

Derivation of Four-Point Integrator for the Solutions of Second-order Oscillatory Problems

Lydia J. Kwari^{a,*}, Joshua Sunday^b and Joel N. Ndam^b

^aDepartment of Mathematics, Federal College of Education, Pankshin 933105, Nigeria

^bDepartment of Mathematics, University of Jos, Jos 930003, Nigeria

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ABSTRACT

The research introduces a new four-point integrator (FPI) for solving second-order differential equations characterized by oscillatory behavior. The proposed FPI is developed through a continuous scheme within a linear multistep framework, utilizing two off-step points to enhance efficiency. Unlike conventional methods that reduce higher-order equations to first-order systems, often resulting in loss of inherent characteristics, the FPI maintains essential equation properties. Through this approach, the FPI enables a self-starting, block-by-block algorithmic implementation that does not require predictor values, simplifying computation. The integrator's properties are rigorously analyzed, confirming its consistency, zero-stability, and convergence, all of which contribute to its reliability in handling oscillatory problems. Additionally, stability analysis, including the region of absolute stability, demonstrates the A-stability of the integrator. Numerical experiments conducted on some second-order problems reveal the FPI's computational accuracy, effectively comparing with established solvers such as ode45, ode15s and other methods. These results highlight the potential of the FPI as a viable alternative for directly solving oscillatory differential equations with damping characteristics, providing computational efficiency and accuracy.

1. Introduction

The research addresses the need for effective numerical methods to solve second-order differential equations, particularly those that exhibit oscillatory characteristics. Standard approaches for solving these equations often involve transforming them into systems of first-order equations, allowing for the use of established numerical methods. However, this transformation can lead to a loss of inherent characteristics and essential properties of the original second-order form, which may affect the accuracy and stability of the solution, especially in oscillatory cases. To overcome these limitations, direct methods that solve second-order differential equations in their original form will be proposed in this research. Direct integration of these equations preserves their structure, potentially improving the fidelity of oscillatory.

In this research, FPI will be derived for the solution of second-order problems exhibiting oscillations.

Let the problem given by,

$$y''(t) = f(t, y(t), y'(t)), \quad (1)$$

be subject to the initial conditions

$$y(t_0) = y_0, y'(t_0) = y_0' \quad (2)$$

on the interval $t \in [t_0, t_N]$, where $f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, N is an integer and m the dimension of equation (1).

Second-order oscillatory problems find applications in numerous areas of human endeavours. These include mechanical vibrations, quantum mechanics, control systems, vibration analysis, aerodynamic drag, population dynamics and planetary motion. Others are weak signal detection and phase lags, (Kwari *et al.*, 2022, 2023; Sunday, Ndam and Kwari, 2023).

The development of a reliable and efficient method that maintains the essential properties of second-order oscillatory differential equations of the form (1) while also being computationally manageable is therefore critical. Various researchers have contributed to this area by proposing high-order hybrid and block methods designed to

*Corresponding author. Tel.: +2347034884482

E-mail address: lydiakwari@gmail.com (Lydia J. Kwari.)

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improve the accuracy and efficiency of solutions for such systems. Previous works have demonstrated the effectiveness of hybrid methods, often implemented in a block structure, for directly solving second-order differential equations without converting them into first-order systems. Gandafa (2017) for instance derived a class of implicit hybrid block algorithms for the simulation of second-order problems. The author also investigated the convergence and stability properties of the methods. Jator and King (2018) formulated an order eleven hybrid method for direct integration of second order oscillatory problems including Hamiltonian systems. The method was implemented in a block mode. Areo, Adeyanju and Kayode (2020) also proposed a class of hybrid block methods for the solutions of problems of the form (1). The authors considered two cases with step number $k = 4$ using interpolation and collocation procedures. Other scholars that also derived methods for direct solution of problems of the form (1) are Adeyeye and Zurmi (2016), Obarhua and Kayode (2020), Yakubu, Yahaya and Lawal (2021), Yakusak and Adeniyi (2022), Sunday *et al.* (2021), Muhammad, Kwanamu and Sunday (2024) and Sunday *et al.* (2024).

The FPI proposed in this research builds on these developments by employing a linear multistep approach with two off-step points within the four-step integration interval, allowing the solution to proceed in a block-by-block, self-starting manner. This design negates the need for additional predictor values, streamlining implementation. The focus on a self-starting mechanism is significant, as many existing methods require an external predictor, which can increase computational overhead and reduce efficiency. The FPI aims to address both computational efficiency and solution accuracy while preserving the oscillatory characteristics of second-order differential equations. Its design targets problems that require consistent, stable methods due to their oscillatory behaviour.

2. Derivation and Implementation of the FPI

2.1 Derivation of the FPI

Let the power series of the form,

$$y(x) = \sum_{j=0}^{r+s-1} \tau_j x^j = \tau_0 + \tau_1 x + \tau_2 x^2 + \dots + \tau_{r+s-1} x^{r+s-1} \quad (3)$$

be the approximate solution $y(x)$ to the second-order oscillatory problem (1), where r is the number of collocation points, s is the number of interpolation points and $h = x_n - x_{n-1}$ is the step-size. A discrete hybrid block integrator formula (called the FPI) of the form,

$$\left. \begin{aligned} IY_m^{(i)} &= \sum_{j=0}^1 h^j e_j y_n^{(j)} + h^{\mu-i} [d_i f(y_n) + b_i f(Y_m)]; i = 0 \\ IY_m^{(i)} &= e_i y_n^{(i)} + h^{\mu-i} [d_i f(y_n) + b_i f(Y_m)]; i = 1 \end{aligned} \right\} \quad (4)$$

where

$$Y_m^{(i)} = [y_{n+i}^{(i)} \quad y_{n+j}^{(i)} \quad \dots \quad y_{n+k}^{(i)}]^T, \quad f(Y_m) = [f_{n+i} \quad f_{n+j} \quad \dots \quad f_{n+k}]^T$$

$$y_n^{(i)} = [y_{n-i}^{(i)} \quad y_{n-j}^{(i)} \quad \dots \quad y_n^{(i)}]^T, \quad f(y_n) = [f_{n-i} \quad f_{n-j} \quad \dots \quad f_n]^T$$

$I = (r-1) \times (r-1)$ is an identity matrix and μ is the order of the differential equation, will be derived. Note that the superscripts in parentheses are treated as derivatives. When $i = 0$, we evaluate the $(r-1) \times (r-1)$ matrices e_0, e_1, d_0 and b_0 and when $i = 1$ (i.e. the first derivative), we evaluate the $(r-1) \times (r-1)$ matrices e_1, d_1 and b_1 . The k -step formulae in equation (4) are applied simultaneously to solve the oscillatory problem (1).

To derive the FPI, two off-grid points were selected in the integration interval $[x_n, x_{n+4}]$, that is $x_{n+\frac{5}{2}}$ and $x_{n+\frac{7}{2}}$

. Interpolating (3) at points $x_{n+l}, l=3, \frac{7}{2}$ and collocating its second derivate at points

$x_{n+m}, m=0, 1, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$ give a system of $(r+s)$ equations of the form,

$$\sum_{j=0}^{r+s-1} \tau_j x^j = y_{n+l}, l=3, \frac{7}{2} \quad (5)$$

$$\sum_{j=2}^{r+s-1} j(j-1)\tau_j x^{j-2} = f_{n+m}, m=0, 1, 2, \frac{5}{2}, 3, \frac{7}{2}, 4 \quad (6)$$

The system of $(r+s)$ equations in (5) and (6) is then written compactly in matrix form

$$XT = U \quad (7)$$

where

$$T = [\tau_0 \ \tau_1 \ \tau_2 \ \tau_3 \ \tau_4 \ \tau_5 \ \tau_6 \ \tau_7 \ \tau_8]^T,$$

$$U = \left[y_{n+3} \ y_{n+\frac{7}{2}} \ f_n \ f_{n+1} \ f_{n+2} \ f_{n+\frac{5}{2}} \ f_{n+3} \ f_{n+\frac{7}{2}} \ f_{n+4} \right]^T$$

and

$$X = \begin{bmatrix} 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\ 1 & x_{n+\frac{7}{2}} & x_{n+\frac{7}{2}}^2 & x_{n+\frac{7}{2}}^3 & x_{n+\frac{7}{2}}^4 & x_{n+\frac{7}{2}}^5 & x_{n+\frac{7}{2}}^6 & x_{n+\frac{7}{2}}^7 & x_{n+\frac{7}{2}}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{5}{2}} & 12x_{n+\frac{5}{2}}^2 & 20x_{n+\frac{5}{2}}^3 & 30x_{n+\frac{5}{2}}^4 & 42x_{n+\frac{5}{2}}^5 & 56x_{n+\frac{5}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{7}{2}} & 12x_{n+\frac{7}{2}}^2 & 20x_{n+\frac{7}{2}}^3 & 30x_{n+\frac{7}{2}}^4 & 42x_{n+\frac{7}{2}}^5 & 56x_{n+\frac{7}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 \end{bmatrix}$$

Solving equation (7) for τ_j 's which are constants to be determined and substituting into the approximate solution (3) gives a continuous FPI of the form,

$$y(t) = \alpha_3(t)y_{n+3} + \alpha_7(t)y_{n+\frac{7}{2}} + h^2 \left(\sum_{j=0}^4 \beta_j(t)f_{n+j} + \beta_5(t)f_{n+\frac{5}{2}} + \beta_7(t)f_{n+\frac{7}{2}} \right), \quad (8)$$

where

$$\begin{aligned}
 \alpha_3(t) &= 7 - 2t \\
 \alpha_7(t) &= 2t - 6 \\
 \beta_0(t) &= \frac{1}{11760} \left(384t^8 - 8192t^7 + 74368t^6 - 373632t^5 + 1129408t^4 \right. \\
 &\quad \left. - 2084096t^3 + 2257920t^2 - 1275591t + 268821 \right) \\
 \beta_1(t) &= -\frac{1}{483840} \left(384t^8 - 7680t^7 + 63616t^6 - 278208t^5 + 665728t^4 \right. \\
 &\quad \left. - 752640t^3 + 816869t - 548079 \right) \\
 \beta_2(t) &= \frac{1}{322560} \left(1920t^8 - 35840t^7 + 271488t^6 - 1053696t^5 + 2134720t^4 \right. \\
 &\quad \left. - 1881600t^3 + 465669t + 412209 \right) \\
 \beta_{\frac{5}{2}}(t) &= -\frac{1}{30240} \left(384t^8 - 6912t^7 + 50176t^6 - 185472t^5 + 356608t^4 \right. \\
 &\quad \left. - 301056t^3 + 126377t - 47355 \right) \\
 \beta_3(t) &= \frac{1}{161280} \left(1920t^8 - 33280t^7 + 232064t^6 - 823872t^5 + 1527680t^4 \right. \\
 &\quad \left. - 1254400t^3 + 397281t + 147357 \right) \\
 \beta_{\frac{7}{2}}(t) &= -\frac{1}{70560} \left(384t^8 - 6400t^7 + 43008t^6 - 147840t^5 + 267008t^4 \right. \\
 &\quad \left. - 215040t^3 + 84033t - 24003 \right) \\
 \beta_4(t) &= \frac{1}{1935360} \left(1920t^8 - 30720t^7 + 199808t^6 - 669312t^5 \right. \\
 &\quad \left. + 1184960t^4 - 940800t^3 + 355885t - 77511 \right)
 \end{aligned} \tag{9}$$

and t is as defined in

$$t = \frac{x - x_n}{h} \tag{10}$$

Solving (8) for the independent solution gives the continuous FPI of the form,

$$y'(t) = y'_n + h \left(\sum_{j=0}^4 \sigma_j(t) f_{n+j} + \sigma_{\frac{5}{2}}(t) f_{n+\frac{5}{2}} + \sigma_{\frac{7}{2}}(t) f_{n+\frac{7}{2}} \right) \tag{11}$$

where

$$\left. \begin{aligned}
 \sigma_0(t) &= \frac{1}{35280} (24t^7 - 448t^6 + 3486t^5 - 14595t^4 + 35294t^3 - 48846t^2 + 35280t) \\
 \sigma_1(t) &= -\frac{1}{7560} (48t^7 - 840t^6 + 5964t^5 - 21735t^4 + 41608t^3 - 35280t^2) \\
 \sigma_2(t) &= \frac{1}{1260} (60t^7 - 980t^6 + 6363t^5 - 20580t^4 + 33355t^3 - 22050t^2) \\
 \sigma_{\frac{5}{2}}(t) &= -\frac{4}{945} (24t^7 - 378t^6 + 2352t^5 - 7245t^4 + 11144t^3 - 7056t^2) \\
 \sigma_3(t) &= \frac{1}{2520} (240t^7 - 3640t^6 + 21756t^5 - 64365t^4 + 95480t^3 - 58800t^2) \\
 \sigma_{\frac{7}{2}}(t) &= -\frac{4}{2205} (24t^7 - 350t^6 + 2016t^5 - 5775t^4 + 8344t^3 - 5040t^2) \\
 \sigma_4(t) &= \frac{1}{15120} (120t^7 - 1680t^6 + 9366t^5 - 26145t^4 + 37030t^3 - 22050t^2)
 \end{aligned} \right\} \quad (12)$$

Evaluating equations (8) and (11) at $t = 1, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$ gives a discrete FPI of the form (4), where

$$Y_m^{(i)} = \begin{bmatrix} y_{n+1}^{(i)} & y_{n+2}^{(i)} & y_{n+\frac{5}{2}}^{(i)} & y_{n+3}^{(i)} & y_{n+\frac{7}{2}}^{(i)} & y_{n+4}^{(i)} \end{bmatrix}^T, \quad y_n^{(i)} = \begin{bmatrix} y_{n-1}^{(i)} & y_{n-2}^{(i)} & y_{n-\frac{5}{2}}^{(i)} & y_{n-3}^{(i)} & y_{n-\frac{7}{2}}^{(i)} & y_n^{(i)} \end{bmatrix}^T$$

$$F(Y_m) = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+\frac{5}{2}} & f_{n+3} & f_{n+\frac{7}{2}} & f_{n+4} \end{bmatrix}^T, \quad f(y_n) = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-\frac{5}{2}} & f_{n-3} & f_{n-\frac{7}{2}} & f_n \end{bmatrix}^T$$

I is a 6×6 identity matrix given by,

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

When $i = 0$, we have

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{3113}{14112} \\ 0 & 0 & 0 & 0 & 0 & \frac{743}{1470} \\ 0 & 0 & 0 & 0 & 0 & \frac{1168075}{1806336} \\ 0 & 0 & 0 & 0 & 0 & \frac{6177}{7840} \\ 0 & 0 & 0 & 0 & 0 & \frac{11417}{12288} \\ 0 & 0 & 0 & 0 & 0 & \frac{472}{441} \end{bmatrix}$$

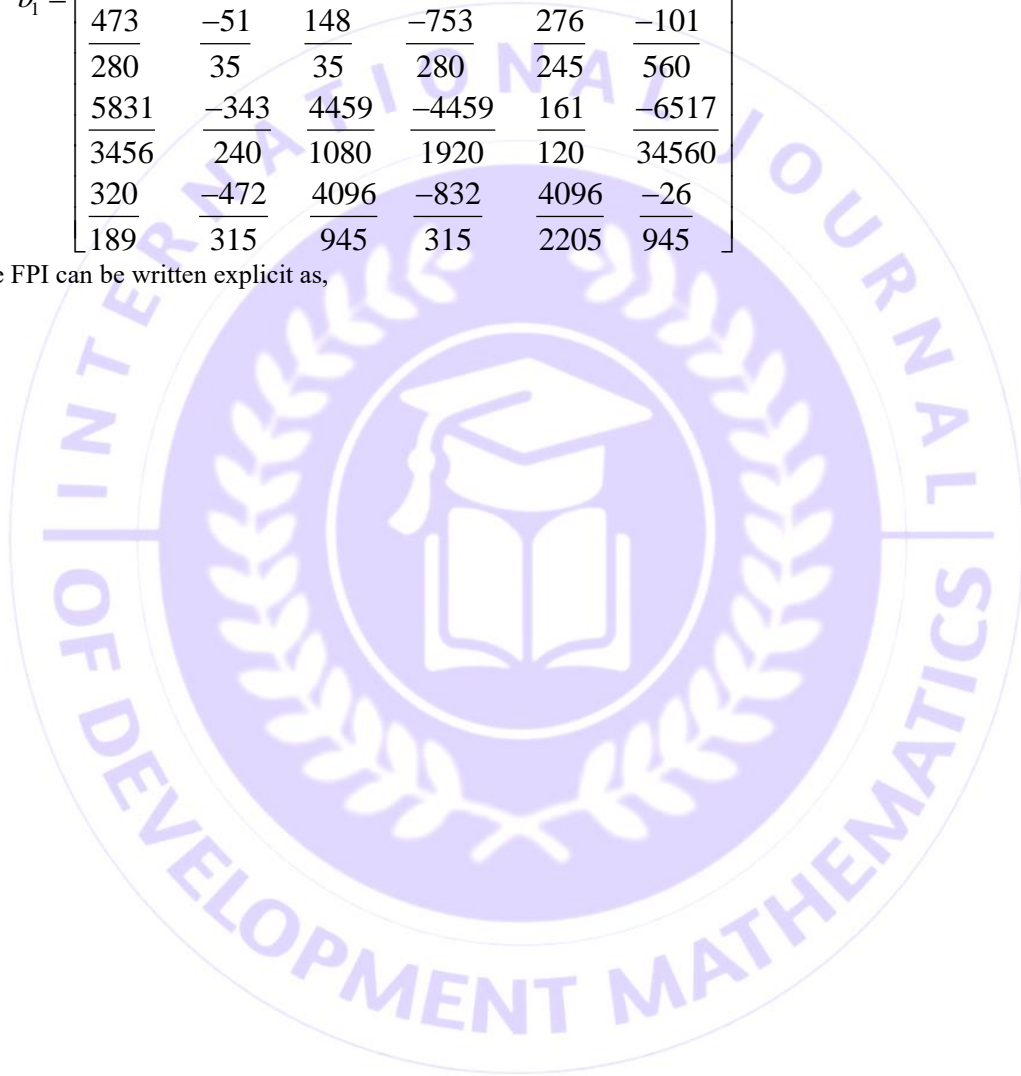
$$b_0 = \begin{bmatrix} \frac{965}{1512} & \frac{-8797}{5040} & \frac{2696}{945} & \frac{-781}{360} & \frac{368}{441} & \frac{-3971}{30240} \\ \frac{424}{189} & \frac{-29}{7} & \frac{6656}{945} & \frac{-568}{105} & \frac{512}{245} & \frac{-89}{270} \\ \frac{85375}{27648} & \frac{-630625}{129024} & \frac{107725}{12096} & \frac{-443125}{64512} & \frac{75125}{28224} & \frac{-325625}{774144} \\ \frac{1101}{280} & \frac{-3141}{560} & \frac{384}{35} & \frac{-465}{56} & \frac{792}{245} & \frac{-573}{1120} \\ \frac{132055}{27648} & \frac{-64827}{10240} & \frac{112847}{8640} & \frac{-146461}{15360} & \frac{245}{64} & \frac{-333739}{552960} \\ \frac{5312}{945} & \frac{-2224}{315} & \frac{2048}{135} & \frac{-3392}{315} & \frac{2048}{441} & \frac{-128}{189} \end{bmatrix}$$

When $i = 1$, we have

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{2039}{7056} \\ 0 & 0 & 0 & 0 & 0 & \frac{83}{294} \\ 0 & 0 & 0 & 0 & 0 & \frac{31895}{112896} \\ 0 & 0 & 0 & 0 & 0 & \frac{1107}{3920} \\ 0 & 0 & 0 & 0 & 0 & \frac{217}{768} \\ 0 & 0 & 0 & 0 & 0 & \frac{622}{2205} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 2047 & -958 & 4636 & -9329 & 3124 & -3359 \\ 1512 & 315 & 945 & 2520 & 2205 & 15120 \\ 1598 & -172 & 3392 & -302 & 832 & -341 \\ 945 & 105 & 945 & 105 & 735 & 1890 \\ 40825 & -2875 & 5965 & -23875 & 4075 & -8875 \\ 24192 & 2016 & 1512 & 8064 & 3528 & 48384 \\ 473 & -51 & 148 & -753 & 276 & -101 \\ 280 & 35 & 35 & 280 & 245 & 560 \\ 5831 & -343 & 4459 & -4459 & 161 & -6517 \\ 3456 & 240 & 1080 & 1920 & 120 & 34560 \\ 320 & -472 & 4096 & -832 & 4096 & -26 \\ 189 & 315 & 945 & 315 & 2205 & 945 \end{bmatrix}$$

The FPI can be written explicit as,



$$\begin{aligned}
 y_{n+1} &= y_n + hy'_n + h^2 \left(\frac{3113}{14112} f_n + \frac{965}{1512} f_{n+1} - \frac{8797}{5040} f_{n+2} + \frac{2696}{945} f_{n+\frac{5}{2}} - \frac{781}{360} f_{n+3} + \frac{368}{441} f_{n+\frac{7}{2}} \right. \\
 &\quad \left. - \frac{3971}{30240} f_{n+4} \right) \\
 y_{n+2} &= y_n + 2hy'_n + h^2 \left(\frac{743}{1470} f_n + \frac{424}{189} f_{n+1} - \frac{29}{7} f_{n+2} + \frac{6656}{945} f_{n+\frac{5}{2}} - \frac{568}{105} f_{n+3} + \frac{512}{245} f_{n+\frac{7}{2}} \right. \\
 &\quad \left. - \frac{89}{270} f_{n+4} \right) \\
 y_{n+\frac{5}{2}} &= y_n + \frac{5}{2} hy'_n + h^2 \left(\frac{1168075}{1806336} f_n + \frac{85375}{27648} f_{n+1} - \frac{630625}{129024} f_{n+2} + \frac{107725}{12096} f_{n+\frac{5}{2}} \right. \\
 &\quad \left. - \frac{443125}{64512} f_{n+3} + \frac{75125}{28224} f_{n+\frac{7}{2}} - \frac{325625}{774144} f_{n+4} \right) \\
 y_{n+3} &= y_n + 3hy'_n + h^2 \left(\frac{6177}{7840} f_n + \frac{1101}{280} f_{n+1} - \frac{3141}{560} f_{n+2} + \frac{384}{35} f_{n+\frac{5}{2}} - \frac{465}{56} f_{n+3} + \frac{792}{245} f_{n+\frac{7}{2}} \right. \\
 &\quad \left. - \frac{573}{1120} f_{n+4} \right) \\
 y_{n+\frac{7}{2}} &= y_n + \frac{7}{2} hy'_n + h^2 \left(\frac{11417}{12288} f_n + \frac{132055}{27648} f_{n+1} - \frac{64827}{10240} f_{n+2} + \frac{112847}{8640} f_{n+\frac{5}{2}} - \frac{146461}{15360} f_{n+3} \right. \\
 &\quad \left. + \frac{245}{64} f_{n+\frac{7}{2}} - \frac{333739}{552960} f_{n+4} \right) \\
 y_{n+4} &= y_n + 4hy'_n + h^2 \left(\frac{472}{441} f_n + \frac{5312}{945} f_{n+1} - \frac{2224}{315} f_{n+2} + \frac{2048}{135} f_{n+\frac{5}{2}} - \frac{3392}{315} f_{n+3} + \frac{2048}{441} f_{n+\frac{7}{2}} \right. \\
 &\quad \left. - \frac{128}{189} f_{n+4} \right)
 \end{aligned}
 \tag{13}$$

and

$$\begin{aligned}
 y'_{n+1} &= y'_n + h \left(\begin{aligned} &\frac{2039}{7056} f_n + \frac{2047}{1512} f_{n+1} - \frac{958}{315} f_{n+2} + \frac{4636}{945} f_{n+\frac{5}{2}} \\ &-\frac{9329}{2520} f_{n+3} + \frac{3124}{2205} f_{n+\frac{7}{2}} - \frac{3359}{15120} f_{n+4} \end{aligned} \right) \\
 y'_{n+2} &= y'_n + h \left(\begin{aligned} &\frac{83}{294} f_n + \frac{1598}{945} f_{n+1} - \frac{172}{105} f_{n+2} + \frac{3392}{945} f_{n+\frac{5}{2}} \\ &-\frac{302}{105} f_{n+3} + \frac{832}{735} f_{n+\frac{7}{2}} - \frac{341}{1890} f_{n+4} \end{aligned} \right) \\
 y'_{n+\frac{5}{2}} &= y'_n + h \left(\begin{aligned} &\frac{31895}{112896} f_n + \frac{40825}{24192} f_{n+1} - \frac{2875}{2016} f_{n+2} + \frac{5965}{1512} f_{n+\frac{5}{2}} \\ &-\frac{23875}{8064} f_{n+3} + \frac{4075}{3528} f_{n+\frac{7}{2}} - \frac{8875}{48384} f_{n+4} \end{aligned} \right) \\
 y'_{n+3} &= y'_n + h \left(\begin{aligned} &\frac{1107}{3920} f_n + \frac{473}{280} f_{n+1} - \frac{51}{35} f_{n+2} + \frac{148}{35} f_{n+\frac{5}{2}} \\ &-\frac{753}{280} f_{n+3} + \frac{276}{245} f_{n+\frac{7}{2}} - \frac{101}{560} f_{n+4} \end{aligned} \right) \\
 y'_{n+\frac{7}{2}} &= y'_n + h \left(\begin{aligned} &\frac{217}{768} f_n + \frac{5831}{3456} f_{n+1} - \frac{343}{240} f_{n+2} + \frac{4459}{1080} f_{n+\frac{5}{2}} \\ &-\frac{4459}{1920} f_{n+3} + \frac{161}{120} f_{n+\frac{7}{2}} - \frac{6517}{34560} f_{n+4} \end{aligned} \right) \\
 y'_{n+4} &= y'_n + h \left(\begin{aligned} &\frac{622}{2205} f_n + \frac{320}{189} f_{n+1} - \frac{472}{315} f_{n+2} + \frac{4096}{945} f_{n+\frac{5}{2}} \\ &-\frac{832}{315} f_{n+3} + \frac{4096}{2205} f_{n+\frac{7}{2}} - \frac{26}{945} f_{n+4} \end{aligned} \right)
 \end{aligned} \tag{14}$$

2.2 Implementation of the FPI

The implementation strategy for deriving the FPI involves a continuous scheme based on a linear multistep approach. The FPI is constructed by incorporating two off-step points within the four-step integration interval, allowing it to operate in a block-by-block, self-starting manner. This design avoids the need for external predictors, thereby simplifying the integrator's application to oscillatory differential equations. In deriving the FPI, interpolation at the selected points and collocation at specified points generate a system of equations, which is then solved for constant parameters. These parameters are substituted back into the interpolated solution, yielding a continuous formulation of the FPI. The integrator's block implementation ensures computational efficiency, as each step is calculated without requiring prior solutions or starting values, enhancing its suitability for high-frequency, oscillatory problems.

3. Convergence and Stability Properties of the FPI

The properties of the proposed FPI will be analysed in this section.

3.1 Order and Error Constant of the FPI

Suppose the linear difference operator L associated with the continuous linear multistep method (LMM) in equation (8) is defined by

$$L\{y(x); h\} = \sum_{j=0}^k [\alpha_j y(x_n + jh) - h^2 \beta_j y''(x_n + jh)] \quad (15)$$

where $y(x)$ is the arbitrary test function that is continuously differentiable in the interval $[a, b]$. Expanding $y(x_n + jh)$ and $y''(x_n + jh)$, $j = 0, 1, 2, \dots, m$ in Taylor series about the point x_n and collecting like terms in h and y gives

$$L\{y(x); h\} = c_0 y_0 + c_1 h y'(x) + c_2 h^2 y^{(2)}(x) + \dots + c_p h^p y^{(p)}(x) \quad (16)$$

The difference operator L and the continuous LMM (8) are said to be of order p if in (16), $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \dots = \bar{c}_p = \bar{c}_{p+1} = 0$, $\bar{c}_{p+2} \neq 0$, Fatunla (1980).

The term $\bar{c}_{p+2} \neq 0$ is called the error constant of the method and it implies that the local truncation error is

$$t_{n+k} = c_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3})$$

Note that, the constants c_p are defined as

$$\left. \begin{aligned} c_0 &= \sum_{j=0}^k \alpha_j \\ c_1 &= \sum_{j=0}^k (j\alpha_j - \beta_j) \\ &\vdots \\ c_p &= \sum_{j=0}^k \left[\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!} j^{p-1} \beta_j \right], p = 2, 3, \dots \end{aligned} \right\}$$

The procedure above gives the order and error constant of the FPI as displayed in Table 1.

Table 1. Order and Error Constant of the FPI

Step Number	Evaluation Point x_{n+j}	Order	Error Constant
$k = 4$	$j = 1$	7	1.7201×10^{-03}
	$j = 2$	7	4.3437×10^{-03}
	$j = \frac{5}{2}$	7	5.5817×10^{-03}
	$j = 3$	7	6.8208×10^{-03}
	$j = \frac{7}{2}$	7	8.0613×10^{-03}
	$j = 4$	7	9.2920×10^{-03}

Therefore, the proposed FPI is of order 7.

3.2 Consistency of the FPI

A linear multistep method is consistent if it has order $p \geq 1$, Fatunla (1980). Thus, the proposed FPI is consistent since it is of order 7.

3.3 Zero-Stability of the FPI

If no root of the characteristic polynomial has a modulus greater than one and every root with modulus one is simple, then such a method is called zero-stable, Dahlquist (1963). This implies that the FPI is zero-stable if its first characteristic polynomial $\rho(z)$ satisfies $|z_d| \leq 1, d = 1, 2, 3, \dots, n$. For the FPI given in equations (13) and (14), the first characteristic polynomial is given by,

$$\rho(z) = z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} z & 0 & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & 0 & -1 \\ 0 & 0 & z & 0 & 0 & -1 \\ 0 & 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & 0 & z-1 \end{vmatrix} = z^5(z-1) \quad (17)$$

Solving equation (17) for z gives $z_1 = z_2 = z_3 = z_4 = z_5 = 0$ and $z_6 = 1$. Thus, the FPI is zero-stable

3.4 Convergence of the FPI

A LMM is convergent if it is both consistent and zero-stable, Dahlquist (1963). The FPI is convergent since it is consistent and zero-stable.

3.5 Region of Absolute Stability of the FPI

A method whose stability region contains the whole left half-plane $\text{Re}(h\lambda) < 0$, is called A-stable, Lambert (1973). Highlighted below are the steps for obtaining the stability polynomial of the FPI.

Step 1: Evaluate the matrix $Aw - e_0 - e_1 - h^2d - h^2b$, where A is an identity matrix e_0, e_1, d and b are matrices obtained from the FPI, w and h are variables.

Step 2: Compute the determinant of the matrix obtained in Step 1.

Step 3: Sort the determinant obtained in Step 2 in terms of the variable h to obtain a polynomial equation $h(w)$.

Step 4: Import the coefficient of w obtained in Step 3 into the Matlab.

Step 5: Execute the command to plot the stability region. End.

On the application of the steps above, the stability polynomial of the FPI in equations (13) and (14) is given by,

$$\begin{aligned} \bar{h}(w) = & -h^{12} \left(\frac{1}{1344} w^5 - \frac{157}{2304} w^6 \right) - h^{10} \left(\frac{279857}{7526400} w^5 - \frac{235661}{67737600} w^6 \right) \\ & + h^8 \left(\frac{173203}{67737600} w^6 - \frac{43561711}{67737600} w^5 \right) - h^6 \left(\frac{17}{9216} w^6 + \frac{302669}{64512} w^5 \right) \quad (18) \\ & + h^4 \left(\frac{767}{26880} w^6 - \frac{1185713}{80640} w^5 \right) - h^2 \left(\frac{167}{672} w^6 + \frac{3903}{224} w^5 \right) + w^6 - 5w^5 \end{aligned}$$

The region of absolute stability of the FPI is shown in Figure 1.

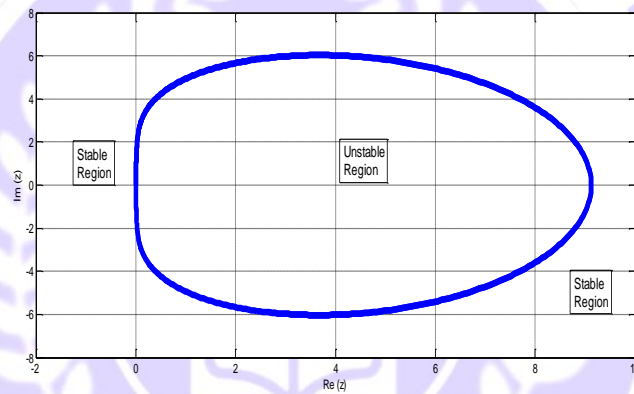


Figure 1. Region of absolute stability of the FPI

From Figure 1, it is obvious that the FPI is A-stable.

4 Results and Discussion

The newly derived FPI will be used in solving some second-order problems exhibiting oscillation. The following abbreviations will be used in the Tables 3-5 and Figures 3-10.

t or x : point of evaluation

h : step-size

NS: number of steps taken

MaxErr: maximum error

AbsErr: absolute error

CPUT: central processing unit time (in seconds)

ode45: inbuilt Matlab solver

ode15s: inbuilt Matlab solver

HBA: hybrid block algorithm of order 10 developed by Gandafa (2017)

DTM: differential transform method by Anemee and Latif (2022)

TSHBM: three-step hybrid block method of order 8 developed by Rasedee *et al.* (2019)

VOVSA: variable-order variable-step algorithm derived by Rasedee *et al.* (2017)

FPI: Newly proposed sixth order four-point integrator

4.1 Numerical Examples

Problem 1

Consider the second-order oscillatory problem

$$y''(x) + y(x) + y^3(x) = 0, \quad (19)$$

subject to the initial conditions,

$$y(0) = 1, y'(0) = 0 \quad (20)$$

defined on the interval $0 \leq x \leq 5$. This oscillator does not have an exact solution. The FPI was used in solving the problem (at $h = 0.1$) and the results obtained were compared with those of the ode15s and VOVSA (with initial stepsize $h = 0.1$ and tolerance 10^{-10}) derived by Rasedee *et al.* (2017). The computations were carried out at the endpoints $x = 0.5, 1.0, 2.0, 3.5$ and 5 . See Table 2 for numerical solution of Problem 1.

Table 2. Numerical Solution of Problem 1

x	FPI	ode15s	VOVSA
0.50	0.768719	0.768806	0.768802
1.00	0.222602	0.222689	0.222692
2.00	-0.859480	-0.859363	-0.859349
3.50	-0.093092	-0.093001	-0.093011
5.00	0.947090	0.947126	0.947130

Problem 2

Consider the second-order oscillatory problem

$$y''(x) + 2y'(x) + y(x) + 8y^3(x) = e^{-3x} \quad (21)$$

subject to the initial conditions,

$$y(0) = \frac{1}{2}, y'(0) = -\frac{1}{2} \quad (22)$$

The exact solution is given by,

$$y(x) = \frac{1}{2}e^{-x} \quad (23)$$

The FPI was also used in solving Problem 2 and the results obtained were compared with that of the TSHBM derived by Rasedee *et al.* (2019). See Table 3 and Figure 2 for numerical results and solution curves respectively.

Table 3. Maximum and average error of Problem 2

Tol	Method	NS	MaxErr	AvrErr	CPUT
10^{-2}	FPI	160	1.1807×10^{-04}	2.0512×10^{-04}	0.8379
	TSHBM	324	1.8129×10^{-01}	7.2336×10^{-02}	-
10^{-4}	FPI	208	2.2312×10^{-08}	3.6433×10^{-08}	1.9901
	TSHBM	331	1.3584×10^{-03}	4.3135×10^{-04}	-
10^{-6}	FPI	256	2.5343×10^{-10}	3.4441×10^{-10}	2.9212
	TSHBM	333	1.9814×10^{-05}	2.2591×10^{-06}	-
10^{-8}	FPI	296	4.7763×10^{-12}	2.8234×10^{-12}	3.6002
	TSHBM	347	1.5230×10^{-07}	6.7281×10^{-08}	-
10^{-10}	FPI	324	1.9412×10^{-15}	3.1231×10^{-15}	4.0271
	TSHBM	376	4.9294×10^{-09}	8.2086×10^{-10}	-

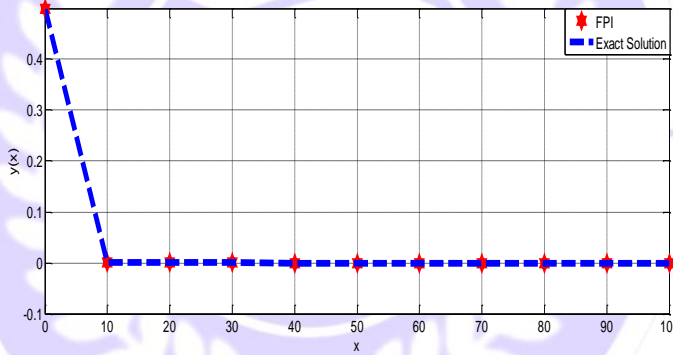


Figure 2. Solution Curves for Problem 2

Problem 3

Consider the second-order oscillatory problem

$$y''(x) + 3y(x) + 2y^3(x) = \cos(x) \sin(2x) \tag{24}$$

subject to the initial conditions,

$$y(0) = 0, y'(0) = 1 \tag{25}$$

The exact solution is given by,

$$y(x) = \sin(x) \tag{26}$$

The FPI was applied in solving Problem 3. The results obtained were again compared with those of the HBA derived by Gandafa (2017) and DTM derived by Anemee and Latif (2022). See Tables 4 for exact and numerical solutions, Table 5 for absolute errors and Figure 3 for solution curves.

Table 4. Exact and numerical solutions for Problem 3

x	Exact Solution	FPI
0.10	0.0998334166468282	0.0998334027690962
0.20	0.1986693307950612	0.1986693033436190
0.30	0.2955202066613396	0.2955201664026088
0.40	0.3894183423086505	0.3894182903676331
0.50	0.4794255386042030	0.4794254763210368
0.60	0.5646424733950354	0.5646424021627543
0.70	0.6442176872376910	0.6442176083417305
0.80	0.7173560908995227	0.7173560053751947
0.90	0.7833269096274833	0.7833268181449937
1.00	0.8414709848078964	0.8414708875874859

Table 5. Comparison of absolute errors of the FPI with those of HBA derived by Gandafa (2017) and DTM derived by Anemee and Latif (2022) for Problem 3

x	FPI	HBA	DTM
0.10	1.387773e-008	1.234156e-006	3.33e-004
0.20	2.745144e-008	4.345261e-006	2.65e-003
0.30	4.025873e-008	6.123145e-006	8.84e-003
0.40	5.194102e-008	7.350192e-006	2.07e-002
0.50	6.228317e-008	8.001627e-006	3.96e-002
0.60	7.123228e-008	1.341672e-005	6.70e-002
0.70	7.889596e-008	3.091272e-005	1.04e-001
0.80	8.552433e-008	5.718293e-005	1.50e-001
0.90	9.148249e-008	7.825512e-005	2.06e-001
1.00	9.722041e-008	9.671528e-005	2.72e-001

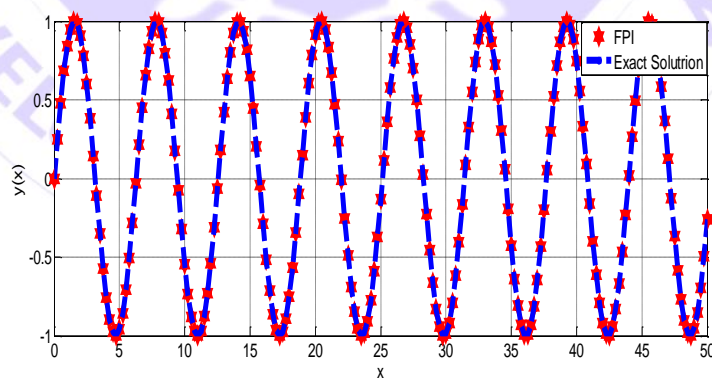


Figure 3. Solution Curves of Problem 3

Problem 4

Consider the nonlinear system below which describes the aerodynamic damping of a pendulum

$$\theta''(t) + \frac{g}{L} \sin \theta + \frac{c_a}{mL} (\theta'(t))^2 \operatorname{sgn}(\theta'(t)) = 0 \quad (27)$$

and the linear system below which describes the viscous damping of a pendulum

$$\theta''(t) + \frac{c_d}{m} (\theta'(t))^2 + \frac{g}{L} \theta = 0 \quad (28)$$

To determine the quadratic damping in equations (27) and (28) respectively, the newly derived FPI is employed in simulating the two equations and the simulation results were compared with that of ode45 adopted by Soraya (2016). The author transformed (27) and (28) into their equivalent first-order systems and then applied Matlab solver (ode45) to simulate the results. Let $g = 981 \text{cms}^{-2}$, $c_a = c_d = 14$, $L = 5 \text{cm}$, $m = 10 \text{g}$, $\theta(0) = 1.57$ and $\theta'(0) = 0$, with the initial value θ being in radians. Also, sgn in equation (27) represents signum function; c_a and c_d are the coefficients of aerodynamic damping and viscous damping respectively. Figure 4 shows a simple pendulum with two forces acting on the mass; that is, the weight mg and the tension T . The net force is found to be $F = mg \sin \theta$.

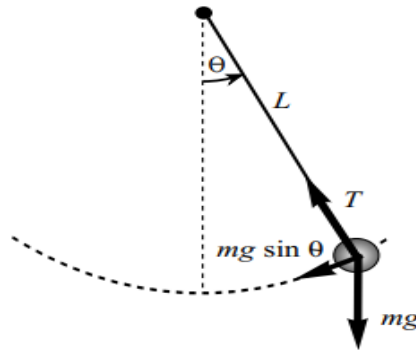


Figure 4. A simple pendulum (Soraya, 2016)

From simulation results presented in Figure 5, it is clear that quadratic damping causes a rapid initial amplitude reduction but is less effective as time increases and amplitude decreases. On the other hand, linear damping is less effective initially but is more effective as time increases. The plots confirm that the FPI is computationally reliable as they agree with the simulation results of ode45.

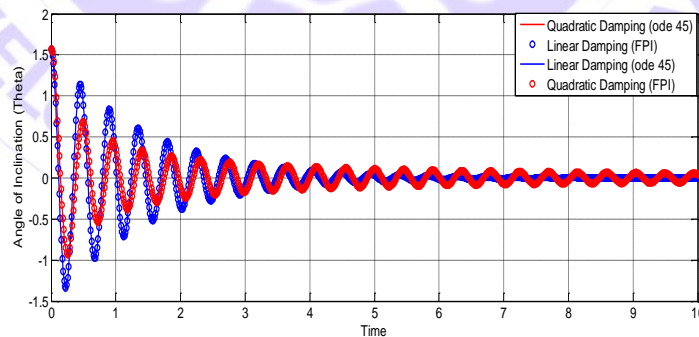


Figure 5. Solution curves of Problem 4 showing quadratic and linear damping

Problem 5

Consider the governing equation for motion of a semi-cylinder (half cylinder) with mass m and radius R rolling on a horizontal plane. This problem is given by the oscillatory equation,

$$\left(\frac{3mR^2}{2} - 2mbR \sin \theta \right) \theta''(t) - mbR \cos \theta (\theta'(t))^2 = wb \cos \theta \quad (29)$$

Equation (29) is solved by generating a solution curve of angle of oscillation θ against time t . The mass centre is at G located at distance $b = \frac{4R}{3\pi}m$ from the cylindrical axis at O . Take $m = 5\text{kg}$, $R = 0.1\text{m}$, $g = 9.81\text{ms}^{-2}$ and $w = mg$. Figure 6 shows a typical half-cylinder on a horizontal plane with an angle of inclination θ .

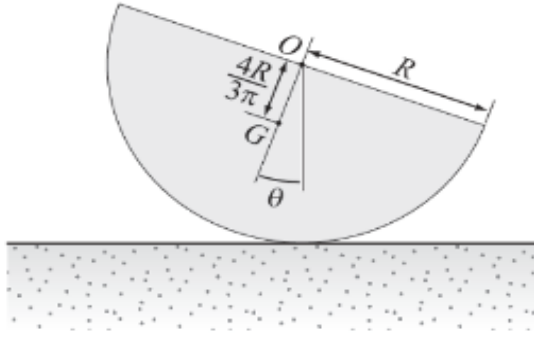


Figure 6. Schema of a Half Cylinder Rolling on a Horizontal Plane (Harihara and Childs, 2020)

Harihara and Childs (2020) simulated Problem 5 by converting the equation into its equivalent first order systems and then employed ode45 solver to generate the solution curve.

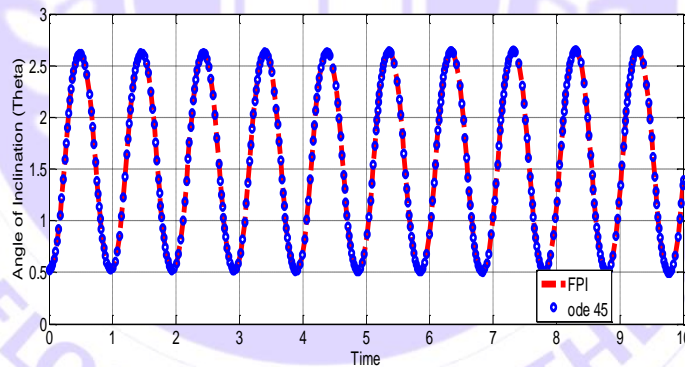


Figure 7. Solution curves for Problem 5 showing the angle of inclination against time

The solution curves obtained in Figure 7 using the newly derived FPI on Problem 5 converge to those of the curve generated by the ode45 solver, showing that the new FPI is computationally reliable.

4.2 Discussion of Results

The FPI, designed to handle problems involving oscillations, was applied in solving five oscillatory problems to evaluate its accuracy, stability, and computational efficiency. The consistency and zero-stability of the FPI were confirmed through theoretical analysis, which showed that the method meets the necessary criteria for convergence. The order of the FPI, being seven, further contributes to its accuracy, as it demonstrates that the integrator can handle higher oscillatory frequencies without loss of precision. Stability analysis, demonstrated through the region of absolute stability, suggests that the FPI is A-stable, confirming its suitability for oscillatory problems, where other methods may struggle with numerical instability.

Comparative results with traditional methods, such as ode45, ode15s and other hybrid approaches, underscore the efficiency of the FPI in terms of computational resources and memory usage. Tables 2-5 and Figures 2,3,5 and 7 display the high precision of the FPI in approximating solutions to oscillatory problems, with error rates consistently lower than those observed in standard solvers. In practical applications, including the modeling of damped oscillators and pendulum dynamics, the results of the FPI were in close agreement with the outcomes generated by established solvers like ode45 and ode15s but were achieved with less computational overhead. This reduced resource demand makes the FPI a promising choice for large-scale simulations and real-time applications where efficiency is critical. The performance of the FPI across various test cases highlights its adaptability to a wide range of oscillatory problems, demonstrating its utility as a reliable tool for both linear and nonlinear second-order differential equations.

5 Conclusion

The newly derived FPI is an effective and computationally efficient method for solving second-order differential equations exhibiting oscillatory behaviour. Through its design, which preserves the original structure of these equations, the FPI avoids the drawbacks associated with traditional methods that require conversion to first-order systems. This direct approach maintains the essential properties of the original equations, ensuring both phase accuracy and amplitude preservation, which are critical for oscillatory solutions. The analysis confirmed the FPI's theoretical strengths, including consistency, zero-stability, and A-stability, which collectively guarantee that the method is both convergent and capable of handling stiff problems. With its seventh-order accuracy, the FPI demonstrates a high level of precision, making it especially suited to problems involving high-frequency oscillations. The self-starting, block-by-block structure of the FPI further enhances computational efficiency by eliminating the need for predictor values. Comparative testing against established solvers, such as ode45, ode15s and other existing methods, shows that the FPI achieves lower error rates and requires fewer computational resources, underscoring its practical applicability in simulations of oscillatory systems. In summary, the FPI is presented as a reliable, high-performance integrator, offering a valuable alternative for researchers and practitioners dealing with complex oscillatory differential equations.

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