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# Numerical Solution of 2D Nonlinear Volterra-Fredholm Integral Equations using Polynomial Collocation Method

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A B S T R A C T

In this research, polynomial collocation method was used to develop and implement numerical solutions of nonlinear two-dimensional (2D) mixed Volterra-Fredholm integral equations. The Integral equation was transform into systems of algebraic equations using standard collocation points with Bernstein polynomial as a basis function and then solves the nonlinear algebraic equations using Newton-Rhapson method. The analysis of the developed method was investigated and the solution was found to be unique and convergent. To illustrate the efficiency, simplicity, and accuracy of the approach, illustrative examples are provided which shows that the method outperforms the other methods.

## 1. Introduction

An equation is considered integral if the unknown function appears inside the integral sign. The various forms of integral equations primarily depend on the equation's kernel and the integration's limits. According to Wazwaz (2011), an integral equation is referred to as a Volterra integral equation if at least one of the limits is variable and a Fredholm integral equation if the limits of integration are fixed. The Fredholm integral equation is characterized by fixed integration limits, whereas the Volterra integral equation exhibits at least one variable integration limit.

An essential tool for modelling a wide range of phenomena and resolving various boundary value issues involving ordinary and partial differential equations is the integral equation. One of the most helpful mathematical fields in both pure and applied mathematics is integral equations, which have numerous applications in science, engineering, etc. (Khuri and Wazwaz, 1996). An equation that combines the Fredholm integral and the Volterra integral in one equation is known as the Volterra-Fredholm Integral equation.

Majority of nonlinear two-dimensional mixed Volterra-Fredholm integral (NL2D MVFIE) equations are difficult to solve analytically. Most approach in literatures could only solve the linear Volterra-Fredholm integral equations in 2D. It is necessary therefore to develop numerical method that can solve the nonlinear 2D mixed Volterra-Fredholm integral equations; in this research polynomial collocation method was used to solve these problems.

Many have developed various methods for solving two-dimensional mixed Volterra-Fredholm integral equations including; multiquadric radial basis functions (Almasied and Meleh, 2014), Two-dimensional Legendre wavelets method (Banifatemi *et al*., 2007), Applications of two-dimensional triangular functions (Maleknejad and

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Behbahani, 2012), series solution methods (Rostam and Karzan, 2015) and many more. In this research work, we will consider two dimensional linear mixed Volterra-Fredholm integral equation of the form:
$m(x,t) = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)m(y,z)^n dydz\, , n \geq 2$ (1)
Where $u(x,t)$ is considered an unknown function to be determined, the functions $f(x,t)$ is analytic on $C\ ([0,1]^2, \mathrm{R}), k(x,t,y,z)$ is analytic on $C$ $([0,1]^4$, R), $u(y, z)$ is a continuous function with respect to $u(y, z)$ and $\lambda$ is a constant coefficient.
In order to apply the Bernstein polynomials in the interval [0,1], $B_{i,n}(x)$ is defined as (Joy, 2000);
$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}\, , i = 0,1,2,3, \dots, n$ (2)
Bernstein polynomials of degree n in the interval [0,1] can also be written in the following equivalent form;
$B_{i,n}(x) = \sum_{p=0}^{n-i} \binom{n}{i} \binom{n-i}{p} (-1)^p x^{i+p}$ (3)
Bernstein polynomial of degree n can be defined recursively by combining two Bernstein polynomials of degree (n-1). That is the kth nth-degree Bernstein polynomial can be written as (Joy, 2000);
$B_{k,n}(x) = (1` - x)B_{k,n-1}(x) + xB_{k-1,n-1}(x), k = 0(1)n,\ n \geq 1$ (4)

## 2. Methodology

In this section, we developed a numerical method using polynomial collocation method for solving two dimensional mixed Volterra-Fredholm integral equations (2D MVFIE). This method is based on collocation approach and also considers linear combination of Bernstein polynomial as our approximated solution. In this section, we will develop a method by reducing the two dimensional mixed Volterra-Fredholm integral equation to system of nonlinear algebraic equations using standard collocation points.

### 2.1 Method of Solution to NL2D MVFIE

Recall that equation (1) with $n \geq 2$ is given by;
$m(x,t) = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)V(m(y,z))dydz$ (5)
Let $m_N(x,t)$ be the approximate solution of equation (5) where
$m_N(x,t) = \sum_{i=0}^N \sum_{j=0}^N c_{i,j} B_{i,N}(x) B_{j,N}(t) = \phi(x,t)C$ (6)
Substituting (6) into (5) we have,
$\phi(x,t)C = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)V(\phi(x,t)C)dydz$ (7)
With
$V\big(m(y,z)\big) = m(y,z)^n \ and\ V(\phi(x,t)C) = (\phi(x,t)C)^n = \beta(y,z)C$ (8)
$\phi(x,t)C = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)(\beta(y,z)C)dydz$ (9)
Collecting like terms we have;
$\phi(x,t)C - \lambda \int_0^t \int_a^b k(x,t,y,z)(\beta(y,z)C)dydz = f(x,t)$ (10)
$\left\{\phi(x,t) - \lambda \int_0^t \int_a^b k(x,t,y,z)\big(\beta(y,z)\big)dydz\right\} C = f(x,t)$ (11)
Collocating equation (11) using standard collocation points at $x = x_i$ and $t = t_j$ with

$$x_i = a + \frac{(b-a)i}{N}, \qquad i = 0(1)N$$

$$t_j = a + \frac{(b-a)j}{N}, \qquad j = 0(1)N$$

$\left\{\phi(x_i,t_j) - \lambda \int_0^t \int_a^b k(x_i,t_j,y,z)\big(\beta(y,z)\big)dydz\right\} C = f(x_i,t_j)$ (12)

$$where\ \ \tau(x_i,t_j) = \left\{\phi(x_i,t_j) - \lambda \int_0^t \int_a^b k(x_i,t_j,y,z)\big(\beta(y,z)\big)dydz\right\}\ and$$

$$C = \left[c_{0,0}, c_{0,1}, c_{0,2}, \dots, c_{0,N}, \dots, c_{N,0}, c_{N,1}, c_{N,2}, \dots, c_{N,N}\right]^n$$

$G(c) = \tau(x_i,t_j)C - f(x_i,t_j) = 0$ (13)

From equation (13), $k(x, t, y, z, m(y, z))$ is a nonlinear function. Consider the interval $[a, b]$ and divide it into a series of sub-intervals $[x_n, x_{n+1}]$ such that $x_0 = a$. In each subinterval $k(x,t,y,z,m(y, z))$ may be linearised as follows; If $k(x,t,y,z,m(y,z))$ is regular, $k(x,t,y,z,m(y,z))$ may be approximated by the few terms of its Taylor series expansion around $(x_n,t_n,y_n,z_n,m_n)$ in the form below:

$$k(x,t,y,z,m(y,z)) = k(x_n,t_n,y_n,z_n,m_n) + (x-x_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial x} + (t-t_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial t} + (y-y_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial y} + (z-z_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial z} + (u-u_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial u} \quad (14)$$

Substituting equation (14) into equation (5) to get

$$m(x,t) = f(x,t) + \lambda\int_0^t\int_a^b\Big\{k(x_n,t_n,y_n,z_n,m_n) + (x-x_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial x} + (t-t_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial t} + (y-y_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial y} + (z-z_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial z} + (u-u_n)\frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial u}\Big\}\,dydz \quad (15)$$

$$m(x,t) = f(x,t) + \lambda\int_0^t\int_a^b(K_n + (x-x_n)A_n + (t-t_n)B_n + (y-y_n)C_n + (z-z_n)D_n + (u-u_n)E_n)dydz \quad (16a)$$

where

$$K_n = k(x_n,t_n,y_n,z_n,m_n)$$
$$A_n = \frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial x}$$
$$B_n = \frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial t}$$
$$C_n = \frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial y}$$
$$D_n = \frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial z}$$
$$E_n = \frac{\partial(k(x_n,t_n,y_n,z_n,m_n))}{\partial u}$$

Therefore, equation (16a) is a linear two dimensional mixed Volterra-Fredholm equation which can be represented as;

$$C_{k+1} = C_k - (G'(C_k)^{-1}G(C_k)) \quad (16b)$$

where

$$C_{k+1} = \begin{pmatrix} C_{0,0,k+1} \\ C_{0,1,k+1} \\ \vdots \\ C_{0,n,k+1} \\ C_{1,0,k+1} \\ C_{1,1,k+1} \\ \vdots \\ C_{1,n,k+1} \\ C_{2,0,k+1} \\ C_{2,1,k+1} \\ \vdots \\ C_{2,n,k+1} \\ \vdots \\ C_{n,0,k+1} \\ C_{n,1,k+1} \\ \vdots \\ C_{n,n,k+1} \end{pmatrix};\; C_k = \begin{pmatrix} C_{0,0,k} \\ C_{0,1,k} \\ \vdots \\ C_{0,n,k} \\ C_{1,0,k} \\ C_{1,1,k} \\ \vdots \\ C_{1,n,k} \\ C_{2,0,k} \\ C_{2,1,k} \\ \vdots \\ C_{2,n,k} \\ \vdots \\ C_{n,0,k} \\ C_{n,1,k} \\ \vdots \\ C_{n,n,k} \end{pmatrix};$$

$$G(C_k) = \begin{pmatrix} g_{0,0}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ g_{0,1}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ \vdots \\ g_{0,n}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ g_{1,0}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ g_{1,1}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ \vdots \\ g_{1,n}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ g_{2,0}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ g_{2,1}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ \vdots \\ g_{2,n}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ \vdots \\ g_{n,0}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ g_{n,1}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \\ \vdots \\ g_{n,n}(c_{0,0,k},\dots,c_{0,n,k},c_{1,0,k},\dots,c_{1,n,k},\dots,c_{n,0,k},\dots,c_{n,n,k}) \end{pmatrix}$$

$$G'(C_k) = \begin{pmatrix} \frac{\partial g_{0,0}}{\partial c_{0,0,k}} & \frac{\partial g_{0,0}}{\partial c_{0,1,k}} & \cdots & \frac{\partial g_{0,0}}{\partial c_{0,n,k}} & \cdots & \frac{\partial g_{0,0}}{\partial c_{n,0,k}} & \frac{\partial g_{0,0}}{\partial c_{n,1,k}} & \cdots & \frac{\partial g_{0,0}}{\partial c_{n,n,k}} \\ \frac{\partial g_{0,1}}{\partial c_{0,0,k}} & \frac{\partial g_{0,1}}{\partial c_{0,1,k}} & \cdots & \frac{\partial g_{0,1}}{\partial c_{0,n,k}} & \cdots & \frac{\partial g_{0,1}}{\partial c_{n,0,k}} & \frac{\partial g_{0,1}}{\partial c_{n,1,k}} & \cdots & \frac{\partial g_{0,1}}{\partial c_{n,n,k}} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial g_{0,n}}{\partial c_{0,0,k}} & \frac{\partial g_{0,n}}{\partial c_{0,1,k}} & & \frac{\partial g_{0,n}}{\partial c_{0,n,k}} & \vdots & \frac{\partial g_{0,n}}{\partial c_{n,0,k}} & \frac{\partial g_{0,n}}{\partial c_{n,1,k}} & & \frac{\partial g_{0,n}}{\partial c_{n,n,k}} \\ \frac{\partial g_{1,0}}{\partial c_{0,0,k}} & \frac{\partial g_{1,0}}{\partial c_{0,1,k}} & \vdots & \frac{\partial g_{1,0}}{\partial c_{0,n,k}} & \cdots & \frac{\partial g_{1,0}}{\partial c_{n,0,k}} & \frac{\partial g_{1,0}}{\partial c_{n,1,k}} & \vdots & \frac{\partial g_{1,0}}{\partial c_{n,n,k}} \\ \frac{\partial g_{1,1}}{\partial c_{0,0,k}} & \frac{\partial g_{1,1}}{\partial c_{0,1,k}} & \cdots & \frac{\partial g_{1,1}}{\partial c_{0,n,k}} & \cdots & \frac{\partial g_{1,1}}{\partial c_{n,0,k}} & \frac{\partial g_{1,1}}{\partial c_{n,1,k}} & \cdots & \frac{\partial g_{1,1}}{\partial c_{n,n,k}} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_{1,n}}{\partial c_{0,0,k}} & \frac{\partial g_{1,n}}{\partial c_{0,1,k}} & \cdots & \frac{\partial g_{1,n}}{\partial c_{0,n,k}} & \cdots & \frac{\partial g_{1,n}}{\partial c_{n,0,k}} & \frac{\partial g_{1,n}}{\partial c_{n,1,k}} & \cdots & \frac{\partial g_{1,n}}{\partial c_{n,n,k}} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_{n,0}}{\partial c_{0,0,k}} & \frac{\partial g_{n,0}}{\partial c_{0,1,k}} & \vdots & \frac{\partial g_{n,0}}{\partial c_{0,n,k}} & \cdots & \frac{\partial g_{n,0}}{\partial c_{n,0,k}} & \frac{\partial g_{n,0}}{\partial c_{n,1,k}} & \cdots & \frac{\partial g_{n,0}}{\partial c_{n,n,k}} \\ \frac{\partial g_{n,1}}{\partial c_{0,0,k}} & \frac{\partial g_{n,1}}{\partial c_{0,1,k}} & \cdots & \frac{\partial g_{n,1}}{\partial c_{0,n,k}} & \cdots & \frac{\partial g_{n,1}}{\partial c_{n,0,k}} & \frac{\partial g_{n,1}}{\partial c_{n,1,k}} & \cdots & \frac{\partial g_{n,1}}{\partial c_{n,n,k}} \\ \frac{\partial g_{n,2}}{\partial c_{0,0,k}} & \frac{\partial g_{n,2}}{\partial c_{0,1,k}} & \vdots & \frac{\partial g_{n,2}}{\partial c_{0,n,k}} & \vdots & \frac{\partial g_{n,2}}{\partial c_{n,0,k}} & \frac{\partial g_{n,2}}{\partial c_{n,1,k}} & \vdots & \frac{\partial g_{n,2}}{\partial c_{n,n,k}} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\ \frac{\partial g_{n,n}}{\partial c_{0,0,k}} & \frac{\partial g_{n,n}}{\partial c_{0,1,k}} & & \frac{\partial g_{n,n}}{\partial c_{0,n,k}} & & \frac{\partial g_{n,n}}{\partial c_{n,0,k}} & \frac{\partial g_{n,n}}{\partial c_{n,1,k}} & & \frac{\partial g_{n,n}}{\partial c_{n,n,k}} \end{pmatrix}$$

The system of nonlinear equations with $(n+1)^2$ unknown coefficients $c_{ij}$ is then solved using MAPLE 18 software with the well known Newton's iterative method. The unknown constants obtained are substituted back into the approximate solution to get the required solution.

## 3. Uniqueness, Convergence and Error Analysis

**Hypothesis**

The following assumptions were made:

$S_1$:Let ($C([a, b] \times [0, d])$, ||. ||) be the space of all continuous functions on the interval [0, $d$] ×[$a$, $b$] with the norm $\|U\|_\infty = \max_{x,t\in[a,b]} |U(x,t)|$

$S_2$ : $U(x,t) \neq 0$

$S3$ : $|k(x,t,y,z)| \leq M$ (M is a positive real number) for all $(x, t) \in [a, b] \times [0, d]$ and

$S4$ : $\forall (x, t) \in [a, b] \times [0, d]$ *and* $\beta = \{(x, t, y, z) : 0 \leq z \leq t \leq d;\ a \leq y \leq x \leq b\}$

With this conditions, we present the uniqueness and convergence theorem

### 3.1 Lemma 1: Uniqueness of solution for NL2D MVFIE

let $M(x,t)$ be an exact solution of equation (1) and $M_{N,N}(x,t)$ be the approximate solution of equation (1) where

$$M_{N,N}(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} c_{i,j} B_{i,N}(x) B_{j,N}(t) \tag{17}$$

Then equation (1) has a unique solution whenever $0 \leq \beta \leq 1$ and $\beta = 1 - L_2 \lambda (b-a)\,t$.

**Proof:** Equation (1) can be written in the form

$$M(x,t) = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)(m(y,z))^n dydz\,, n \geq 2 \tag{18}$$

such that the nonlinear term $F(M)$ is Lipschitz continuous with $|F(M) - F(V)| \leq L_2\, |U - V|$. Let $M_{N,N}{}^*$ and $M_{N,N}{}^{**}$ be any two different approximate solutions of equation (5) then;

$$M_{N,N}{}^*(x,t) - M_{N,N}{}^{**}(x,t) = f(x,t) + \lambda \int_0^t \int_a^b F(x,t,y,z,M_{N,N}{}^*(y,z))^n dydz - f(x,t) - \lambda \int_0^t \int_a^b F(x,t,y,z,M_{N,N}{}^{**}(y,z))^n dydz \tag{19}$$

$$\left|M_{N,N}{}^*(x,t) - M_{N,N}{}^{**}(x,t)\right| = \left|\lambda \int_0^t \int_a^b F(x,t,y,z,M_{N,N}{}^*(y,z))^n dydz - \lambda \int_0^t \int_a^b F(x,t,y,z,M_{N,N}{}^{**}(y,z))^n dydz\right| \tag{20}$$

$$\left|M_{N,N}{}^*(x,t) - M_{N,N}{}^{**}(x,t)\right| \leq |\lambda| \left|\int_0^t \int_a^b F(x,t,y,z,M_{N,N}{}^*(y,z))^n - \int_0^t \int_a^b F(x,t,y,z,M_{N,N}{}^{**}(y,z))^n\right| dydz \tag{21}$$

$$\left|M_{N,N}{}^*(x,t) - M_{N,N}{}^{**}(x,t)\right| \leq |\lambda| \int_0^t \int_a^b \left|F(M_{N,N}{}^*) - F(M_{N,N}{}^{**})\right| dydz \tag{22}$$

$$\left|M_{N,N}{}^* - M_{N,N}{}^{**}\right| \leq |\lambda| L_1 \int_0^t \int_a^b \left|M_{N,N}{}^* - M_{N,N}{}^{**}\right| dydz \tag{23}$$

$$\left|M_{N,N}{}^* - M_{N,N}{}^{**}\right| - |\lambda| L_1 (b-a)t \left|\left|M_{N,N}{}^* - M_{N,N}{}^{**}\right|\right| dydz \leq 0 \tag{24}$$

$$\{1 - |\lambda| L_1 (b-a)t\} \left|M_{N,N}{}^* - M_{N,N}{}^{**}\right| \leq 0 \tag{25}$$

Where $\beta = \{1 - |\lambda| L_1 (b-a)t\}$

$$\beta \left|M_{N,N}{}^* - M_{N,N}{}^{**}\right| \leq 0 \tag{26}$$

As $0 \leq \beta \leq 1$, $\left|M_{N,N}{}^* - M_{N,N}{}^{**}\right| = 0$ which implies $M_{N,N}{}^* = M_{N,N}{}^{**}$ . Hence, the uniqueness proof is complete.■

### 3.2 Lemma 2: Convergence of the method for NL2D MVFIE

Let $M(x, t)^e$ be an exact solution of equation (1.5) and $M_{N,N}{}^a\,(x, t)$ be the approximate solution of equation (1) where

$$M_{N,N}^a(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} c_{i,j} B_{i,N}(x) B_{j,N}(t)$$

Then the solution of 2D-NLMVFIE by using Bernstein polynomial as a basis function is unique and convergent if $0 \leq \eta_2 \leq 1$.

**Proof:**

The convergence is proved using definition of norms and $S_1 - S_4$ we have;

$$\left\|M(x,t)^e - M_{N,N}^a(x,t)\right\|_\infty = \max_{x,t\in[a,b]} \left|M(x,t)^e - M_{N,N}^a(x,t)\right| \tag{27}$$

$$\left\|M(x,t)^e - M_{N,N}^a(x,t)\right\|_\infty = \max_{x,t\in[a,b]} \left|\lambda \int_0^t \int_a^b k(x,t,y,z)M(y,z)^e dydz - \lambda \int_0^t \int_a^b k(x,t,y,z)M_{N,N}^a(y,z)dydz\right| \quad (28)$$

$$\left\|M(x,t)^e - M_{N,N}^a(x,t)\right\|_\infty \le |\lambda| \max_{x,t\in[a,b]} \int_0^t \int_a^b |k(x,t,y,z)| \left|M(y,z)^e - M_{N,N}^a(y,z)dydz\right| \quad (29)$$

$$\left\|M(x,t)^e - M_{N,N}^a(x,t)\right\|_\infty \le |\lambda||M\beta| \left\|M(y,z)^e - M_{N,N}^a(y,z)\right\|_\infty \quad (30)$$

$$\{1-|\lambda||M\beta|\}\left\|M(y,z)^e - M_{N,N}^a(y,z)\right\|_\infty \le 0 \quad (31)$$

Where $\eta_2 = (|\lambda||M\beta|)$

$$\{1-\eta_2\}\left\|M(y,z)^e - M_{N,N}^a(y,z)\right\|_\infty \le 0 \quad (32)$$

Then, if $0 \le \eta_2 \le 1$ and $N \to \infty$, then $\left\|M(x,t)^e - M_{N,N}^a(x,t)\right\|_\infty = 0$. This shows that our method converges.■

### 3.3 Lemma 3: Error Bound for NL2D MVFIE

In establishing the error bound of this method, we substituted the approximate solution into equation (1) to give;

$$M_{N,N}^\omega(x,t) = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)M_{N,N}^\omega(y,z)dydz \quad (33)$$

and the exact solution give by

$$M(x,t) = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)M(y,z)dydz \quad (34)$$

$$M_{N,N}^\omega(x,t) - M(x,t) = e_N^\omega(x,t) \quad (35)$$

$$\left|M_{N,N}^\omega(x,t) - M(x,t)\right| = \left|\lambda \int_0^t \int_a^b k(x,t,y,z)M_{N,N}^\omega(y,z)dydz - \lambda \int_0^t \int_a^b k(x,t,y,z)M(y,z)dydz\right| \quad (36)$$

$$\frac{\left|M_{N,N}^\omega(x,t)-M(x,t)\right|}{\left|M_{N,N}^\omega(y,z)-M(y,z)\right|} \le \frac{|\lambda| \int_0^t \int_a^b |k(x,t,y,z)|\left|M_{N,N}^\omega(y,z)-M(y,z)\right|dydz}{\left|M_{N,N}^\omega(y,z)-M(y,z)\right|} \quad (37)$$

$$\frac{|e_{N,N}^\omega(x,t)|}{|e_{N,N}^\omega(y,z)|} \le |\lambda| \int_0^t \int_a^b |k(x,t,y,z)|dydz \quad (38)$$

$$\frac{\|e_{N,N}^\omega(x,t)\|_\infty}{\|e_{N,N}^\omega(y,z)\|_\infty} \le |\lambda| M_\delta \beta_\delta \quad (39)$$

Therefore the error is bounded and hence the method is convergent.■

**Theorem1**

Let $M(x.t)$ be the solution of equation (1) then the solution is

$$M_N(x,t) = \tau(x,t)\left(C_k - \left(G'(C_k)^{-1}G(C_k)\right)\right); \qquad i,j = 0(1)N$$

Where $\tau(x_i,t_j) = \left\{\phi(x_i,t_j) - \lambda \int_0^t \int_a^b k(x_i,t_j,y,z)\left(\beta(y,z)\right)dydz\right\}, G(c) = \tau(x_i,t_j)C - f(x_i,t_j) = 0$

**Proof:** The approximate solution of equation (1) is

$$M_{N,N}(x,t) = \sum_{i=0}^N \sum_{j=0}^N c_{i,j} B_{i,N}(x) B_{j,N}(t) = \phi(x,t)C \quad (40)$$

From equation (16b); $C = C_{k+1} = C_k - (G'(C_k)^{-1}G(C_k))$

Where $\tau(x_i,t_j) = \left\{\phi(x_i,t_j) - \lambda \int_0^t \int_a^b k(x_i,t_j,y,z)\left(\beta(y,z)\right)dydz\right\}$

Substituting for C in the approximate solution gives;

$$M_N(x,t) = \tau(x,t)\left(C_k - \left(G'(C_k)^{-1}G(C_k)\right)\right); \qquad i,j = 0(1)N$$

## 4. Numerical Examples

In this research, numerical examples are used to test the accuracy and efficiency of the method and are presented in tables except where it gives exact solution. All computations are done with the help of MAPLE 18 software. Let $U_N(x,t)$ and $U(x,t)|$ be the approximate and exact solution respectively then $Error_N = |U_N(x,t) - U(x,t)|$.

Table 1: Notations

| | *description* |
|---|---|
| $Error_P$ | absolute Error with Power Series |
| $Error_B$ | absolute Error with Bernstein Polynomial |
| $Error_{MB}$ | absolute Error From Maleknejad and Behbahani (2012) |

### 4.1 Problem 1

Maleknejad and Behbahani (2012) considered a nonlinear two-dimensional mixed Volterra Fredholm integral equation of the second kind;

$$v(x,y) = 16\int_0^t\int_0^1 e^{x+y+s+t}\, v(s,t)^3 ds dt + e^{5x+y} + e^{4+x+y} - e^{4+5x+y} \quad (41)$$

Which has an exact solution given as $v(x,y) = e^{x+y}$ in the interval $(x, y) = [0,1]$.

#### 4.1.1 Case 1: Using Bernstein Polynomial Basis Function

Let the approximate solution of equation (41) for $N = 2$ be

$$v_N(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j} B_{i,2}(x) B_{j,2}(y) \quad (42)$$

Substituting (42) into (41) and Collocating it using standard collocation points at $x = x_i$ and $y = y_j$ gives;

$$\sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j} B_{i,2}(x_i) B_{j,2}(y_j) - 16\int_0^t\int_0^1 e^{x_i+y_j+s+t}\left(\sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j} B_{i,2}(s) B_{j,2}(t)\right)^3 ds dt - e^{5x_i+y_j} - e^{4+x_i+y_j} + e^{4+5x_i+y_j} = 0 \quad (43)$$

The method was implemented using MAPLE 18 software and $v_2(x, y)$ was obtained as;

$$v_2(x,y) = 1 + 1.438301628y + 2.718281828y^2 + 1.160283052x + 0.5677667425xy + 4.854987024xy^2 - 7.993100758x^2 + 1.343860256x^2y + 9.456921547x^2y^2$$

#### 4.1.2 Case 2: Using Power Series

Let the approximate solution of equation (56) for $N = 2$ be

$$v_N(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j} x^i y^j \quad (44)$$

Substituting (44) into (41) and Collocating it using standard collocation points at $x = x_p$ and $y = y_q$ gives;

$$\sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j} {x^i}_p {y^j}_q - 16\int_0^t\int_0^1 e^{x_p+y_q+s+t}\left(\sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j} s^i t^j\right)^3 ds dt - e^{5x_p+y_q} - e^{4+x_p+y_q} + e^{4+5x_p+y_q} = 0 \quad (45)$$

The method was implemented using MAPLE 18 software and $v_2(x,y)$ was obtained as;

$$v_2(x,y) = +0.8766032560y + 0.8416785720y^2 + 0.9197895568x + 0.7415878752xy + 0.8277060982xy^2 + 0.7640749104x^2 + 0.7290816218x^2y + 0.6070281808x^2y^2$$

Table 2: Result of Absolute Error of Problem 1

| (x,t) | Exact | Bernstein | Power Series | $ERROR_B$ | $ERROR_P$ |
|---|---|---|---|---|---|
| (0,0) | 1.0000000000 | 1.0000000000 | 1.0000000000 | 0.0000000000 | 0.0000000000 |
| (0.1,0.1) | 1.221402758 | 1.219932484 | 1.204730187 | $1.470274\times10^{-3}$ | $1.6672571\times10^{-2}$ |
| (0.2,0.2) | 1.491824698 | 1.396156701 | 1.466597763 | $9.5667997\times10^{-2}$ | $2.5226935\times10^{-2}$ |
| (0.3,0.3) | 1.822118800 | 1.599910649 | 1.797128763 | $2.22208151\times10^{-1}$ | $2.4990037\times10^{-2}$ |
| (0.4,0.4) | 2.225540928 | 1.925128941 | 2.209306078 | $3.00411987\times10^{-1}$ | $1.6234850\times10^{-2}$ |
| (0.5,0.5) | 2.718281828 | 2.488442800 | 2.717569472 | $2.29839028\times10^{-1}$ | $7.12356\times10^{-4}$ |
| (0.6,0.6) | 3.320116923 | 3.429180064 | 3.337815576 | $1.09063141\times10^{-1}$ | $1.7698653\times10^{-2}$ |
| (0.7,0.7) | 4.055199967 | 4.909365185 | 4.087397888 | $8.54165218\times10^{-1}$ | $3.2197921\times10^{-2}$ |
| (0.8,0.8) | 4.953032424 | 7.113719217 | 4.985126774 | 2.160686793 | $3.2094350\times10^{-2}$ |
| (0.9,0.9) | 6.049647464 | 10.24965983 | 6.051269467 | 4.200012366 | $1.622003\times10^{-3}$ |

Table 3: Comparison of Absolute Error of Problem 1

| (x,t) | Exact | $ERROR_{MB}$ | $ERROR_P$ |
|---|---|---|---|
| (0,0) | 1.0000000000 | 0.000000000 | 0.0000000000 |
| (0.1,0.1) | 1.221402758 | $1.4853\times10^{-2}$ | $1.6672571\times10^{-2}$ |
| (0.2,0.2) | 1.491824698 | $5.9763\times10^{-2}$ | $2.5226935\times10^{-2}$ |
| (0.3,0.3) | 1.822118800 | $7.8689\times10^{-2}$ | $2.4990037\times10^{-2}$ |
| (0.4,0.4) | 2.225540928 | $6.5780\times10^{-2}$ | $1.6234850\times10^{-2}$ |
| (0.5,0.5) | 2.718281828 | $9.7749\times10^{-2}$ | $7.1235600\times10^{-4}$ |
| (0.6,0.6) | 3.320116923 | $8.3208\times10^{-2}$ | $1.7698653\times10^{-2}$ |
| (0.7,0.7) | 4.055199967 | $1.3690\times10^{-1}$ | $3.2197921\times10^{-2}$ |
| (0.8,0.8) | 4.953032424 | $1.6545\times10^{-1}$ | $3.2094350\times10^{-2}$ |
| (0.9,0.9) | 6.049647464 | $1.6955\times10^{-1}$ | $1.622003\times10^{-3}$ |

### 4.2 Problem 2

Maleknejad and Behbahani (2012) considered a nonlinear two-dimensional mixed Volterra Fredholm integral equation of the second kind

$$v(x,y) = x^2e^{2y} - \frac{1}{5}x^5 + y^2 + \int_0^t\int_0^1 y^2e^{-4s}\,v(s,t)^2dtds \qquad (46)$$

Which has an exact solution given as $v(x,y) = x^2e^{2y}$ in the interval $(x, y)$ = [0, 1].

#### 4.2.1 Case 1: Using Bernstein Polynomial Basis Function

Let the approximate solution of equation (46) for N=3 gives;

$$v_N(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}B_{i,3}(x)B_{j,3}(y) \qquad (47)$$

Substituting (47) into (46) and collocating it using standard collocation points at $x = x_i$ and $y = y_j$ gives;

$$\sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}B_{i,3}(x_i)B_{j,3}(y_j) = {x_i}^2e^{2y_j} - \frac{1}{5}{x_i}^5 + {y_j}^2 + \int_0^t\int_0^1 {y_j}^2e^{-4s}\left\{\sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}B_{i,3}(x_i)B_{j,3}(y_j)\right\}^2dtds \qquad (48)$$

The method was implemented using MAPLE 18 software and $v_3(x,y)$ was obtained as;

$$\begin{aligned} v_3(x,y) = {} & -3.190793703\times10^{-15}y + 1.000000000y^2 + 6.986227354\times10^{-14}y^3 \\ & - 0.8888888953x + 2.736773415\times10^{-8}xy + 0.1140531893xy^2 \\ & + 2.241353073\times10^{-8}xy^3 + 1.444444447x^2 + 2.347156852x^2y \\ & + 0.4009377897x^2y^2 + 3.830645346x^2y^3 - .5555555571x^3 \\ & + 7.534280090\times10^{-8}x^3y - 0.5447817326x^3y^2 + 7.526188695\times10^{-8}x^3y^3 \end{aligned}$$

#### 4.2.2 Case 2: Using Power Series

Let the approximate solution of equation (46) for $N = 3$ be

$$v_N(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}x^iy^j \qquad (49)$$

Substituting (49) into (46) and collocating it using standard collocation points at $x = x_i$ and $y = y_j$ gives;

$$\sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}{x^i}_i{y^j}_j = {x_i}^2e^{2y_j} - \frac{1}{5}{x_i}^5 + {y_j}^2 + \int_0^t\int_0^1 {y_j}^2e^{-4s}\left\{\sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}x^iy^j\right\}^2dtds \qquad (50)$$

The method was implemented using MAPLE 18 software and $v_3(x,y)$ was obtained as;

$V_3(x,y)=1.68\times10^{-7}t^3x^3+3.830645243t^3x^2-0.174201385t^2x^3+4.8\times10^{-8}t^3x+0.88908673t^2x^2+1.11\times10^{-7}tx^3+5.0\times10^{-10}t^3+0.128525512t^2x+2.347156812tx^2+0.9999999994t^2+3.52\times10^{-8}xt+9.000000000\times10^{-11}t-0.5555555560x^3+1.444444446x^2-0.8888888937x$

Table 4: Table of Absolute Error of Problem 2

| (x,t) | *Exact* | *ERROR*$_B$ | *ERROR*$_P$ | *ERROR*$_{MB}$ |
|---|---|---|---|---|
| (0,0) | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |
| (0.1,0.1) | 0.01221402758 | $5.38712819\times10^{-3}$ | $5.32503815\times10^{-3}$ | $1.5119\times10^{-2}$ |
| (0.2,0.2) | 0.05967298792 | $3.828062871\times10^{-3}$ | $3.742212312\times10^{-3}$ | $1.0238\times10^{-2}$ |
| (0.3,0.3) | 0.1639906920 | $9.72728297\times10^{-3}$ | $9.32189981\times10^{-3}$ | $1.0782\times10^{-2}$ |
| (0.4,0.4) | 0.3560865485 | $1.825597361\times10^{-2}$ | $1.703628606\times10^{-2}$ | $1.6751\times10^{-2}$ |
| (0.5,0.5) | 0.6795704570 | $2.969438683\times10^{-2}$ | $2.683668704\times10^{-2}$ | $2.7193\times10^{-2}$ |

## 5. Discussion

In this section numerical finding from the solved examples using the proposed numerical approach was discussed. The results obtained for Problem 1 and Problem 2 were compared with the exact solution and some existing works in the literature. The numerical result obtained shows that our method is better than the method proposed by Maleknejad and Behbahani (2012).

From Problem 1 at N=2 and various values of (x,y), the result obtained performs favourably than the result presented by Maleknejad and Behbahani (2012). From Table 3 at $(x,y) = (0.5,0.5)$ and $(x,y) = (0.9,0.9)$, the result of the absolute errors gives $Error_{MB} = 9.7749\times10^{-2}$, $Error_B = 7.12356\times10^{-4}$ and $Error_{MB} = 1.6955\times10^{-1}$, $Error_B = 1.622003\times10^{-3}$ respectively, which further shows the consistency and reliability of the method. The absolute errors obtained from Problem 2 at $N = 3$ shows that the result obtained performs better than the result presented by Maleknejad and Behbahani (2012). From Table 4 at $(x,y) = (0.1,0.1)$ and $(x,y) = (0.3,0.3)$, the result of absolute errors are $Error_B = 5.32503815\times10^{-3}$, $Error_{MB} = 1.5119\times10^{-2}$ and

$Error_B = 9.32189981 \times 10^{-3}$, $Error_{MB} = 1.0782 \times 10^{-2}$ respectively. This shows that our method is reliable and efficient. It has been observed and examined that when the values of N increases the error decreases and the approximate solution converges rapidly to the exact solution, the values of N were chosen arbitrarily and for simplicity.

## 6. Conclusion

In this work, Bernstein polynomial collocation method was used to solve NL2D MVFIE. Standard collocation points were used to transform the integral into system of nonlinear integral equations which is then solved using Newton-Rhapson method. The uniqueness theorems, convergence of solution and error analysis were presented. It was discovered that the developed method is simple, reliable and effective. Maple 18 software is used for all computations in this work. The accuracy of the method is demonstrated by considering some numerical examples. This research can be extended to cover two dimensional mixed Volterra-Fredholm integro-differential equations (2D MVFIDE), two dimensional mixed Volterra-Fredholm partial differential equation (2D MVFPDE) , system of two dimensional mixed Volterra-Fredholm integro-differential equations, system of two dimensional mixed Volterra-Fredholm integral equations, system of two dimensional mixed Volterra-Fredholm partial differential equations etc.

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