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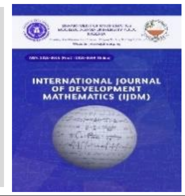

On Simple and Regular Permutation Groups of Degree 3p - Numerical Approach

Tijjani I Gandi[a], Sabo Hamma[b] and Mshelia I Bello[b]

[a]Nigerian Meteorological Agency (NiMet) Abuja, Nigeria.

[b]Department of Mathematics, Abubakar Tafawa Balewa University Bauchi, Nigeria.

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ABSTRACT

Let G be a dihedral group of degree 3p that are not p-groups. Several survey articles in the research space written about the implications of the classification of finite simple group for permutation groups and related areas. Determination of the properties of permutation groups as simple, regular, nilpotency, transitivity, primitivity and solvability of various categories of groups have been carried out with a view to classify finite permutation groups of certain degree is still of interest to many mathematicians. Our result showed that all Groups of degree 3p that are not p-groups are not simple and irregular. We used some theorems and a computer programme – Groups, Algorithms and Programming (GAP) to enhance and validate our findings.

1. Introduction

Permutation group is a group G of whose elements are permutations of a given set Ω and whose group operation is composition of permutations in G which are bijective functions from the set Ω to itself (Praeger and Schneider, 2018b). The group of all permutations of a set is the symmetric group and the term permutation group is usually restricted to mean a subgroup of the symmetric group (Wielandt, 1964). Although it is often less beneficial to study group within this framework, permutation groups are still quite important and not only appear in many other branches of mathematics, but also form an active field of research today. Transitive and Primitive finite permutation groups can be thought of as the building blocks of finite permutation groups, and questions about finite permutation groups can often be reduced to the primitive case (Fawcett, 2009). Thus, it would be very important to know the structure of these groups. Therefore, we see that normal subgroup plays a crucial role in obtaining structural results of groups. Finite Permutation groups which includes direct product of pairs of permutation groups have been generated for research purposes using various approaches, see Kantor (1979) and Praeger (1984). Determinations of such properties as primitivity, solubility, nilpotency and regularity of various categories of groups have been carried out with the view to classify finite permutation groups by various authors. Thus, the search for an easier way to classify permutation groups of various degrees without using the Classification of the Finite Simple Groups (CFSG) continues (Fengler, 2018). After the publication of the Classification of the Finite Simple Groups (CFSG) only four theorems bordering on whether specific categories of groups are primitive, regular, nilpotent and soluble were necessary to classify permutation groups (Aschbacher and Smith, 2004). As there did not exist a uniform method to classify permutation groups based on their degrees according to (Fengler, 2018), many papers dealt with various approaches to examine the structure of these groups. Some of the authors used theoretical methods to determine some properties of the groups whereas others computed some of them on computers. It is against this background that we are carrying out this research by considering the Simplicity and Regularity of finite permutation groups of degrees 3p using numerical approach.

2. Preliminaries

The following definition and results will be required

2.1 Simple group (Galois, 1832)

A group *G* is called simple if the trivial groups are the only normal subgroups in *G*. Every group has two normal

* Corresponding author. Tel.: +2347035213522
E-mail address: tijjaniibrahimgiade@gmail.com (Tijjani I. Gandi.)
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subgroups namely the group itself and the identity {1}.

2.2 Dihedral Group (Gandi, 2019)

A dihedral group D_n is a symmetric group for an n-sided regular polygon for $n>2$. Dihedral group are non-abelian permutation groups with group order 2n. We can mathematically write dihedral group $D_n = \{x, y | x^n = y^2 = 1, yx = x^{n-1}y = x^{-1}y\}$

2.3 Semiregular and Regular Group (Audu, 2000)

A permutation group G is called semi-regular if one is the only element of G which fixes each point. In other word, G is semi-regular when $G_\alpha=1$ for each $\alpha \in G$. A transitive semi-regular is called a regular group. Thus, the group G={(1),(12)(34),(14)(23),(13)(24)} is a regular group. Clearly subgroups of semi-group are semi-regular; 1 is semi-regular. As we get that in a semi –regular group G, orbits have the same size, namely $|G|$, and hence, the order of G divides the degree of G. Furthermore, in a regular group G we have that $|G| = |\alpha^G| = |\Omega|$, $\alpha \epsilon \Omega$ and so the order and the degree of G coincide.

3. Methodology

We now provide relevant result that will form a basis in our study of permutation groups of degree 3p and where necessary provide their proofs.

3.1 Theorem (Sylow's 1872)

Let G be a finite group. If $|G| = p^r m \; and \; (p, m) = 1$, then

1. There is at least one Sylow p-subgroup H of G.
2. If B is any p-subgroup of G, then $X \subseteq x^{-1}Hx \; for \; some \; x \in G$.
3. If K is any Sylow p-subgroup of G, $H_1 = g^{-1}Hg \; for \; some \; g \in G$.
4. If n is the number of Sylow p-subgroups of G, then n divides m and n ≡ 1mod p.

3.2 Corollary (Gregory, 2018)

A Sylow p-subgroup of a group G is normal if and only if it is unique.

Proof

Suppose that a Sylow p-subgroup H of a group G is unique. Since all Sylow p-subgroups are conjugate to H, the uniqueness of H implies that $H = g^{-1}Hg$ for all $g \in G$, that is H is normal in G. Conversely, suppose H is normal in G, then $g^{-1}Hg = H$ for all $g \in G$. Let k be any other Sylow p-subgroup of G, then $K = g^{-1}Hg$ for some $g \in G$ that is $K = H$.

3.3 Theorem (Ma'u, 2015)

If |G| = pq where p,q are distinct primes such that q≠1mod p, then G is not simple. That is a group G is simple if and only if no proper subgroup is normal.

Proof:

By Sylow's theorem, the number N_p of distinct Sylow p-Subgroup of G is a divisor of q and it is ≡ 1mod p. Since q is a prime, N_p is either 1 or q and since q ≠ 1mod p it follows that, $N_p = 1$. Thus G has unique Sylow p-Subgroup say H which is normal in G. Therefore, G is not simple.

3.4 Lemma (Wielandt, 1964)

Let G be a primitive permutation group having no regular normal subgroups. If $G_\alpha (\alpha \in \Omega)$ is simple then G is simple.

Proof

Let $N \triangleleft G$ with $N \neq 1$. Since G is primitive, N is transitive. Now G_α is simple and $N_\alpha = (N \cap G_\alpha) \triangleleft G_\alpha$. Hence either $N_\alpha = 1$ or $N_\alpha = G_\alpha$. If $N_\alpha = 1$, then N is a regular normal subgroup of G, a contradiction. Thus $N_\alpha = G_\alpha$ and since N is transitive, we get that $|N : N_\alpha| = |\Omega| = |G : G_\alpha|$. Hence $N = G$ is simple

3.5 Theorem (Audu, 2000)
Let G be a transitive abelian group. Then, the group G is regular.

Proof

Fix $\alpha \in \Omega$. If $\beta \in \Omega \ni g \in G$ with $\alpha^G = \beta$.Now $G_\alpha = {G_\alpha}^g = (G_\alpha)^g = g^{-1}(G_\alpha)g = G_\alpha$ (since G is abelian). As α, β are arbitrary, we get that $G_\alpha = 1$ Since G is transitive it is regular.

3.6 Lemma (Passman, 1968)
Suppose G is a dihedral group of any order, then G is transitive.

Proof

For given α_i, α_j as any two vertices of the regular polygon with i < j, we readily see that $(\alpha_1\alpha_2 \ldots \alpha_i \ldots \alpha_j \ldots \alpha_n)^{j-i}$ is the rotation about the centre of the polygon through angle $2\pi^c / n$, (where n is the number of edges of the polygon) which take α_i to α_j. As such G is transitive.

4. Results
The following are the main results on the dihedral group of degree 3p

4.1 Proposition
Let G be a dihedral group of degree 3p, where p is an odd prime number greater than 3 (P>3). Then G is not simple and the stabilizer of the point $x \in \Omega$, G_α is of order 2. And hence G is not regular.

Proof:
The regular polygon of degree 3p is given by

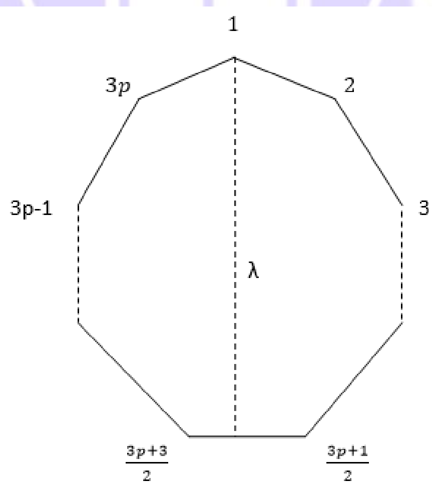

Figure 1: Diagram for Dihedral Groups of Degree 3p

We have that $|G| = 2.3p, \Omega = 3p \text{ and } \alpha \epsilon \Omega$.
The rotation in G is given as

$$x = \left(1,2,3, \ldots, \tfrac{3p+1}{2}, \tfrac{3p+3}{2}, \ldots, 3p-1, 3p\right)$$

And the reflection λ in G via vertical axis of symmetry fixing point 1 is given by

$$y = (2,3p)(3,3p-1)\ldots\left(\frac{3p+1}{2},\frac{3p+3}{2}\right)$$

Now the distinct elements of the group G is given by

$$G = \{1, x, x^2, \ldots, x^{3p-1}, y, xy, \ldots, x^{3p-1}y\}.$$

Now, the number of Sylow 2-subgroups in the group G namely $N_2 = |Syl_2(G)| \equiv 1 mod 2\ and\ divides\ 3p$.Clearly $N_2 = 1, 3, p$ or 3p. it follows that $Syl_2(G)$ is not unique and hence not normal in G.

Similarly, the number of Sylow p-subgroups in the group G Namely $N_p = |Syl_p(G)| \equiv 1 mod p$ and divide 6. clearly $N_p = 1\ or\ 6$. Thus, it follows that $Syl_p(G)$ is not unique and hence not normal in G. Also, the number of Sylow 3-subgroups in the group G namely $N_3 = |Syl_3(G)| \equiv 1 mod 3$ and divide 2p. Clearly $N_3 = 1$. It follows that $Syl_3(G)$ is unique and consequently normal in G by Corollary 3.0.2. As such the group G is not simple.

Again. we see that the stabilizer of the point 1 in G is given by $G_1 = \{(1), y\}\ \forall p > 3$, i.e $|G_\alpha| = 2$. AS $|G_1| \neq 1$ that G is not semi regular.

As we conclude G is transitive and not semiregular, then as such G is not regular. Hence the proof.

Example (1)

Let G be the dihedral group of degree 3p, where $p = 5$ is acting on $\Omega = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$. Then G is not Simple and not regular.

Now the order of dihedral group of degree 15 (D_{15}) is given as $|D_{15}| = 2.3.5$

The Sylow 2-subgroups of D_{15} have order 2 as illustrated bellow:

$H_1 = \{(1), (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9)\}$
$H_2 = \{(1), (1,13)(2,12)(3,11)(4,10)(5,9)(6,8)(14,15)\}$
$H_3 = \{(1), (1,10)(2,9)(3,8)(4,7)(5,6)(11,15)(12,14)\}$
$H_4 = \{(1), (1,7)(2,6)(3,5)(8,15)(9,14)(10,13)(11,12)\}$
$H_5 = \{(1), (1,4)(2,3)(5,15)(6,14)(7,13)(8,12)(9,11)\}$
$H_6 = \{(1), (1,15)(2,14)(3,13)(4,12)(5,11)(6,10)(7,9)\}$
$H_7 = \{(1), (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)(13,15)\}$
$H_8 = \{(1), (1,9)(2,8)(3,7)(4,6)(10,15)(11,14)(12,13)\}$
$H_9 = \{(1), (1,6)(2,5)(3,4)(7,15)(8,14)(9,13)(10,12)\}$
$H_{10} = \{(1), (1,3)(4,15)(5,14)(6,13)(7,12)(8,11)(9,10)\}$
$H_{11} = \{(1), (1,14)(2,13)(3,12)(4,11)(5,10)(6,9)(7,8)\}$
$H_{12} = \{(1), (1,11)(2,10)(3,9)(4,8)(5,7)(12,15)(13,14)\}$
$H_{13} = \{(1), (1,8)(2,7)(3,6)(4,5)(9,15)(10,14)(11,13)\}$
$H_{14} = \{(1), (1,5)(2,4)(6,15)(7,14)(8,13)(9,12)(10,11)\}$
$H_{15} = \{(1), (1,2)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10)\}$

We note that in accordance with Sylow theorems there are 15 such subgroups since $15 \equiv 1 mod 2$ and 15 divides 15. It follows that $Syl_2(D_{15})$ is not unique and hence not a normal subgroup of D_{15}.

The Sylow 3-subgroups of D_{15} has order 3. We readily see that D_{15} has a unique Sylow 3-subgroup given by

$$J_1 = \left\{\begin{matrix}(1), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(8,10,15), (1,11,6)(2,12,7)(3,13,8) \\ (4,14,9)(5,15,10)\end{matrix}\right\}$$

It clearly follows that $Syl_3(D_{15})$ say J is unique and consequently normal in D_{15} by Corollary 3.0.2.

Similarly, the Sylow 5-subgroups of D_{15} has order 5. We readily see that D_{15} has a unique Sylow 5-subgroup given by

$$K = \left\{\begin{matrix}(1), (1,7,13,4,10)(2,8,14,5,11)(3,9,15,6,12), (1,13,10,7,4)(2,14,11,8,5) \\ (3,15,12,9,6), (1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,15)(3,12,6,15,9)\end{matrix}\right\}$$

It clearly follows that $Syl_5(D_{15})$ say K is unique and consequently normal in D_{15}.

Routing calculation shows that $Syl_2(G)$ is not unique and $Syl_5(G)$ is unique by corollary 3.0.2. As such D_{15} is not simple.

Next, since $|G| = 2.3.5$ and $\Omega = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$ is the set of the point of G. It follows from lemma 3.0.6, G is Transitive as the orbit $\alpha^G = \Omega\ \forall\ \alpha\ \in \Omega$.

Also the stabilizer of the point 1 in G is given by $G_1 = \{(1), (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9)\}$. As $|G| \neq 1, \forall\ p > 3$ which shows that G is not semiregular. Therefore by virtue of theorem 3.0.5 G is not regular.

Comparison with a Standard Programmed

We now test our result with a standard programmed (GAP)

```
gap> D15:=DihedralGroup(IsGroup,30);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15),(2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9) ])
gap> Order(D15);30
gap> IsTransitive(D15);true
gap> IsSemiRegular(D15);false
gap> IsSimple(D15);false
gap> IsRegular(D15);false
gap> S2:=SylowSubgroup(D15,2);
Group([ (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9) ])
gap> Order(S2);2
gap> IsNormal(D15,S2);false
gap> S3:=SylowSubgroup(D15,3);
Group([ (1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10) ])
gap> Order(S3);3
gap> IsNormal(D15,S3);true
gap> S5:=SylowSubgroup(D15,5);
Group([ (1,10,4,13,7)(2,11,5,14,8)(3,12,6,15,9) ])
gap> Order(S5);5
gap> IsNormal(D15,S5);true
```

5. Conclusion

This study showed that permutation group of degree 3p, where p is prime number greater than 3 is not simple and irregular. We used a standard computational group theory technique GAP 4.11.1 (2021) in carryout all the computations. This research can be extended for one or more groups properties such as solubility, primitivity etc of same algebraic structures especially with the aid of computer programs

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