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The Analysis of Hypo-Exponential and Hyper-Exponential Distributions in Solving Performance Measures for General Phase Type Distributions

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ABSTRACT

The importance of phase-type distribution in modeling activities cannot be under emphasized when both a distribution's initial and second moments are accessible, or when the sequence of data points for computing moments is the information available. In continuous time process for an absorbing finite state Markov chain, the phase-type distribution can be thought of as the of the time until absorption, and it is widely used in queueing theories and other fields of applied probabilities. The common phase-type distributions are generalized Erlang, Coxian, Hypo-exponential, and Hyper-exponential distributions. In this study, performance measures of phase-type distribution using Hypo-exponential, and Hyper-exponential distributions have been looked into, in order to provide meaningful study into the probability function, mean, k^{th} moment, variance, Laplace Stieltjes transform and squared coefficient of variation of phase type distribution. The study started by considering the tractability and memory less properties of exponential distribution, and since these properties are not enough, we examined the journey through a series of exponential phases to arrive at performance measures. Illustrative examples are demonstrated for various cases to arrive at various values for probability functions, Laplace Stieltjes transform, squared coefficient of variation, k^{th} moment, mean and variance for the phase type distribution.

1. Introduction

The exponential distribution is very important due to both its tractability and memory-less characteristics in performance modeling, but for modeling random variables with lower unpredictability, exponential distribution might not be enough, and this makes the exponential distribution not sufficient. To overcome this problem while modelling general distributions, the phase type distribution is recommended. Phase-type distributions is very useful when the distribution with known mean and variance is to be formed, and the name phase-type distributions came to be due to fact that, processes can be seen as the movement via a series of exponential steps. Phase type distribution has it major applications in queueing theories and applied probabilities with the used of generalized Erlang, Coxian, Hypo-exponential, and Hyper-exponential distributions. The useful technique in phase type distribution when the added network is represented by flow-equivalent servers instead of subsystems is found in Marie (1980). The principles, laws of phase type and most cited introductory article is (Neuts, 1981) and some Important theoretical concepts on phase-type is established (Cumani, 1982; O'Cinneide, 1989). A reliable recursive method for calculating the probability vector of the steady state is suggested (Ramaswami *et al.*, 1980; Ramaswami, 1988). The Hessenberg matrix computation of the exponential in the evaluation of the Padé approximation for phase type (Aalen and Sidje, 1993). The simulation of phase type distribution in a unique way is found in (William, 2009) and, some new results on Markov chain connected to a distribution of phase types in (Christian and Stephane, 2010)

Furthermore, the second order recurrence relation with constant coefficient, limiting behaviour and recursion process to arrive at performance measures is considered (Agboola, 2011), while the introduction of phase type in survival analysis is analysed on how a phenomenon such as a disease, moves through different phases, and calculated

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hazard rates and densities of phase-type distributions using Markov chain to affirmed that hazard rates are asymptotically constant due to quasi-stationarity (Aalen, 2014) and a phase-type distribution approximation function so as to Steer clear of inverse matrix calculations is introduced (Belen *et al.,* 2020)

The direct equation approaches for the stationary distribution of Markov chains to yield a far more accurate response in less time once a predetermined number of clearly defined stages have been achieved (Agboola, 2021), and the likelihood of transitioning to one or more of the closed communicative classes from transitory states and the probability of the absorption matrix is established (Agboola and Ayoade, 2021).
The block iterative approach only needs one iteration to produce the solution for the stationary distribution of Markov chains (Agboola and Ayinde, 2022), and a partial characterization of the set of busy period durations which are presented by an r-phases Coxian distribution (Osogami and Harchol-Balter, 2002). The phase-type model is extended to accommodate competing risks using a few data points and the Coxian competing risks model (Bo Henry, 2022). The multi-state processes models are illustrated when the dimension of the state space is greater than one and obtained the proportional hazards specification. (Martin, 2023) In addition, the general approach for the two-layer censored data, using the canonical form of a cyclic phase-type distributions (APHDs) and the expectation algorithm to compute the estimate by maximum likelihood (Yudong and Zhi-Scheng, 2023). A phase-type distribution that is not homogenous for the cumulative hazard rate, reliability function, and hazard rate using the maximum likelihood, and the characteristic function (Acal *et al*., 2024). The phase-type distributions in coalescent models showing the states in the ancestral process are represented by stages in the phase-type distribution and concluded that a mathematical foundation for coalescent theory is made possible by phase-type distributions (*Hobolth et al., 2024).*

However, this paper's primary goal is to examine how phase-type technique might be altered to incorporate performance measure of phase type distribution using hypo- exponential and hyper – exponential distributions, in order to evaluate the mean, k^{th} moment, variance, Laplace Stieltjes transform and phase type distributions' squared coefficient of variation.

Notation

$E(Y)$, *expected value of random variable Y; μ, service time parameter;* σ_y^2, *variance;* $f_Y(y)$, density function for a random variable Y; $F_Y(y)$, *distribution function for a random variable Y;* $L_y(s)$, *Laplace transform of random variable Y;* p_k, likelihood that only the first k service phases will be executed before the process ends; $E[Y^k]$, k^{th} moment of a random variable Y; $\propto_i$, the likelihood of going from state i to state $(i+1)$; c_y^2, squared coefficient of variation for random variable Y; R_i, $i = 1, 2, 3, \dots, k$, initial probabilities; r_{ij}, $i, j = 1, 2, 3, \dots, k$, routine probability.

2. METHODOLOGY

The study area emphasized the analysis of exponential distribution, two –exponential service phases processes, and general phase distribution, with the evaluation of hypo-exponential and hyper-exponential distributions to arrive at performance parameters, mean, variance, k^{th} moment, Laplace transform and squared coefficient of variation for general stage type distribution

2.1 Two Exponential Service Phase Distribution

We started by looking at a random variable Y that represents a customer's service time at a service center in order to analyze the exponential distribution, which has a single exponential phase. This is the amount of time that the client spends getting service; it excludes any waiting time that may have occurred. We assume that this service time is exponentially distributed with parameter $\mu > 0$. This is shown graphically in Figure 1, where a circle containing the exponential distribution's parameter represents the single exponential phase. In order to service customers, they enter the phase from the left, stay there for a period of time that is exponentially distributed with parameter μ, and then leave to the right. Assuming we have the random variable Y, which has an exponential distribution with mean $E(Y) = \frac{1}{\mu}$ *and variance* $\sigma_y^2 = \frac{1}{\mu^2}$, to represent the time it takes for clients to arrive at a service center.

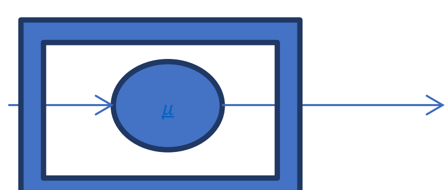


Figure 1: An Exponential Service Phase

The Figure 1 indicates that, One exponential phase might be used to represent the service rendered to a user, while the figure 2 indicates that, the service time can be expressed by a second exponential phase.

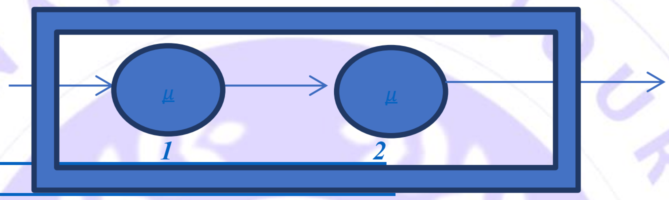


Figure 2: Two Tandem Exponential Service Phases

With random variable Y the customer receives service which is exponentially distributed with parameter μ as the customer enters the servicing process. At the completion of service's stage, the customer enters the second stage when the exponentially distributed service time with parameter μ is begins with. At the second stage completion, another's customer service time is begins with by service process. Since both service stages are the same exponentially distributed with parameter μ, and they are independent. Then, two independent servers are not containing in the servicing process, but consist of one service provider operating in one or more stages at some point. In order to examine this instance, we will imagine that each phase's probability density function is provided by given by

$$f_Y(y) = \mu\, e^{-\mu y}, \qquad (1)$$

The time selected at random from $f_Y(y)$ is first spends by the customer. After the time completion, another amount of time chosen independently from $f_Y(y)$ is again spends. After the completion of this second time chosen, the new customer starts to receive service immediately the one in service departs. The customer's overall time distribution in the service is now looked into, and this is taking to be the sum of two identically distributed random variables, X which is independent exponential random variables.

Taking the randomly exponentially distributed variable with parameter μ to be Y.

Then

$$X = Y + Y. \qquad (2)$$

Therefore, using the convolution theorem relating to two random variables that are independent,

The convolution theorem states that

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y)\, f_Y(x-y)dy$$

$$f_X(x) = \int_0^x \mu\, e^{-\mu y}\, \mu\, e^{-\mu(x-y)}\, dy$$

$$f_X(x) = \mu^2 e^{-\mu y} \int_0^x dy = \begin{cases} \mu^2 x e^{-\mu y}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad (3)$$

The equation (3) represents the frequency density function for an Erlang-2, E_2 distribution, while the equation (4) below represents its cumulative distribution

$$F_X(x) = 1 - e^{-\mu x} - \mu x e^{-\mu x} = 1 - e^{-\mu x}\{1 + \mu x\}, \quad x \geq 0. \qquad (4)$$

The density function can equally be computed using Laplace transforms, by multiplying the Laplace transform of the various phases by the Laplace transform of the frequency density function for the entire service time.

Therefore, the Laplace transform to determine the overall duration of service distribution is

$$L_X(s) = \int_0^\infty e^{-sx} f_X(x)dx$$

and each stage of the exponential phases' Laplace transform is

$$L_y(s) = \int_0^\infty e^{-sy} f_Y(y)dy = \left(\frac{\mu}{s+\mu}\right)$$

Then

$$L_X(s) = E\left[e^{-s\{y_1+y_2\}}\right] = E\left[e^{-sy_1}\right] \times= E\left[e^{-sy_2}\right] = \left(\frac{\mu}{s+\mu}\right)^2 \quad (5)$$

To find the function of x whose transform is $\left(\frac{\mu}{s+\mu}\right)^2$, the Laplace inversion theorem is being used.

Since the Laplace transform of $\frac{1}{(s+a)^{r+1}}$, i..e. $L_X\left(\frac{1}{(s+a)^{r+1}}\right) = \frac{x^r}{r!} e^{-ax}$.

By representing a = μ and r = 1, we arrived at inversion of $L_X(s)$ to obtain

$$f_X(x) = \mu^2 x e^{-\mu y}, \qquad x \geq 0$$

Likewise, we may obtain the mean value and higher moments from the Laplace transform as

$$E[X^k] = (-1)^k \frac{d^k}{ds^k} L_X(s)\Big|_{s=0}, \quad for\ k = 1, 2, \cdots$$

$$E[X] = \frac{d}{ds} L_X(s)\Big|_{s=0} = -\mu^2 \frac{d}{ds}\left(\frac{1}{s+\mu}\right)^2 = \mu^2 \frac{d}{ds}(s+\mu)^{-2}\Big|_{s=0} = \frac{2}{\mu}. \quad (6)$$

$$\sigma_X^2 = \left(\frac{1}{\mu}\right)^2 + \left(\frac{1}{\mu}\right)^2 = \frac{2}{\mu^2}. \quad (7)$$

2.2 The Hypo Exponential Distribution

An Erlang-r distribution, which is a series of exponential phases, is preferable to an exponentially distributed random variable for modeling random variables with lower unpredictability while preserving the advantageous mathematical characteristics of the exponential distribution. An acceptable choice for the mean of each phase can yield any given value μ for the mean, and if there are r phases, then the mean of each should equal $r\mu$. However, the choice of variance is constrained when there is just one set of phases. For integer i, only values that result in squared coefficients of variation equal to $(1/i)$ are feasible. We discovered that a combination of Erlang distributions is one way to get around this problem. Another method is to adjust the Erlang-r distribution's phase-type representation by allowing each phase to have a distinct parameter; that is, we allowed phase i to offer exponentially distributed service with parameter μ_i which may vary for each phase. This results in the hypo exponential distribution, which is depicted in Figure 3 below.

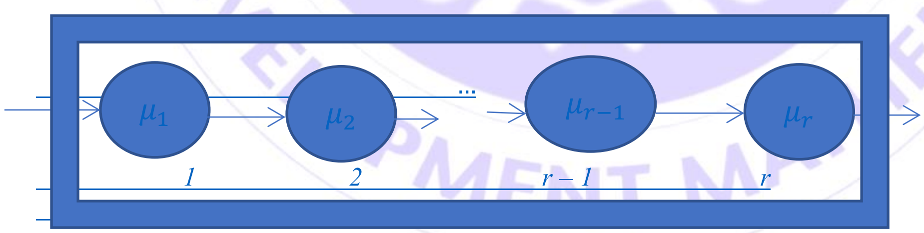


Figure 3: Hypo exponential Distribution

Firstly, we examine the case with two exponential phases. Assume that $Y = X_1 + X_2$ and that X_1 and X_2 are two independent random variables with exponential distributions and parameters μ_1 and μ_2, respectively. Next, the convolution of the two exponential distributions yields the probability density function of Y, We have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(y-x)dx$$

$$= \int_{-\infty}^{\infty} \mu_1 e^{-\mu_1 y} \mu_2 e^{-\mu_2(y-x)} dx$$

$$= (\mu_1 \mu_2) e^{-\mu_1 y} \int_{-\infty}^{\infty} e^{-\mu_2 (y-x)}\, dx$$

$$f_Y(y) = \frac{\mu_1 \mu_2}{(\mu_1 - \mu_2)} e^{-(\mu_2)y} - e^{-(\mu_1)y}, \quad \forall x \geq 0. \qquad (8)$$

The cumulative distribution function associated with it is presented by

$$F_Y(y) = 1 - \left\{ \frac{\mu_1}{(\mu_1 - \mu_2)} e^{-(\mu_1)y} \right\} + \left\{ \frac{\mu_2}{(\mu_1 - \mu_2)} e^{-(\mu_2)y} \right\}, \quad y \geq 0. \qquad (9)$$

Its squared coefficient of variation, variance, and expectation are provided by

$$E(y) = \frac{1}{\mu_1} + \frac{1}{\mu_2}, \quad \sigma_Y^2 = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}, \quad C_Y^2 = \frac{\mu_1^2 + \mu_2^2}{(\mu_1 - \mu_2)^2} < 1 \qquad (10)$$

Its Laplace transform given by

$$L_Y(s) = \frac{\mu_1}{(s - \mu_1)} \frac{\mu_2}{(s - \mu_2)} \qquad (11)$$

From equation (11), for more than two phases, the probability density function of an r-phase hypo exponential random variable Y, the Laplace transform is

$$L_Y(s) = \frac{\mu_1}{(s - \mu_1)} \frac{\mu_2}{(s - \mu_2)} \cdots \frac{\mu_r}{(s - \mu_r)} \qquad (12)$$

The convolution of r exponential densities, each with a unique parameter μ_i yields the density function $f_Y(y)$, which is given by

$$f_Y(y) = \sum_{i=1}^{r} \propto_i \mu_i e^{-\mu_i y}, \quad y \geq 0.$$

where

$$\propto_i = \begin{cases} \prod_{j-1,\ j\neq i}^{r} \frac{\mu_j}{\mu_j - \mu_i}. \\ 0, \quad elsewhere \end{cases} \qquad (13)$$

The following are the squared coefficient of variation, variance, and expectation:

$$E(y) = \sum_{i=1}^{r} \frac{1}{\mu_i}, \quad \sigma^2(y) = \sum_{i=1}^{r} \frac{1}{\mu_i^2}, \quad and \quad C_y^2 = \frac{\sum_{i=1}^{r} \frac{1}{\mu_i^2}}{\left(\sum_{i=1}^{r} \frac{1}{\mu_i}\right)^2} \leq 1. \qquad (14)$$

To show that C_y^2 cannot be greater than 1, we use the fact that for real $a_i \geq o$, $\sum_i a_i^2 \leq (\sum_i a_i)^2$, since the right-hand side contains the left-hand side plus the sum of all the nonnegative cross-terms.

Now taking

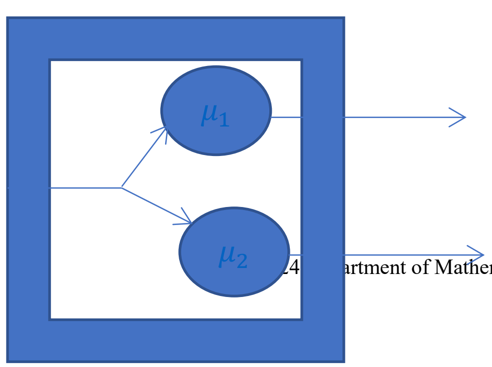
$$a_i = \frac{1}{\mu_i},$$

Implies

$$C_y^2 = \frac{\sum_{i=1}^{r} \frac{1}{\mu_i^2}}{\left(\sum_{i=1}^{r} \frac{1}{\mu_i}\right)^2} \leq 1. \qquad (15)$$

The Hyper exponential distribution

Our aim is finding a phase-type arrangement that yields more coefficients of variation than the exponential. We examine the configuration shown in Figure 4, where the chance of taking the top phase is represented by α_1, and the likelihood of taking the lower phase is represented by $\alpha_2 = 1 - \alpha_1$. A customer entering service with probability α_1, receive service that is exponentially distributed with parameter μ_1 and then leave the server, or, if such a distribution is used to model a service facility, a customer entering service with probability α_2, receive service that is exponentially distributed with parameter μ_2 and then leave the server. This implies neither phase can be active at the same time; only one customer can be receiving service at any given time.



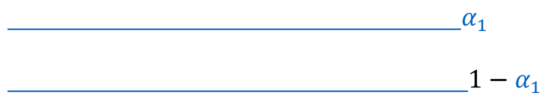


Figure 6: Two Exponential Phases in Parallel

The density function of a customer's service time is provided by

$$f_Y(y) = \begin{cases} \propto_1 \mu_1 e^{-\mu_1 y} + \propto_2 \mu_2 e^{-\mu_2 y}, & y \geq 0 \\ 0, & y < 0 \end{cases} \qquad (16)$$

Whereas, the function of the cumulative distribution is

$$F_Y(y) = \propto_1 (1 - e^{-\mu_1 y}) + \propto_2 (1 - e^{-\mu_2 y}), \quad y \geq 0 \qquad (17)$$

Its Laplace transform is

$$L_Y(s) = \propto_1 \frac{\mu_1}{(s-\mu_1)} + \propto_2 \frac{\mu_2}{(s-\mu_2)} \qquad (18)$$

The initial and subsequent moments are provided by

$$E[Y] = \frac{\propto_1}{\mu_1} + \frac{\propto_2}{\mu_2} \text{ and } E[Y^2] = 2\frac{\propto_1}{\mu_1^2} + 2\frac{\propto_2}{\mu_2^2}. \qquad (19)$$

The variance can be calculated as $E[Y^2] - E[Y]^2$ and the squared coefficient of variation can be obtained as

$$C_y^2 = \frac{E[Y^2]-E[Y]^2}{E[Y]^2} = \frac{E[Y^2]}{E[Y]^2} - 1 = \frac{2\frac{\propto_1}{\mu_1^2} + 2\frac{\propto_2}{\mu_2^2}}{\left(\frac{\propto_1}{\mu_1} + \frac{\propto_2}{\mu_2}\right)^2} - 1 \geq 1. \qquad (20)$$

In the broader context of numerous parallel phases, it is demonstrated that the squared coefficient of variation is not less than 1.

By expanding the number of phases it contains, this is now extended to a phase-type service facility. With branching probability $\sum_{i=1}^{r} \alpha_i = 1$., and r parallel phases The case is depicted in Figure 5.

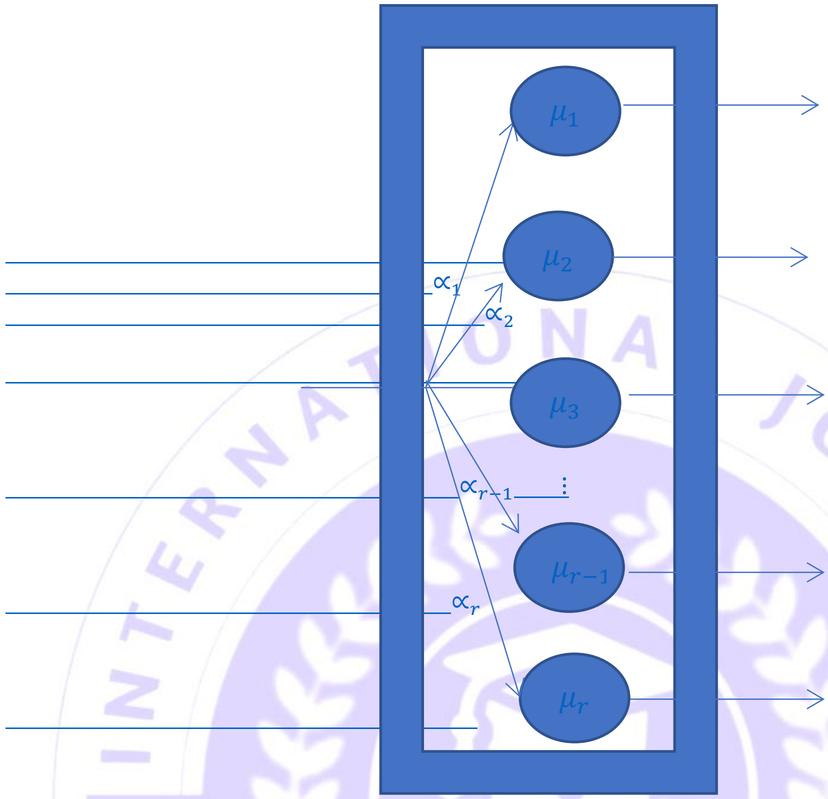


Figure 5: Multiple exponential phases in parallel

The density function and Laplace transform for this phase type distribution are given by

$$f_Y(y) = \sum_{i=1}^{r} \propto_i \mu_i \, e^{-\mu_i y}, \qquad y \geq 0.$$

and

$$L_y(s) = \sum_{i=1}^{r} \propto_i \frac{\mu_i}{(s - \mu_i)}$$

This distribution is referred to as hyper-exponential distribution. It's first two moment are

$E[Y] = \sum_{i=1}^{r} \frac{\propto_i}{\mu_i}$ and $E[Y^2] = 2\sum_{i=1}^{r} \frac{\propto_i}{\mu_i^2}$

And the square of variation is given as

$$C_y^2 = \frac{E[Y^2]}{(E[Y])^2} - 1 = \frac{2\sum_{i=1}^{r}\frac{\propto_i}{\mu_i^2}}{\left(\sum_{i=1}^{r}\frac{\propto_i}{\mu_i}\right)^2} - 1$$

To prove that this square coefficient of variation is greater than one

$$\left(\sum_{i=1}^{r}\frac{\propto_i}{\mu_i}\right)^2 \leq \sum_{i=1}^{r}\frac{\propto_i}{\mu_i^2}$$

By Cauchy –Schewatz inequality, which for real a_i and b_i states that

$$\left(\sum_{i=1}^{r} a_i b_i\right)^2 \leq \sum_{i=1}^{r}(a_i)^2 \sum_{i=1}^{r}(b_i)^2$$

Substituting $a_i = \sqrt{\propto_i}$ and $b_i = \sqrt{\frac{\propto_i}{u_i}}$ implies that

$$\left(\sum_{i=1}^{r}\frac{\propto_i}{\mu_i}\right)^2 = \left(\sum_{i=1}^{r}\sqrt{a_i}\frac{\sqrt{a_i}}{\mu_i}\right)^2 \leq \left(\sum_{r}^{2} a_i\right)^2\left(\sum_{r}^{r}\frac{\sqrt{a_i}}{\mu_i}\right)^2 = \left(\sum_{r}^{2} a_i\right)\left(\sum_{r}^{r}\frac{\sqrt{a_i}}{\mu_i}\right)$$

Using Cauchy Schewatz

$$\sum_{i=1}^{r}\sqrt{a_i}\frac{\sqrt{a_i}}{\mu_i} = \sum_{i=1}^{r}\frac{a_i}{\mu_i^2}$$

Since

$$\sum_{i=1}^{r}\propto_i = 1$$

Therefore

$$C_y^2 \leq 1$$

3. RESULTS AND DISCUSSION

This section presented the Illustrative examples and solutions for performance measures for phase type distribution using hypo-exponential and hyper-exponential distributions to obtain the expectation, k^{th} moment, variance, and squared coefficient of variation of Z, as well as its probability density function.

Illustrative Example 1

Given the two-phase hyper-exponential random variable X with parameters $\propto_1 = 0.3$, $\mu_1 = 4$, and $\mu_2 = \frac{1}{3}$.

To find the performance measures, expected value, standard deviation, and squared coefficient of variation of the random variable X.

Solution:

$$E[Y] = \frac{\propto_1}{\mu_1} + \frac{\propto_2}{\mu_2} = \frac{0.3}{4} + \frac{0.7}{1/3} = 0.075 + 2.1 = 2.175.$$

$$E[Y^2] = 2\frac{\propto_1}{\mu_1^2} + 2\frac{\propto_2}{\mu_2^2} = 2\frac{0.3}{16} + 2\frac{0.7}{\frac{1}{9}} = 0.0375 + 12.6 = 12.6375$$

$$C_y^2 = \frac{12.6375 - (2.175)^2}{(2.175)^2} = \frac{12.675}{4.7306} - 1 = 1.6794.$$

This shows that the coefficient of variation is greater than one.

Illustrative Example 2

Suppose a random variable with the parameters $\mu_1 = 2,\ \mu_2 = 3,\ \mu_3 = 4$ represented as three successive exponential phases. To determine the random variable X probability density function, expectation, variance, and squared coefficient of variation. Since the three exponential phases are independent of one another, the variance is thus equal to the sum of the variances of each phase, and the expectation of Y is simply equal to the sum of the expectations of each phase.

Hence

$$E(y) = \sum_{i=1}^{3} \frac{1}{\mu_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

$$\sigma^2(y) = \sum_{i=1}^{r} \frac{1}{\mu_I^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{61}{144}$$

$$C_y^2 = \frac{\sum_{i=1}^{r} \frac{1}{\mu_I^2}}{\left(\sum_{i=1}^{r} \frac{1}{\mu_i}\right)^2} = \frac{61}{144} \times \frac{144}{169} = \frac{61}{169} = 0.361 \leq 1$$

When $i = 1$,

$$\propto_1 = \prod_{j=1,\ j\neq i}^{r} \frac{\mu_j}{\mu_j - \mu_1} = \frac{\mu_2}{\mu_2 - \mu_1} \times \frac{\mu_3}{\mu_3 - \mu_1} = \frac{3}{1} \times \frac{4}{2} = 6$$

When $i = 2$,

$$\propto_2 = \prod_{j=1,\ j\neq i}^{r} \frac{\mu_j}{\mu_j - \mu_2} = \frac{\mu_1}{\mu_1 - \mu_2} \times \frac{\mu_3}{\mu_3 - \mu_2} = \frac{2}{-1} \times \frac{4}{1} = -8$$

When $i = 3$,

$$\propto_3 = \prod_{j=1,\ j\neq i}^{r} \frac{\mu_j}{\mu_j - \mu_3} = \frac{\mu_2}{\mu_3 - \mu_3} \times \frac{\mu_1}{\mu_1 - \mu_3} = \frac{2}{-2} \times \frac{3}{-1} = 3$$

It follows then that

$$f_Y(y) = \sum_{i=1}^{r} \propto_i \mu_i\, e^{-\mu_i y} = 12e^{-2y} + 12e^{-2y} + 12e^{-2y} - 24e^{-3y} + 12e^{-4y}, y \geq 0.$$

If coefficients of variation larger than 1 are needed, neither the hypo-exponential distribution nor a combination of Erlang distributions can be applied, and instead of using phases in series as we have up to this point, we switch to phases in parallel.

4. Conclusion

In this study, performance measures of phase type distribution using Hypo-exponential, and Hyper-exponential distributions have been looked into, in order to provide meaningful study into the probability function, mean, kth

moment, variance, Laplace Stieltjes transform and squared coefficient of variation of phase type distribution. We begin from the tractability and memory less properties of exponential distribution, and since these properties are not enough, w e examined the journey through a series of exponential phases. by the use of matrix and vector operations to arrive at performance measures. Illustrative examples are demonstrated for various cases to arrive at various values for probability functions, Laplace Stieltjes transform, squared coefficient of variation, k^{th} moment, mean and variance for the phase type distribution.

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