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An Approximate Technique for Solving Fractional Order Hirota-Satsuma Equation

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ABSTRACT

In this investigation, the technique of the Laplace decomposition method (LDM) was derived through the incorporation of the Laplace transform and the Adomian polynomial to obtain the approximate solution of the nonlinear partial differential fractional Hirota-Satsuma equation in the Caputo sense. The technique was validated by comparing our results with the literature and further comparison was made with the exact solution for the classical form. The approximate results and graphical depictions show that the scheme is efficient and the implementation is straightforward. Therefore, the scheme can be adopted to solve problems of nonlinear fractional order Hirota-Satsuma equations arising from science and engineering.

1. Introduction

The Hirota-Satsuma coupled Kdv is a first order partial differential equation formulated by (Hirota and Satsuma, 1981), while the generalized form was introduced by (Wu *et al.,* 1999) and several analytical and numerical approaches have been devised to solve these equations (Ali *et al.,* 2023, Kangagil and Ayaz, 2010; Ganji and Rafei, 2006; Biazar, Hosseini and Gbolamin, 2009; Maturi, 2012). Generally, fractional differential equations (FDE) depict a universal form of both ordinary and partial differential equations (Miller and Ross, 1993; Almeida, Bastos and Monteiro, 2016; Podlubny, 1999; Oldham and Spanier, 1974). The wide applications of FDE in sciences, socio sciences and engineering have confirmed that several systems in nature can be described through its nonlinear form, such as biotechnology, chaos theory, dielectric polarization, electrochemical processes, electrodynamics, fluid flow, financial dynamical systems, nanotechnology, modeling of disease outbreaks, random walks, signal and image processing, tumor-immune surveillance, viscoelasticity materials, and other various disciplines (Baleanu, Guvenc and Machado, 2010; Zhang, *et al.,* 2012; Kumar, Seadawy and Joardar, 2018; Salahshour *et al.,* 2015; Baleanu, Wu and Zeng, 2017; Ruzhansky *et al.,* 2017; Pu, 2007; Nasrolahpour, 2013; Mainardi, 2010; Hilfer and Anton, 1995; Jain, Agarwal and Kilicman, 2018; Saoudi *et al.,* 2018; El-Sayed and Agarwal, 2019; Agarwal e*t al.,* 2018; Nigmatullina and Agarwal, 2019; Qureshi and Yusuf, 2019; Qureshi and Yusuf, 2019; Rekhviashvili, *et al.,* 2019). It is notably known that coupled systems of fractional equations are difficult to solve than its classical form because its operators are defined as integrals and as a result, several methods such as Adomian decomposition method (ADM), homotopy perturbation method (HPM), differential transform method (DTM), Laplace transform method (LTM), homotopy analysis method (HAM), variational iteration method (VIM) (Abbasbandy, 2007; Akinboro, Alao and Akinpelu, 2014; Guo-Zhong *et al.,* 2010; Sontakke, Shaikh and Nisar, 2018; Kaya, 2004; Abazari and Abazari, 2012; Jibran *et al.,* 2018; Arife, Vanani and Yildirim, 2011; Fan, 2001; Ganji, Ganji and Rostamiyan, 2009; Saadeh, Alayed and Qazza, 2022) and modifications like reduced differential transformation method (RDTM), extended tanh-function method, homotopy perturbation transformation method (HPTM), Laplace variational iteration method (LVIM), Laplace decomposition method (LDM), homotopy analysis decomposition method (HADM), q-homotopy analysis method (q-HAM), q-homotopy analysis transformation method (q-HATM), Laplace residual power series method (LRPSM), fractional reduced differential transform method (FRDTM), Aboodh transform-weighted residual based method (AT-WRM), modified Laplace decomposition method (Sahu and Jena, 2024, Oderinu, Owolabi and

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Taiwo, 2023; Abera and Mebrate, 2023; Luca, 2023; Shah *et al.,* 2019; Benkhettou *et al.,* 2023; Goswami and Alqahtani, 2016; Alao *et al.,* 2022; Darzi and Agheli, 2018; Abuasad *et al.,* 2021; Khan *et al.,* 2020; Akinola *et al.,* 2022; Alao *et al.,* 2019; Prajapatia, Rakesh and Pankaj, 2016; Akinyemi and Iyiola, 2020; Botmart *et al.,* 2022) have been considered to obtain the solution to these sets of equations either numerically, analytically, or in its closed form. This paper aims to extend the application of the Laplace decomposition method by incorporating the Laplace transform and Adomian polynomial to solve the generalized fractional Hirota-Satsuma coupled Kdv equation, as it will continue to be of interest to researchers in decades ahead owing to its accuracy and less computational time.

2. Basic Definitions

Definition 1 *The definition of Riemann-Liouville integral of a fractional function f with order* $\alpha \geq 0$ *as represented* (Botmart *et al.,* 2022)

$$I_y^\alpha f(y) = \begin{cases} f(y) & \alpha = 0 \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^y (y-v)^{\alpha-1} f(v)dv & \alpha > 0. \end{cases}$$

Definition 2 *The definition of Caputo sense integral of a fractional order derivative of function f as expressed* (Khan *et al.,* 2020)

$$D_y^\alpha f(y) = \frac{1}{\Gamma(p-\alpha)}\int_0^y (y-v)^{p-\alpha-1} f^p(v)dv,$$

for $p-1 < \alpha \leq p$ with $p \in \mathcal{N}$ and $f \in C_t, t \geq -1$.

Lemma 1 *if* $p-1 < \alpha \leq p$ *with* $p \in N$ *and* $f \in C_t, t \geq -1$ (Khan *et al.,* 2020) then

$$I^\alpha I^a f(y) = I^{\alpha+a} f(y), \quad a, \alpha \geq 0.$$

$$I^\alpha y^\psi = \frac{\Gamma(\psi+1)}{\Gamma(\alpha+\psi+1)} y^{\alpha+\psi}, \quad \alpha > 0, \quad \psi+1 > 0, y > 0.$$

$$I^\alpha D^\alpha f(y) = f(y) - \sum_{i=0}^{p-1} f^{(i)}(0^+)\frac{y^i}{i!}, \quad y > 0, \quad p-1 < \alpha \leq p.$$

Definition 3 *The Laplace transformation of* $f(t)$ *for positive* t *is expressed as* (Khan *et al.,* 2020)

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt.$$

Definition 4 *The convolution theorem for Laplace transformation is expressed as* (Botmart *et al.,* 2022)

$$L[f_1 * f_2] = L[f_1(t)] * L[f_2(t)],$$

where $f_1 * f_2$ represent the convolution between f_1 and f_2,

$$(f_1 * f_2)(t) = \int_0^t f_1(\tau) f_2(t-\tau)d\tau.$$

The Laplace transformation of fractional derivative is

$$L(D_t^\alpha f(t)) = s^\alpha F(s) - \sum_{i=0}^{p-1} s^{\alpha-1-i} f^{(i)}(0), \, p-1 < \alpha \leq p,$$

where $F(s)$ is the transformation of $f(t)$.

Definition 5 *The function* $E_\alpha(p)$ *for* $\alpha > 0$ *called Mittag-Leffler is denoted as* (Botmart *et al.,* 2022)

$$E_{\alpha}(p)=\sum_{m=0}^{\infty}\frac{p^{m}}{\Gamma(\alpha m+1)},\quad \alpha>0,\quad p\in\mathcal{C}.$$

3. Procedure of LDM for fractional partial differential equations

Consider the fractional partial differential equation (Jincun and Hong, 2013)

$$D^{\alpha}u(y,t)+Lu(y,t)+Nu(y,t)=g(y,t),\quad y,t\geq 0,\quad p-1<\alpha\leq p, \tag{1}$$

where D^{α}, L, N and g represent Caputo operator, linear term, nonlinear term and source term respectively with the initial condition

$$u(y,0)=f(y). \tag{2}$$

Applying the Laplace transform to (1),

$$s^{\alpha}\mathcal{L}[u(y,t)]-s^{(\alpha-1)}u(y,0)=\mathcal{L}[g(y,t)]-\mathcal{L}[Lu(y,t+Nu(y,t))], \tag{3}$$

$$\mathcal{L}[u(y,t)]=\frac{f(y)}{s}-\frac{1}{s^{\alpha}}\mathcal{L}[Lu(y,t)+Nu(y,t)+g(y,t)].$$

The solution obtained is shown by the infinite series

$$u(y,t)=\sum_{i=0}^{\infty}u_{i}(y,t), \tag{4}$$

and the nonlinear expressions are decomposed using Adomian polynomial

$$Nu(y,t)=\sum_{i=0}^{\infty}A_{i}, \tag{5}$$

where

$$A_{i}=\frac{1}{i!}\left[\frac{d^{i}}{d\lambda^{i}}[N\sum_{i=0}^{\infty}(u_{i}\lambda^{i})]\right]_{\lambda=0},i=0,1,2,... \tag{6}$$

Substitution of (4) and (5) into (3), give

$$\mathcal{L}[\sum_{i=0}^{\infty}u(y,t)]=\frac{f(y)}{s}+\frac{1}{s^{\alpha}}\mathcal{L}[g(y,t)]-\frac{1}{s^{\alpha}}\mathcal{L}[L\sum_{i=0}^{\infty}u_{i}(y,t)+\sum_{i=0}^{\infty}A_{i}], \tag{7}$$

$$\mathcal{L}[u_{0}(y,t)]=\frac{u(y,0)}{s}+\frac{1}{s^{\alpha}}\mathcal{L}[g(y,t)], \tag{8}$$

$$\mathcal{L}[u_{1}(y,t)]=-\frac{1}{s^{\alpha}}\mathcal{L}[Lu_{0}(y,t)+A_{0}]. \tag{9}$$

Generally,

$$\mathcal{L}[u_{i+1}(y,t)]=-\frac{1}{s^{\alpha}}\mathcal{L}[Lu_{i}(y,t)+A_{i}],\quad i\geq 1. \tag{10}$$

Taking the inverse Laplace transformation of (8) and (10) respectively yield (Botmart *et al.,* 2022)

$$u_{0}(y,t)=f(y)+g(y,t)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{11}$$

$$u_{i+1}(y,t)=-\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[Lu_{i}(y,t)+A_{i}]]. \tag{12}$$

4. Existence and Uniqueness of Solution

Theorem 1: The Banach Fixed-Point Theorem(Contraction Mapping Theorem)

Let $(\xi, \|.\|)$ be a complete metric space and let $G:\xi \to \xi$ be a contraction, there exist a constant $k \in [0,1)$ such that $\|G(x)-G(y)\| \le k\|x-y\|, \ \forall x, y \in \xi.$ Then G has a unique fixed point $x^* \in \xi$ such that $G(x^*) = x^*$, and the sequence $\{x_n\}$ generated by the iterative process $x_{n+1} = G(x_n) \to x^*$ (Jachymski, Józwik and Terepeta, 2024).

4.1 Norm Definition and Contraction Property

To apply theorem 1, We need to prove that the operator G is a contraction. Let define the supremum norm on the space of LDM as $\|M(s,x)\|_\infty = \sup_{x\in\Omega} |M(s,x)|.$ Now, for successive iterations $M_n(s,x)$ and $M_l(s,x)$, the difference between their images under G is given by :

$$\|G(M_n)-G(M_l)\|_\infty = \sup_{x\in\Omega}\left|\frac{F(s,x,M_n(s,x))}{s^\alpha} - \frac{F(s,x,M_l(s,x))}{s^\alpha}\right|.$$

Using Lipschitz continuity of the function $f(t,x,m)$ which satisfies the condition:

$|f(t,x,m_1) - f(t,x,m_2)| \le L|m_1 - m_2|$, we can establish that:

$$\|G(M_n)-G(M_l)\|_\infty \le \frac{L}{s^\alpha}\sup_{x\in\Omega}|M_n(s,x)-M_l(s,x)|.$$

This leads to the following contraction inequality: $\|G(M_n)-G(M_l)\|_\infty \le \frac{L}{1+L}\|M_n - M_l\|_\infty$. Since $\frac{L}{1+L} < 1$, we have prove that G is a contraction mapping. Since G is a contraction, there exists a unique fixed point $M^*(s,x)$, which is the Laplace decomposition of the unique solution $m(t,x)$ to the fractional Hirota-Satsuma equation.

5. Application of LDM

Consider the fractional Hirota-Satsuma coupled KdV equation (Akinyemi and Iyiola, 2020; Jincun and Hong, 2013)

$$\begin{aligned}
D_t^\alpha u(y,t) &= \frac{1}{2}u_{yyy} - 3uu_y + 3(vw)_y, \\
D_t^\alpha v(y,t) &= -v_{yyy} + 3uv_y, \\
D_t^\alpha w(y,t) &= -w_{yyy} + 3uw_y, \quad 1 \ge \alpha > 0, \quad t \ge 0,
\end{aligned} \tag{13}$$

with conditions

$$\begin{aligned}
u(y,0) &= \frac{1}{3}(\beta - 2c^2) + 2c^2 tanh^2(cy), \\
v(y,0) &= \frac{-4c^2 a_0(\beta + c^2)}{3a_1^2} + \frac{4c^2(\beta + c^2)}{3a_1} tanh(cy), \\
w(x,0) &= a_0 + a_1 tanh(cy),
\end{aligned} \tag{14}$$

where a_0, a_1, c and β are constants. The analytical solution of (13) subject to (14) when $\alpha = 1$ in (Prajapatia, Rakesh and Pankaj, 2016; Jincun and Hong, 2013) is

$$u(y,t)=\frac{1}{3}(\beta-2c^2)+2c^2 tanh^2(c(y+\beta t)),$$
$$v(y,t)=\frac{-4c^2a_0(\beta+c^2)}{3a_1^2}+\frac{4c^2(\beta+c^2)}{3a_1}tanh(c(y+\beta t)), \tag{15}$$
$$w(y,t)=a_0+a_1 tanh(c(y+\beta t)).$$

Taking Laplace transformation of (13) with conditions (14), we obtain

$$s^{\alpha}\mathcal{L}[u(y,t)]=\sum_{i=0}^{\infty}s^{\alpha-i-1}u^{(i)}(y,0)+\mathcal{L}[\frac{1}{2}u_{yyy}-3uu_y+3(vw)_y],$$
$$s^{\alpha}\mathcal{L}[v(y,t)]=\sum_{i=0}^{\infty}s^{\alpha-i-1}v^{(i)}(y,0)+\mathcal{L}[-v_{yyy}+3uv_y], \tag{16}$$
$$s^{\alpha}\mathcal{L}[w(y,t)]=\sum_{i=0}^{\infty}s^{\alpha-i-1}w^{(i)}(y,0)+\mathcal{L}[-w_{yyy}+3uw_y].$$

Choosing i=0 in (16) gives;

$$\mathcal{L}[u(y,t)]=\frac{1}{s}u^{(0)}(y,0)+\frac{1}{s^{\alpha}}\mathcal{L}[\frac{1}{2}u_{yyy}-3uu_y+3(vw)_y],$$
$$\mathcal{L}[v(y,t)]=\frac{1}{s}v^{(0)}(y,0)+\frac{1}{s^{\alpha}}\mathcal{L}[-v_{yyy}+3uv_y], \tag{17}$$
$$\mathcal{L}[w(y,t)]=\frac{1}{s}w^{(0)}(y,0)+\frac{1}{s^{\alpha}}\mathcal{L}[-w_{yyy}+3uw_y].$$

Introducing the inverse Laplace transformation on (17) and simplifying to give

$$u(y,t)=\mathcal{L}^{-1}[\frac{1}{s}u^{(0)}(y,0)]+\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[\frac{1}{2}u_{yyy}-3uu_y+3(vw)_y]],$$
$$v(y,t)=\mathcal{L}^{-1}[\frac{1}{s}v^{(0)}(y,0)]+\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[-v_{yyy}+3uv_y]], \tag{18}$$
$$w(y,t)=\mathcal{L}^{-1}[\frac{1}{s}w^{(0)}(y,0)]+\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[-w_{yyy}+3uw_y]].$$

$$\sum_{i=0}^{\infty}u_i(y,t)=\mathcal{L}^{-1}[\frac{1}{s}u(y,0)]+\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[\frac{1}{2}u_{yyy}-3A_i+3(B_i)_y]],$$
$$\sum_{i=0}^{\infty}v_i(y,t)=\mathcal{L}^{-1}[\frac{1}{s}v(y,0)]+\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[-v_{yyy}+3C_i]], \tag{19}$$
$$\sum_{i=0}^{\infty}w_i(y,t)=\mathcal{L}^{-1}[\frac{1}{s}w(y,0)]+\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[-w_{yyy}+3D_i]],$$

where A_i, B_i, C_i and D_i are the nonlinear terms to be decomposed using Adomian polynomial expressed as

$$\begin{aligned} A_i &= u_i u_{iy} = \frac{1}{i!}\frac{d^i}{d\lambda^i}[\sum_{i=0}^{\infty}(u_i\lambda^i)(u_{iy}\lambda^i)]_{\lambda=0}, \\ A_0 &= u_0 u_{0y}, \\ A_1 &= u_1 u_{0y} + u_0 u_{1y}, \\ A_2 &= u_2 u_{0y} + u_1 u_{1y} + u_0 u_{2y} \quad for \quad i = 0,1,2,... \end{aligned} \tag{20}$$

$$\begin{aligned} B_i &= (v_i w_i) = \frac{1}{i!}\frac{d^i}{d\lambda^i}[\sum_{i=0}^{\infty}(v_i\lambda^i)(w_i\lambda^i)]_{\lambda=0}, \\ B_0 &= v_0 w_0, \\ B_1 &= v_1 w_0 + v_0 w_1, \\ B_2 &= v_2 w_0 + v_1 w_1 + v_0 w_2 \quad for \quad i = 0,1,2,... \end{aligned} \tag{21}$$

$$\begin{aligned} C_i &= u_i v_{iy} = \frac{1}{i!}\frac{d^i}{d\lambda^i}[\sum_{i=0}^{\infty}(u_i\lambda^i)(v_{iy}\lambda^i)]_{\lambda=0}, \\ C_0 &= u_0 v_{0y}, \\ C_1 &= u_1 v_{0y} + u_0 v_{1y}, \\ C_2 &= u_2 v_{0y} + u_1 v_{1y} + u_0 v_{2y} \quad for \quad i = 0,1,2,... \end{aligned} \tag{22}$$

$$\begin{aligned} D_i &= u_i w_{iy} = \frac{1}{i!}\frac{d^i}{d\lambda^i}[\sum_{i=0}^{\infty}(u_i\lambda^i)(w_{iy}\lambda^i)]_{\lambda=0}, \\ D_0 &= u_0 w_{0y}, \\ D_1 &= u_1 w_{0y} + u_0 w_{1y}, \\ D_2 &= u_2 w_{0y} + u_1 w_{1y} + u_0 w_{2y} \quad for \quad i = 0,1,2,... \end{aligned} \tag{23}$$

From (19),

$$\begin{aligned} u(y,0) &= \frac{1}{3}(\beta - 2c^2) + 2c^2 tanh^2(cy), \\ v(y,0) &= \frac{-4c^2 a_0(\beta + c^2)}{3a_1^2} + \frac{4c^2(\beta + c^2)}{3a_1} tanh(cy), \\ w(x,0) &= a_0 + a_1 tanh(cy). \end{aligned} \tag{24}$$

Decomposing (20) and substituting into (19), the recursive relations becomes,

$$u_{i+1}(y,t)=\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[\frac{1}{2}u_{iyyy}-3A_i+3(B_i)_y]],$$
$$v_{i+1}(y,t)=\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[-v_{iyyy}+3C_i]], \tag{25}$$
$$w_{i+1}(y,t)=\mathcal{L}^{-1}[\frac{1}{s^{\alpha}}\mathcal{L}[-w_{iyyy}+3D_i]].$$

Substituting $i=0$ in (22) and simplifying gives,

$$u_1=\frac{4t^{\alpha}}{\Gamma(\alpha+1)}\left[c^3\beta\tanh(cy)\mathrm{sech}(\mathrm{cy})^2\right],$$
$$v_1=\frac{4t^{\alpha}}{3\Gamma(\alpha+1)}\left[\frac{c^3\beta(c^2+\beta)}{a_1\cosh(cy)^2}\right], \tag{26}$$
$$w_1=\frac{t^{\alpha}}{\Gamma(\alpha+1)}\left[a_1c\beta\mathrm{sech}(\mathrm{cy})^2\right].$$

substituting $i=1$ in (22) and simplifying gives,

$$u_2=-4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left[c^4\beta^2(2\cosh(cy)^2-3)\mathrm{sech}(\mathrm{cy})^4\right],$$
$$v_2=-\frac{8t^{2\alpha}}{3\Gamma(2\alpha+1)}\left[\frac{c^4\beta^2(c^2+\beta)}{a_1\cosh(cy)^2}\tanh(cy)\right], \tag{27}$$
$$w_2=-\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}\left[a_1c^2\beta^2\tanh(cy)\mathrm{sech}(\mathrm{cy})^2\right].$$

substituting $i=2$ in (22) and simplifying gives,

$$u_3=\frac{16\sinh(cy)c^5\beta^2t^{3\alpha}}{\cosh(cy)^7\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}[\Gamma(\alpha+1)^2\beta\cosh(cy)^4-11\Gamma(\alpha+1)^2c^2\cosh(cy)^2-$$
$$2\Gamma(\alpha+1)^2\beta\cosh(cy)^2+6\Gamma(2\alpha+1)c^2\cosh(cy)^2+18c^2\Gamma(\alpha+1)^2-9c^2\Gamma(2\alpha+1)],$$
$$v_3=\frac{-32c^7\beta^2(c^2+\beta)t^{3\alpha}}{3\cosh(cy)^6a_1\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}[2\Gamma(\alpha+1)^2\cosh(cy)^4-9\Gamma(\alpha+1)^2\cosh(cy)^2+ \tag{28}$$
$$3\Gamma(2\alpha+1)\cosh(cy)^2+6\Gamma(\alpha+1)^2-3\Gamma(2\alpha+1)],$$
$$w_3=\frac{2a_1c^3\beta^2t^{3\alpha}}{\cosh(cy)^6\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}[2\Gamma(\alpha+1)^2\beta\cosh(cy)^4+24\Gamma(\alpha+1)^2c^2\cosh(cy)^2-$$
$$3\Gamma(\alpha+1)^2\beta\cosh(cy)^2-12\Gamma(2\alpha+1)c^2\cosh(cy)^2-24\Gamma(\alpha+1)^2c^2+12c^2\Gamma(2\alpha+1)].$$

Thus, the series solution becomes;

$$u(y,t)=\sum_{i=0}^{3}u_i(y,t),$$
$$v(y,t)=\sum_{i=0}^{3}v_i(y,t), \tag{29}$$
$$w(y,t)=\sum_{i=0}^{3}w_i(y,t).$$

Then,

$$u(y,t)=\frac{1}{3}(\beta-2c^2)+2c^2tanh^2(cy)+\frac{4t^\alpha}{\Gamma(\alpha+1)}[c^3\beta\tanh(cy)\text{sech}(\text{cy})^2]-$$
$$\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}[4c^4\beta^2(2\cosh(cy)^2-3)\text{sech}(\text{cy})^4]+$$
$$\frac{16\sinh(cy)c^5\beta^2t^{3\alpha}}{\cosh(cy)^7\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}[\Gamma(\alpha+1)^2\beta\cosh(cy)^4-11\Gamma(\alpha+1)^2c^2\cosh(cy)^2-$$
$$2\Gamma(\alpha+1)^2\beta\cosh(cy)^2+6\Gamma(2\alpha+1)c^2\cosh(cy)^2+18c^2\Gamma(\alpha+1)^2-9c^2\Gamma(2\alpha+1)],$$
$$v(y,t)=\frac{-4c^2a_0(\beta+c^2)}{3a_1^2}+\frac{4c^2(\beta+c^2)}{3a_1}tanh(cy)+\frac{4t^\alpha}{3\Gamma(\alpha+1)}\frac{c^3\beta(c^2+\beta)}{a_1\cosh(cy)^2}-$$
$$\frac{8t^{2\alpha}}{3\Gamma(2\alpha+1)}[\frac{c^4\beta^2(c^2+\beta)}{a_1\cosh(cy)^2}\tanh(cy)]$$
$$\frac{-32c^7\beta^2(c^2+\beta)t^{3\alpha}}{3\cosh(cy)^6a_1\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}[2\Gamma(\alpha+1)^2\cosh(cy)^4-9\Gamma(\alpha+1)^2\cosh(cy)^2+$$
$$3\Gamma(2\alpha+1)\cosh(cy)^2+6\Gamma(\alpha+1)^2-3\Gamma(2\alpha+1)],$$
$$w(y,t)=a_0+a_1tanh(cy)+\frac{t^\alpha}{\Gamma(\alpha+1)}[a_1c\beta\text{sech}(\text{cy})^2]-\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}[a_1c^2\beta^2\tanh(cy)\text{sech}(\text{cy})^2],$$
$$\frac{-32c^7\beta^2(c^2+\beta)t^{3\alpha}}{3\cosh(cy)^6a_1\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}[2\Gamma(\alpha+1)^2\cosh(cy)^4-9\Gamma(\alpha+1)^2\cosh(cy)^2+$$
$$3\Gamma(2\alpha+1)\cosh(cy)^2+6\Gamma(\alpha+1)^2-3\Gamma(2\alpha+1)]. \tag{30}$$

The classical solution was obtained at $\alpha=1$ as

$$u(y,t)=\frac{1}{3}(\beta-2c^2)+2c^2tanh^2(cy)+4c^3\beta\tanh(cy)\text{sech}(\text{cy})^2t-$$
$$2[c^4\beta^2(2\cosh(cy)^2-3)\text{sech}(\text{cy})^4]t^2+\frac{8}{3}[\beta\cosh(cy)^2+c^2-2\beta]\tanh(cy)\text{sech}(\text{cy})^4t^3,$$
$$v(y,t)=\frac{4c^2(\beta+c^2)}{3a_1}[-\frac{a_0}{a_1}+\tanh(cy)+\frac{c\beta}{\cosh(cy)^2}t-$$

$$\frac{c^2\beta^2}{\cosh(cy)^2}\tanh(cy)t^2-\frac{4}{3}(c^5\beta^2\frac{(2\cosh(cy)^2-3)}{\cosh(cy)^4})t^3],$$

$$w(y,t)=a_0+a_1tanh(cy)+a_1c\beta\text{sech}(\text{cy})^2t-a_1(c\beta)^2\tanh(cy)\text{sech}(\text{cy})^2t^2+$$
$$\frac{1}{3}[a_1(c\beta)^3(2\cosh(cy)^2-3)\text{sech}(\text{cy})^4]t^3. \quad (31)$$

6. Discussion of Results

The computation of this study was carried out using Maple 18 software. The scheme for solving the nonlinear coupled Hirota Satsuma equation was derived, and thereafter, various results were presented. The validation of the method was done by making comparion with the literature(Akinyemi and Iyiola, 2020) as shown in Table 1. Tables 2-4 present the solution of the classical form of the Hirota-Satsuma equation in comparison with the exact solutions of u(y,t), v(y,t), and w(y,t), respectively, and it was observed that there is good agreement as shown in the tables. Similarly, the closed form solutions for u(y,t), v(y,t), and w(y,t) for $\alpha=0.5$ and $\alpha=0.75$ are shown in tables 5 and 6, respectively. Figures 1-3 depict the graphical representations of the results obtained by LDM and their respective exact solutions. Variation of α shows the memory effect of the fractional order problems as presented in figure 4. The results when $\alpha=1$ agree with the exact solutions.

Table 1: Validation of results for $\alpha=1,\ a_0=\beta=1.5,\ a_1=c=0.1.$

y	t	u(y,t)		−v(y,t)		w(y,t)	
		(Akinyemi and Iyiola, 2020)	LDM	(Akinyemi and Iyiola, 2020)	LDM	(Akinyemi and Iyiola, 2020)	LDM
0.25	0.2	0.49339370	0.49339371	3.00893782	3.00893782	1.50549446	1.50549446
0.50		0.49346080	0.49346079	3.00392761	3.00392761	1.50798297	1.50798298
0.75		0.49355223	0.49355222	2.99893736	2.99893735	1.51046158	1.51046158
0.25	0.4	0.49347730	0.49347714	3.00292783	3.00292776	1.50847956	1.50847959
0.50		0.49357356	0.49357336	2.99794233	2.99794233	1.51095579	1.51095585
0.75		0.49369362	0.49369346	2.99298407	2.99298403	1.51341851	1.51341856
0.25	0.6	0.49359636	0.49359635	2.99694857	2.99694823	1.51144938	1.51144954
0.50		0.49372112	0.49372111	2.99199664	2.99199668	1.51390895	1.51390921
0.75		0.49386895	0.49386815	2.98707902	2.98707942	1.51635148	1.51635153

Table 2: The LDM solution of u(y,t) for three iterations in comparison with exact solution when $c=0.1, a_0=a_1=\alpha=\beta=1$.

t	y	u(y,t)	Exact	Error
0.1	0	0.3266686666	0.3266686665	1.00×10^{-10}
	0.5	0.3267384957	0.3267384942	1.50×10^{-9}
	1.0	0.3269067305	0.3269067278	2.70×10^{-9}
0.3	0	0.3266846666	0.3266846558	1.08×10^{-8}
	0.5	0.3267941691	0.3267941224	4.67×10^{-8}
	1.0	0.3270009755	0.3270008946	8.09×10^{-8}
0.5	0	0.3267166666	0.3267165834	8.32×10^{-8}
	0.5	0.3268655890	0.3268653408	2.48×10^{-7}

	1.0	0.3271104056	0.3271100017	4.04×10^{-7}

Table 3: The LDM solution of v(y,t) for three iterations in comparison with exact solution when $c=0.1, a_0=a_1=\alpha=\beta=1$.

t	y	v(y,t)	Exact	Error
0.1	0	-0.0133319998	-0.0133320045	4.67×10^{-9}
	0.5	-0.0126596303	-0.0126596349	2.62×10^{-9}
	1.0	-0.0119912748	-0.0119912793	4.47×10^{-9}
0.3	0	-0.0130626618	-0.0130627878	1.26×10^{-7}
	0.5	-0.0123915014	-0.0123916258	1.24×10^{-7}
	1.0	-0.0117256756	-0.0117257959	1.20×10^{-7}
0.5	0	-0.0127933109	-0.0127938939	5.83×10^{-7}
	0.5	-0.0121238966	-0.0121244710	5.74×10^{-7}
	1.0	-0.0114611270	-0.0114616816	5.55×10^{-7}

Table 4: The LDM solution of w(y,t) for three iterations in comparison with exact solution when $c=0.1, a_0=a_1=\alpha=\beta=1$.

t	y	w(y,t)	Exact	LDM
0.1	0	1.009999667	1.009999667	-
	0.5	1.059928104	1.059928104	-
	1.0	1.109558470	1.109558470	-
0.3	0	1.029991000	1.029991003	3.00×10^{-9}
	0.5	1.079829740	1.079829769	2.90×10^{-8}
	1.0	1.129272529	1.129272584	5.50×10^{-8}
0.5	0	1.049958333	1.049958375	4.20×10^{-8}
	0.5	1.099667748	1.099667995	2.47×10^{-7}
	1.0	1.148884592	1.148885034	4.42×10^{-7}

Table 5: The LDM results for u(y,t), v(y,t) and w(y,t) for three iterations when $\alpha=0.5, c=0.1, a_0=a_1=\beta=1$.

y	t	u(y,t)	v(y,t)	w(y,t)
0.1	0	0.3267066666	-0.0129861170	1.035634905
	0.5	0.3268271297	-0.0123158858	1.085405041
	1.0	0.3270442496	-0.0116513529	1.134753002
0.3	0	0.3267866666	-0.0126342414	1.061556657
	0.5	0.3269576347	-0.0119675730	1.111064294
	1.0	0.3272227026	-0.0113082891	1.160028602
0.5	0	0.3268666666	-0.0123918956	1.079256533
	0.5	0.3270715935	-0.0117285168	1.128522848
	1.0	0.3273684012	-0.0110736674	1.177166584

Table 6: The LDM solution of u(y,t), v(y,t) and w(y,t) for three iterations when $\alpha = 0.75, c = 0.1, a_0 = a_1 = \beta = 1$.

y	t	u(y,t)	v(y,t)	w(y,t)
0.1	0	0.3266761819	-0.0132060999	1.019344428
	0.5	0.3267645558	-0.0125342966	1.069230848
	1.0	0.3269508196	-0.0118671249	1.118773446
0.3	0	0.3267161097	-0.0128726802	1.044053562
	0.5	0.3268532494	-0.0122030491	1.093779186
	1.0	0.3270865295	-0.0115396706	1.143041563
0.5	0	0.3267730512	-0.0125953283	1.064531817
	0.5	0.3269502316	-0.0119283003	1.114065307
	1.0	0.3272216481	-0.0112688583	1.163038829

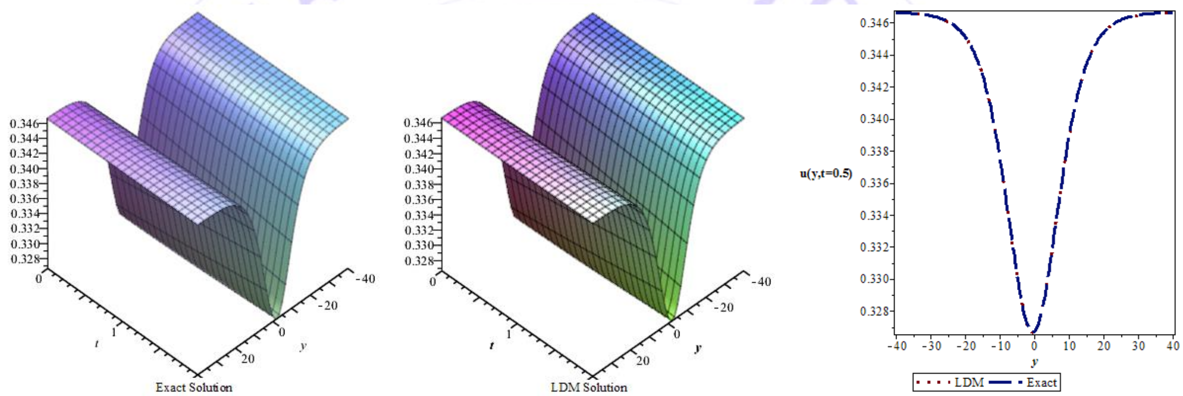

Figure 1: Exact and LDM solutions of u(y,t) for $c = 0.1, \alpha = \beta = c_0 = c_1 = 1$.

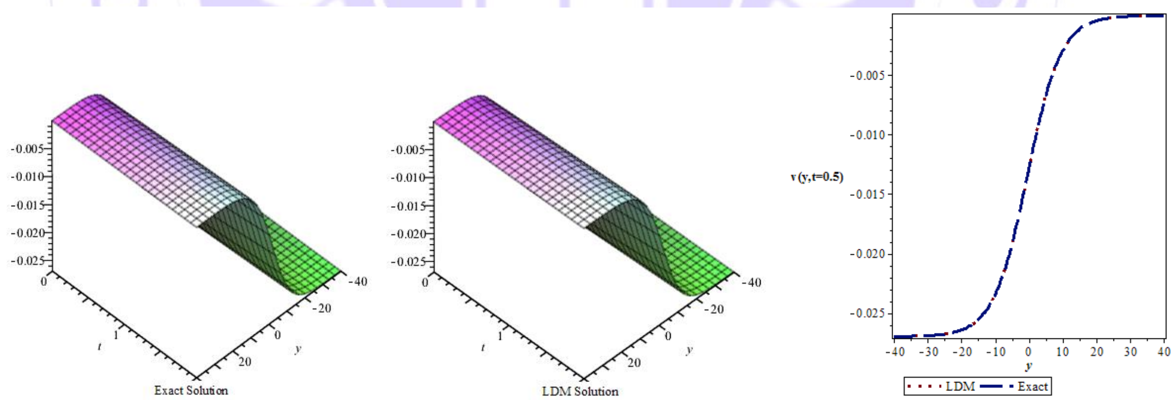

Figure 2: Exact and LDM solutions of v(y,t) for $c = 0.1, \alpha = \beta = a_0 = a_1 = 1$.

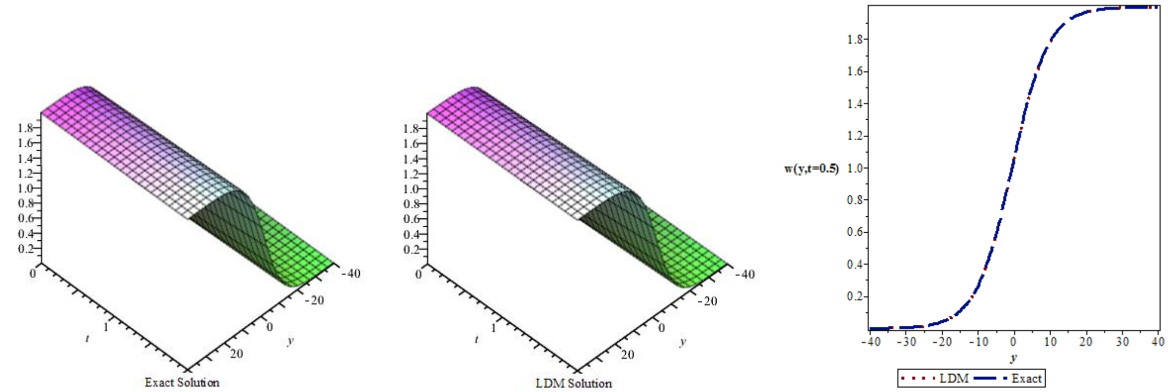

Figure 3: Exact and LDM solutions of w(y,t) for $c = 0.1, \alpha = \beta = a_0 = a_1 = 1$.

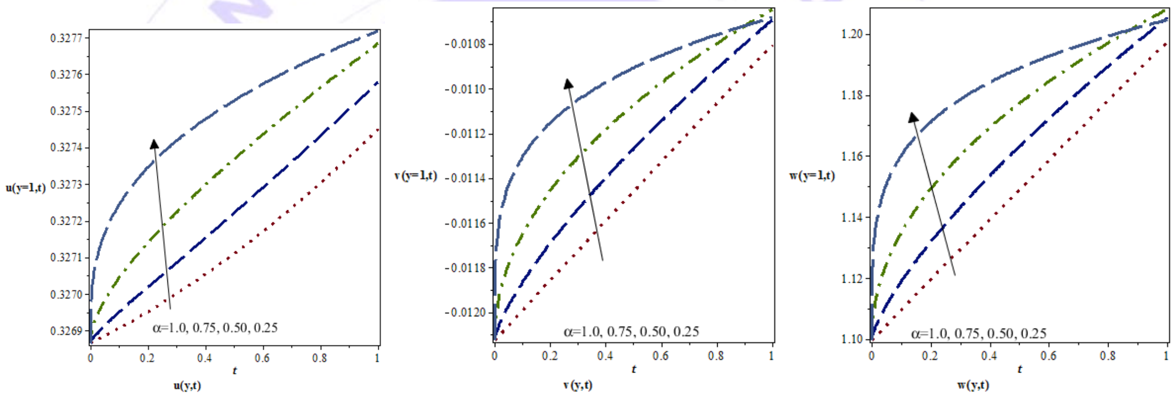

Figure 4: Effect of α when $c = 0.1, a_0 = a_1 = \beta = 1$ using LDM.

6. Conclusions

In this study, the Laplace decomposition method (LDM) was applied to obtain the solution of the fractional order coupled Hirota-Satsuma KdV equation. The tabular and graphical representations of the analytical and numerical solutions depict the accuracy of the LDM for solving various orders of the time-fractional Hirota-Satsuma equation with few iterations. Conclusively, the scheme can be extended to solve fractional-order partial differential equations in different dimensions arising from science and engineering due to its accuracy and less computational time.

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