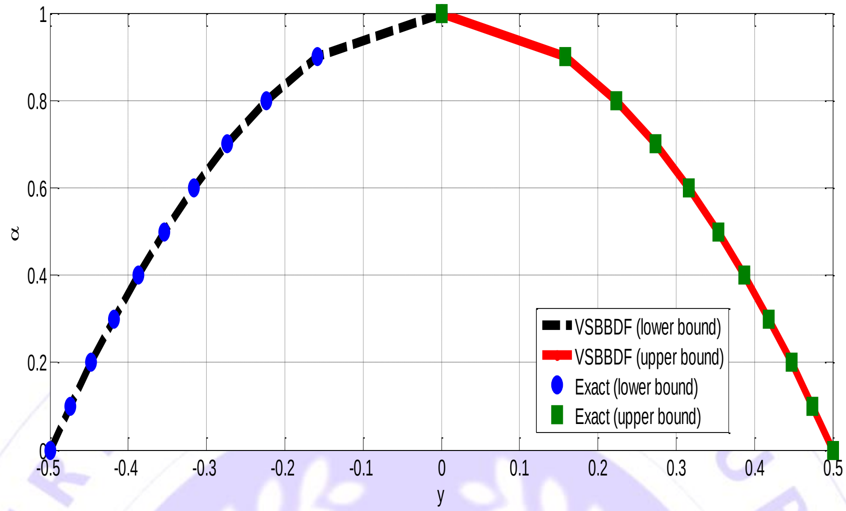

Figure 4. Solution curves for exact and approximate solutions of Problem 3 at $t = 1$

The results obtained validate the effectiveness of the VSBBDF for solving nonlinear FDEs. In comparison with established methods like the IBM, HBM and RKM4, VSBBDF demonstrated superior computational efficiency and accuracy. For Problem 1, which does not have an exact solution, the VSBBDF produced results closely matching that of RKM4 across various α-cut levels, while requiring fewer steps and less computational time, highlighting its reliability in handling FDEs. In the case of Problems 2 and 3, where exact solutions were available, the VSBBDF again outperformed IBM and HBM by producing more accurate results with smaller execution times and fewer function evaluations. Convergence analysis showed that the VSBBDF is consistent, zero-stable and convergent. The stability region analysis further confirmed that the method maintains stability across a range of step-size ratios, particularly at halved step sizes, which expand the stability region significantly. Overall, the results underscore the ability of the VSBBDF to handle nonlinear FDEs effectively, making it a powerful tool for applications requiring computational precision under uncertainty. By leveraging self-adjusting formulae and dynamic stepsize adjustments, VSBBDF proves to be both resource-efficient and highly adaptable, marking it as a valuable addition to computational techniques for FDEs. The examples discussed illustrate its robustness and precision, cementing its role in advancing fuzzy numerical analysis.

7 Conclusion

This research demonstrates that the VSBBDF is a reliable and efficient method for solving nonlinear FDEs. By adapting the step-size dynamically, VSBBDF reduces computational cost while maintaining high accuracy, making it suitable for applications requiring precision under uncertainty. Comparative analyses with other methods, illustrate the strengths of the VSBBDF in stability, convergence, and reduced execution time. These properties make it an excellent choice for modeling complex systems in uncertain environments, such as engineering, biological systems, and finance. The method's adaptability to fuzzy parameters confirms its robustness, and its efficiency provides a significant advantage in handling complex FDEs. Overall, the proposed VSBBDF advances the numerical solution of FDEs, highlighting the method's potential for broader application in fields dealing with imprecise data and nonlinear dynamics.

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