International Journal of Development Mathematics Vol 2 Issue 1 Page No. 075 - 087

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals

Numerical Method for Solving Differential Equations for Epidemiological and Biological Models

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ARTICLE INFO

Article history:
Received 20 December 2024
Received in revised form 10 March 2025
Accepted 20 March 2025

Keywords:
Differential equations, Epidemiological models, Biological modeling, Power series approximation, Numerical accuracy, Computational efficiency.

MSC 2020 Subject classification:
34A30, 34A34

ABSTRACT

This paper presents a novel block hybrid method designed to improve accuracy and efficiency in solving differential equations, with specific applications in epidemiological and biological models. The new method were derived using a power series polynomial via interpolation and collocation procedure. The basic properties of the new method was analyzed numerically and it is obvious that the method is of uniform order nine, consistency, zero-stability, convergent. We further obtain the absolute stability through the stability polynomial showing to be A-stable. Numerical experiments demonstrate the method's effectiveness, with smaller absolute errors than existing methods across various models, including the SIR model and growth models in population dynamics. The results affirm the potential of the new method for high-precision applications in epidemiology, biology and related fields, marking an advancement in differential equation modeling.

1 Introduction

Numerical methods are also crucial in applied fields such as science, social sciences, engineering and medicine, where exact solutions are often infeasible. These methods have practical applications, such as predicting spacecraft trajectories, testing product safety and assessing risks for insurance companies often through simulations that require solving PDEs (Adesanya, Onsachi and Odekunle, 2017). Although numerical analysts aim to approximate solutions as closely to exact solutions, they often face challenges in achieving exact solutions, which can be computationally intensive or constrained by time (Khalsaraei and Shokri, 2020; Kyagya, Raymond and Sabo, 2021; Ojo and Samuel, 2025). Applications of numerical methods are widespread, from multidimensional root-finding to industry predictions, computing, and even profit-loss calculations, with theoretical and practical implications in fields like engineering, medicine, and business (Akinfenwa *et al.* 2020; Kwanamu, Skwame and Sabo, 2021; Ishaq *et al.* 2024).

In epidemiology, Kermack and McKendrick's 1927 SIR model is a foundational example of differential equations modeling infectious disease spread. In this model, populations are divided into susceptible (S), infective (I), and recovered (R) groups, representing disease dynamics through oscillatory differential equations (Raymond and Sabo, 2023). Scholars continue to develop numerical methods for simulating such models, with newer techniques addressing the limitations of traditional methods. For instance, block hybrid methods using power series polynomials via interpolation and collocation have been proposed to enhance the accuracy and efficiency of simulations for epidemiological models, which remain challenging due to the complex nature of disease transmission (Rizky, Mochammad and Sri, 2021; Raymond and Sabo, 2023; Bello *et al.* 2023; Udo and Awoyemi, 2017).

The Predictor-Corrector Method has several limitations that affect the numerical methods' efficiency and applicability in numerical computations. One primary drawback is its dependence on an initial prediction step, which can introduce errors that propagate through subsequent calculations. If the predictor provides an inaccurate estimate, the corrector may require multiple iterations to refine the solution, increasing computational cost and processing time. This issue is particularly evident in solving stiff equations, where poor predictions can result in slow convergence or even divergence, ultimately reducing the method's stability (Adesanya, Odekunle and Alkali, 2012). Furthermore, the correction step often employs implicit methods, necessitating iterative techniques such as Newton's method, which

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https://doi.org/10.62054/ijdm/0201.06

can further increase computational complexity, particularly when applied to nonlinear or high-dimensional systems (Tafida, 2015).

Another major setback of the Predictor-Corrector Method is its inefficiency in large-scale computations due to repeated function evaluations. Both the prediction and correction phases require solving equations at each step, making the method computationally demanding compared to fully implicit approaches that minimize redundant calculations. This inefficiency is especially problematic in real-time applications where computational speed is critical (Kumleng *et al*., 2017). Additionally, selecting an appropriate step size is crucial for maintaining stability and accuracy; however, an improper choice can lead to unnecessary computational overhead or loss of precision. In many cases, alternative numerical methods such as implicit Runge-Kutta or Rosenbrock methods are preferred for their superior stability and efficiency in handling stiff and large-scale problems (Tafida, 2015). Furthermore, some implementations rely on lower-order predictors, limiting the accuracy and effectiveness of the method (Adewale and Sabo, 2024). These challenges underscore the need for further refinements in numerical techniques to enhance performance and applicability across various fields (Abdulrahman and Fatima, 2016).

To address the limitations inherent in the traditional Predictor-Corrector Method, the Block Method was developed as an innovative solution (Raymond and Sabo, 2023). This numerical technique partitions the time interval into smaller blocks, solving discrete schemes at multiple off-grid points simultaneously within each block. By employing implicit integration schemes such as backward Euler or Rosenbrock methods within each block, the Block Method effectively manages stiff and highly nonlinear systems while distributing the computational load more evenly (Omar and Adeyeye, 2016). The primary advantages of block method is to enhanced efficiency as it divide into smaller partitions to reduces the computational burden (Khalsaraei and Shokri, 2020). The Block Method not only overcomes some of the computational and stability challenges of the Predictor-Corrector Method but also opens avenues for further research into more robust and scalable numerical techniques for solving complex differential equations.
This research considers the simulation of first order oscillatory differential equation of the form

$$\frac{dy}{dx} = y,\ y(0) = y_0,\ x \in [a,b] \tag{1}$$

where $f:\Re \times \Re^m \to \Re^m,\ y,\ y_0 \in \Re^m$,
f satisfy the Lipchitz condition.

2 Methodology

Let the power series polynomial be

$$y(x) = \sum_{i=0}^{r+s-1} a_i \left(\frac{x-x_n}{h}\right)^i \tag{2}$$

where r and s represent distinct numbers of interpolation and collocation points, respectively.
The one-step block hybrid methods with eight partitions will be derive by using the partitioned points

$$\left\{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\right\}.$$

Using equation (1) with $r=1,\ s=9$, we have a polynomial of degree $r+s-1=9$ as

$$y(x) = \sum_{i=0}^{9} a_i \left(\frac{x-x_n}{h}\right)^i \tag{3}$$

Differentiating (2) once, to get

$$y'(x) = \sum_{i=0}^{9} i a_i \left(\frac{x-x_n}{h}\right)^{i-1} \tag{4}$$

Substituting (3 and (4) into (1) to gives

$$\sum_{i=0}^{9} a_i \left(\frac{x-x_n}{h}\right)^i = f(x) \tag{5}$$

$$\sum_{i=0}^{9} i a_i \left(\frac{x - x_n}{h} \right)^{i-1} = f(x, y) \tag{6}$$

Now interpolating (5) at $x_{n+r}, r = 1$ and collocating (6) at $x_{n+s}, s = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1$. This leads to a system of equations expressed in matrix form $AX = U$ as

$$\begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 \\ 0 & 1 & 2x_{n+\frac{1}{8}} & 3x^2_{n+\frac{1}{8}} & 4x^3_{n+\frac{1}{8}} & 5x^4_{n+\frac{1}{8}} & 6x^5_{n+\frac{1}{8}} & 7x^6_{n+\frac{1}{8}} & 8x^7_{n+\frac{1}{8}} & 9x^8_{n+\frac{1}{8}} \\ 0 & 1 & 2x_{n+\frac{1}{4}} & 3x^2_{n+\frac{1}{4}} & 4x^3_{n+\frac{1}{4}} & 5x^4_{n+\frac{1}{4}} & 6x^5_{n+\frac{1}{4}} & 7x^6_{n+\frac{1}{4}} & 8x^7_{n+\frac{1}{4}} & 9x^8_{n+\frac{1}{4}} \\ 0 & 1 & 2x_{n+\frac{3}{8}} & 3x^2_{n+\frac{3}{8}} & 4x^3_{n+\frac{3}{8}} & 5x^4_{n+\frac{3}{8}} & 6x^5_{n+\frac{3}{8}} & 7x^6_{n+\frac{3}{8}} & 8x^7_{n+\frac{3}{8}} & 9x^8_{n+\frac{3}{8}} \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & 7x^6_{n+\frac{1}{2}} & 8x^7_{n+\frac{1}{2}} & 9x^8_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+\frac{5}{8}} & 3x^2_{n+\frac{5}{8}} & 4x^3_{n+\frac{5}{8}} & 5x^4_{n+\frac{5}{8}} & 6x^5_{n+\frac{5}{8}} & 7x^6_{n+\frac{5}{8}} & 8x^7_{n+\frac{5}{8}} & 9x^8_{n+\frac{5}{8}} \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{3}{4}} & 4x^3_{n+\frac{3}{4}} & 5x^4_{n+\frac{3}{4}} & 6x^5_{n+\frac{3}{4}} & 7x^6_{n+\frac{3}{4}} & 8x^7_{n+\frac{3}{4}} & 9x^8_{n+\frac{3}{4}} \\ 0 & 1 & 2x_{n+\frac{7}{8}} & 3x^2_{n+\frac{7}{8}} & 4x^3_{n+\frac{7}{8}} & 5x^4_{n+\frac{7}{8}} & 6x^5_{n+\frac{7}{8}} & 7x^6_{n+\frac{7}{8}} & 8x^7_{n+\frac{7}{8}} & 9x^8_{n+\frac{7}{8}} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 & 9x_{n+1}^8 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{bmatrix} \tag{7}$$

Employing the Gaussian elimination method on equation (7) yields the coefficients $a_i, (i = 0(1)9)$. These coefficients are subsequently substituted into equation (1) to obtain the implicit continuous hybrid method in the form:

$$y(x) = \alpha_1 y_{n+1} + h\left(\beta_0 f_{n+0} + \beta_{\frac{1}{8}} f_{n+\frac{1}{8}} + \beta_{\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{\frac{3}{8}} f_{n+\frac{3}{8}} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{\frac{5}{8}} f_{n+\frac{5}{8}} + \beta_{\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_{\frac{7}{8}} f_{n+\frac{7}{8}} + \beta_1 f_{n+1} \right) \tag{8}$$

Where the unknown coefficients of α and β are represented as respectively given as:

$\alpha_1 = 1$

$$\beta_0 = -\frac{989}{28350} + t - \frac{761}{70}t^2 + \frac{59062}{945}t^3 - \frac{1068}{5}t^4 + \frac{34208}{75}t^5 - \frac{3072}{5}t^6 + \frac{53248}{105}t^7 - \frac{8192}{35}t^8 + \frac{131072}{2835}t^8$$

$$\beta_{\frac{1}{8}} = -\frac{2944}{14175} + 32t^2 - \frac{30784}{105}t^3 + \frac{11168}{9}t^4 - \frac{673792}{225}t^5 + \frac{117760}{27}t^6 - \frac{1196032}{315}t^7 + \frac{16384}{9}t^8 - \frac{1048576}{2835}t^8$$

$$\beta_{\frac{1}{4}} = \frac{464}{14175} - 56t^2 + \frac{3312}{5}t^3 - \frac{146824}{45}t^4 + \frac{1956992}{225}t^5 - \frac{366592}{27}t^6 + \frac{3915776}{315}t^7 - \frac{278528}{45}t^8 - \frac{524288}{405}t^8$$

$$\beta_{\frac{3}{8}} = -\frac{5248}{14175} + \frac{224}{3}t^2 - \frac{128192}{135}t^3 + \frac{25504}{5}t^4 - \frac{1097728}{75}t^5 + \frac{72704}{3}t^6 - \frac{2441216}{105}t^7 + \frac{180224}{15}t^8 - \frac{1048576}{405}t^8$$

$$\beta_{\frac{1}{2}} = \frac{464}{2835} - 70t^2 + \frac{2764}{3}t^3 - \frac{46624}{9}t^4 + \frac{703552}{45}t^5 - \frac{733184}{27}t^6 + \frac{1712128}{63}t^7 - \frac{131072}{9}t^8 - \frac{262144}{81}t^8$$

$$\beta_{\frac{5}{8}} = -\frac{5248}{14175} + \frac{224}{5}t^2 - \frac{3008}{5}t^3 + \frac{156512}{45}t^4 - \frac{2443264}{225}t^5 + \frac{2642944}{135}t^6 - \frac{6406144}{315}t^7 + \frac{507904}{45}t^8 - \frac{1048576}{405}t^8$$

$$\beta_{\frac{3}{4}} = \frac{464}{14175} - \frac{56}{3}t^2 + \frac{34288}{135}t^3 - 1496t^4 + \frac{358784}{75}t^5 - \frac{26624}{3}t^6 + \frac{999424}{105}t^7 - \frac{16384}{3}t^8 - \frac{524288}{405}t^8$$

$$\beta_{\frac{7}{8}} = -\frac{2944}{14175} + \frac{32}{7}t^2 - \frac{6592}{105}t^3 + \frac{16864}{45}t^4 - \frac{274432}{225}t^5 + \frac{62464}{27}t^6 - \frac{114688}{45}t^7 + \frac{475136}{315}t^8 - \frac{1048576}{2835}t^8$$

$$\beta_1 = -\frac{989}{28350} - \frac{1}{2}t^2 + \frac{242}{35}t^3 - \frac{1876}{45}t^4 + \frac{30944}{225}t^5 - \frac{7168}{27}t^6 + \frac{94208}{315}t^7 - \frac{8192}{45}t^8 - \frac{131072}{2835}t^8$$

where

$$t = \frac{x - x_n}{h}$$

evaluating (8) at non-interpolating points to yield,

$$y_n = y_{n+1} - \frac{989}{28350} h\, f_n - \frac{2944}{14175} h\, f_{n+\frac{1}{8}} + \frac{464}{14175} h\, f_{n+\frac{1}{4}} - \frac{5248}{14175} h\, f_{n+\frac{3}{8}} + \frac{454}{2835} h\, f_{n+\frac{1}{2}} - \frac{5248}{14175} h\, f_{n+\frac{5}{8}} + \frac{464}{14175} h\, f_{n+\frac{3}{4}} - \frac{2944}{14175} h\, f_{n+\frac{7}{8}} - \frac{989}{28350} h\, f_{n+1} \quad (9)$$

$$y_{n+\frac{1}{8}} = y_{n+1} - \frac{8183}{4147200} h\, f_n - \frac{111587}{2073600} h\, f_{n+\frac{1}{8}} - \frac{261023}{2073600} h\, f_{n+\frac{1}{4}} - \frac{368039}{2073600} h\, f_{n+\frac{3}{8}} - \frac{343}{25920} h\, f_{n+\frac{1}{2}} - \frac{542969}{2073600} h\, f_{n+\frac{5}{8}} - \frac{24353}{2073600} h\, f_{n+\frac{3}{4}} - \frac{408317}{2073600} h\, f_{n+\frac{7}{8}} - \frac{149527}{4147200} h\, f_{n+1} \quad (10)$$

$$y_{n+\frac{1}{4}} = y_{n+1} + \frac{9}{11200} h\, f_n - \frac{9}{1400} h\, f_{n+\frac{1}{8}} - \frac{79}{5600} h\, f_{n+\frac{1}{4}} - \frac{333}{1400} h\, f_{n+\frac{3}{8}} - \frac{9}{280} h\, f_{n+\frac{1}{2}} - \frac{403}{1400} h\, f_{n+\frac{5}{8}} - \frac{9}{5600} h\, f_{n+\frac{3}{4}} - \frac{279}{1400} h\, f_{n+\frac{7}{8}} - \frac{401}{11200} h\, f_{n+1} \quad (11)$$

$$y_{n+\frac{3}{8}} = y_{n+1} + \frac{175}{165888} h\, f_n - \frac{5725}{580608} h\, f_{n+\frac{1}{8}} + \frac{24575}{580608} h\, f_{n+\frac{1}{4}} - \frac{85465}{580608} h\, f_{n+\frac{3}{8}} + \frac{125}{36288} h\, f_{n+\frac{1}{2}} - \frac{159175}{580608} h\, f_{n+\frac{5}{8}} - \frac{3775}{580608} h\, f_{n+\frac{3}{4}} - \frac{115075}{580608} h\, f_{n+\frac{7}{8}} - \frac{41705}{1161216} h\, f_{n+1} \quad (12)$$

$$y_{n+\frac{1}{2}} = y_{n+1} \frac{107}{113400} h\, f_n - \frac{112}{14175} h\, f_{n+\frac{1}{8}} + \frac{989}{28350} h\, f_{n+\frac{1}{4}} - \frac{1154}{14175} h\, f_{n+\frac{3}{8}} + \frac{227}{2835} h\, f_{n+\frac{1}{2}} - \frac{4094}{14175} h\, f_{n+\frac{5}{8}} - \frac{61}{28350} h\, f_{n+\frac{3}{4}} - \frac{2822}{14175} h\, f_{n+\frac{7}{8}} - \frac{4063}{113400} h\, f_{n+1} \quad (13)$$

$$y_{n+\frac{5}{8}} = y_{n+1} \frac{369}{358400} h\, f_n - \frac{243}{25600} h\, f_{n+\frac{1}{8}} + \frac{7031}{179200} h\, f_{n+\frac{1}{4}} - \frac{17217}{179200} h\, f_{n+\frac{3}{8}} + \frac{351}{2240} h\, f_{n+\frac{1}{2}} - \frac{39967}{179200} h\, f_{n+\frac{5}{8}} - \frac{1719}{179200} h\, f_{n+\frac{3}{4}} - \frac{35451}{179200} h\, f_{n+\frac{7}{8}} - \frac{12881}{358400} h\, f_{n+1} \quad (14)$$

$$y_{n+\frac{1}{2}} = y_{n+1} + \frac{119}{129600} h\, f_n - \frac{953}{113400} h\, f_{n+\frac{1}{8}} + \frac{15577}{453600} h\, f_{n+\frac{1}{4}} - \frac{9341}{113400} h\, f_{n+\frac{3}{8}} + \frac{2903}{22680} h\, f_{n+\frac{1}{2}} - \frac{15011}{113400} h\, f_{n+\frac{5}{8}} + \frac{21247}{453600} h\, f_{n+\frac{3}{4}} - \frac{22823}{113400} h\, f_{n+\frac{7}{8}} - \frac{32377}{907200} h\, f_{n+1} \quad (15)$$

$$y_{n+\frac{7}{8}} = y_{n+1} + \frac{33953}{29030400} h\, f_n - \frac{156437}{14515200} h\, f_{n+\frac{1}{8}} - \frac{645607}{14515200} h\, f_{n+\frac{1}{4}} - \frac{1573169}{14515200} h\, f_{n+\frac{3}{8}} - \frac{31457}{181440} h\, f_{n+\frac{1}{2}} - \frac{2797679}{14515200} h\, f_{n+\frac{5}{8}} - \frac{2302297}{14515200} h\, f_{n+\frac{3}{4}} - \frac{2233547}{14515200} h\, f_{n+\frac{7}{8}} - \frac{1070017}{29030400} h\, f_{n+1} \quad (16)$$

To make y_{n+1} the subject of equation (9), we isolate it on one side of the equation

$$y_{n+1} = y_n + \frac{989}{28350} h\, f_n + \frac{2944}{14175} h\, f_{n+\frac{1}{8}} - \frac{464}{14175} h\, f_{n+\frac{1}{4}} + \frac{5248}{14175} h\, f_{n+\frac{3}{8}} - \frac{454}{2835} h\, f_{n+\frac{1}{2}} + \frac{5248}{14175} h\, f_{n+\frac{5}{8}} - \frac{464}{14175} h\, f_{n+\frac{3}{4}} + \frac{2944}{14175} h\, f_{n+\frac{7}{8}} + \frac{989}{28350} h\, f_{n+1} \quad (17)$$

To obtain the new block hybrid method, we substitute equation (17) into equations (10) to (16)

$$\left.\begin{aligned}
y_{n+\frac{1}{8}} &= -y_n + \frac{1070017}{29030400} h\, f_n + \frac{2233547}{14515200} h\, f_{n+\frac{1}{8}} - \frac{2302297}{14515200} h\, f_{n+\frac{1}{4}} + \frac{2797679}{14515200} h\, f_{n+\frac{3}{8}} - \frac{31457}{181440} h\, f_{n+\frac{1}{2}} + \frac{1573169}{14515200} h\, f_{n+\frac{5}{8}} - \frac{645607}{14515200} h\, f_{n+\frac{3}{4}} + \frac{156437}{14515200} h\, f_{n+\frac{7}{8}} - \frac{33953}{29030400} h\, f_{n+1} \\
y_{n+\frac{1}{4}} &= -y_n + \frac{32377}{907200} h\, f_n + \frac{22823}{113400} h\, f_{n+\frac{1}{8}} - \frac{21247}{453600} h\, f_{n+\frac{1}{4}} + \frac{15011}{113400} h\, f_{n+\frac{3}{8}} - \frac{2903}{22680} h\, f_{n+\frac{1}{2}} + \frac{9341}{113400} h\, f_{n+\frac{5}{8}} - \frac{15577}{453600} h\, f_{n+\frac{3}{4}} + \frac{953}{113400} h\, f_{n+\frac{7}{8}} - \frac{119}{129600} h\, f_{n+1} \\
y_{n+\frac{3}{8}} &= -y_n + \frac{12881}{358400} h\, f_n + \frac{35451}{179200} h\, f_{n+\frac{1}{8}} - \frac{1719}{179200} h\, f_{n+\frac{1}{4}} + \frac{39967}{179200} h\, f_{n+\frac{3}{8}} - \frac{351}{2240} h\, f_{n+\frac{1}{2}} + \frac{17217}{179200} h\, f_{n+\frac{5}{8}} - \frac{7031}{179200} h\, f_{n+\frac{3}{4}} + \frac{243}{25600} h\, f_{n+\frac{7}{8}} - \frac{369}{358400} h\, f_{n+1} \\
y_{n+\frac{1}{2}} &= -y_n + \frac{4063}{113400} h\, f_n + \frac{2822}{14175} h\, f_{n+\frac{1}{8}} - \frac{61}{28350} h\, f_{n+\frac{1}{4}} + \frac{4094}{14175} h\, f_{n+\frac{3}{8}} - \frac{227}{2835} h\, f_{n+\frac{1}{2}} + \frac{1154}{14175} h\, f_{n+\frac{5}{8}} - \frac{989}{28350} h\, f_{n+\frac{3}{4}} + \frac{122}{14175} h\, f_{n+\frac{7}{8}} - \frac{107}{113400} h\, f_{n+1} \\
y_{n+\frac{5}{8}} &= -y_n + \frac{41705}{1161216} h\, f_n + \frac{115075}{580608} h\, f_{n+\frac{1}{8}} - \frac{3775}{580608} h\, f_{n+\frac{1}{4}} + \frac{159175}{580608} h\, f_{n+\frac{3}{8}} - \frac{125}{36288} h\, f_{n+\frac{1}{2}} + \frac{85465}{580608} h\, f_{n+\frac{5}{8}} - \frac{24575}{580608} h\, f_{n+\frac{3}{4}} + \frac{5725}{580608} h\, f_{n+\frac{7}{8}} - \frac{175}{165888} h\, f_{n+1} \\
y_{n+\frac{3}{4}} &= -y_n + \frac{403}{11200} h\, f_n + \frac{279}{1400} h\, f_{n+\frac{1}{8}} - \frac{9}{5600} h\, f_{n+\frac{1}{4}} + \frac{403}{1400} h\, f_{n+\frac{3}{8}} - \frac{9}{280} h\, f_{n+\frac{1}{2}} + \frac{333}{1400} h\, f_{n+\frac{5}{8}} - \frac{79}{5600} h\, f_{n+\frac{3}{4}} + \frac{9}{1400} h\, f_{n+\frac{7}{8}} - \frac{9}{11200} h\, f_{n+1} \\
y_{n+\frac{7}{8}} &= -y_n + \frac{149527}{4147200} h\, f_n + \frac{408317}{2073600} h\, f_{n+\frac{1}{8}} - \frac{24353}{2073600} h\, f_{n+\frac{1}{4}} + \frac{542969}{2073600} h\, f_{n+\frac{3}{8}} - \frac{343}{25920} h\, f_{n+\frac{1}{2}} + \frac{368039}{2073600} h\, f_{n+\frac{5}{8}} - \frac{261023}{2073600} h\, f_{n+\frac{3}{4}} + \frac{111587}{2073600} h\, f_{n+\frac{7}{8}} - \frac{8183}{4147200} h\, f_{n+1} \\
y_{n+1} &= -y_n + \frac{989}{28350} h\, f_n + \frac{2944}{14175} h\, f_{n+\frac{1}{8}} - \frac{464}{14175} h\, f_{n+\frac{1}{4}} + \frac{5248}{14175} h\, f_{n+\frac{3}{8}} - \frac{454}{2835} h\, f_{n+\frac{1}{2}} + \frac{5248}{14175} h\, f_{n+\frac{5}{8}} - \frac{464}{14175} h\, f_{n+\frac{3}{4}} + \frac{2944}{14175} h\, f_{n+\frac{7}{8}} - \frac{989}{28350} h\, f_{n+1}
\end{aligned}\right\} \quad (18)$$

3 Analysis of Basic Properties of the new Methods

The analysis of the basic properties of the new method are analyzed numerically, such as order and error constant, consistency, zero-stability and the region of absolute stability (Raymond and Sabo, 2023 and Adewale and Sabo, 2024).

3.1 Order and Error Constant

The linear operator linked with the new block hybrid method is explicitly described as:

$$L[y(x);h] = \sum_{i=0}^{1} \alpha_i y(x+ih) + h^1 \beta_i y'(x+ih) \quad (19)$$

where $y(x)$ is an arbitrary function that is continuous and differentiable on $[a,b]$. Using the Taylor series at point x to expand $y(x+ih)$ and $y'(x+ih)$ gives

$$L\,[y(x),h]=c_0y(x)+c_1hy^l(x)+\cdots+c_qh^qy^q(x)+c_{q+1}h^{q+1}y^{q+1}(x)+\cdots$$

Using the linear operator on the new method, we have

$$\sum_{j=0}^{\infty}\frac{\left(\frac{1}{9}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{9449717}{65318400}\left(\frac{1}{9}\right)^j+\frac{1408913}{8164800}\left(\frac{2}{9}\right)^j-\frac{200029}{816480}\left(\frac{1}{3}\right)^j+\frac{8641823}{32659200}\left(\frac{4}{9}\right)^j-\frac{6755041}{32659200}\left(\frac{5}{9}\right)^j+\frac{462127}{4082400}\left(\frac{2}{3}\right)^j-\frac{335983}{8164800}\left(\frac{7}{9}\right)^j+\frac{116687}{13063680}\left(\frac{8}{9}\right)^j-\frac{8183}{9331200}(1)^j\right]$$
$$\sum_{j=0}^{\infty}\frac{\left(\frac{2}{9}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{37829}{204120}\left(\frac{1}{9}\right)^j+\frac{34369}{510300}\left(\frac{2}{9}\right)^j-\frac{45331}{255150}\left(\frac{1}{3}\right)^j+\frac{103987}{510300}\left(\frac{4}{9}\right)^j-\frac{83291}{510300}\left(\frac{5}{9}\right)^j+\frac{9239}{102060}\left(\frac{2}{3}\right)^j-\frac{8467}{255150}\left(\frac{7}{9}\right)^j+\frac{3697}{510300}\left(\frac{8}{9}\right)^j-\frac{1}{1400}(1)^j\right]$$
$$\sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{16381}{89600}\left(\frac{1}{9}\right)^j+\frac{31}{1600}\left(\frac{2}{9}\right)^j-\frac{13273}{50400}\left(\frac{1}{3}\right)^j+\frac{2123}{8960}\left(\frac{4}{9}\right)^j-\frac{8201}{44800}\left(\frac{5}{9}\right)^j+\frac{5039}{50400}\left(\frac{2}{3}\right)^j-\frac{407}{11200}\left(\frac{7}{9}\right)^j+\frac{101}{12800}\left(\frac{8}{9}\right)^j-\frac{25}{32256}(1)^j\right]$$
$$\sum_{j=0}^{\infty}\frac{\left(\frac{8}{9}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{3346}{18225}\left(\frac{1}{9}\right)^j+\frac{628}{25515}\left(\frac{2}{9}\right)^j-\frac{40648}{127575}\left(\frac{1}{3}\right)^j+\frac{20924}{8960}\left(\frac{4}{9}\right)^j-\frac{21076}{127575}\left(\frac{5}{9}\right)^j+\frac{11852}{127575}\left(\frac{2}{3}\right)^j-\frac{872}{25515}\left(\frac{7}{9}\right)^j+\frac{953}{127575}\left(\frac{8}{9}\right)^j-\frac{94}{127575}(1)^j\right]$$
$$\sum_{j=0}^{\infty}\frac{(1)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{15741}{89600}\left(\frac{1}{9}\right)^j-\frac{27}{2240}\left(\frac{2}{9}\right)^j-\frac{1209}{5600}\left(\frac{1}{3}\right)^j-\frac{2889}{4800}\left(\frac{4}{9}\right)^j-\frac{2889}{4800}\left(\frac{5}{9}\right)^j-\frac{1209}{5600}\left(\frac{2}{3}\right)^j-\frac{27}{2240}\left(\frac{7}{9}\right)^j-\frac{15741}{89600}\left(\frac{8}{9}\right)^j-\frac{2857}{89600}(1)^j\right]$$
$$\sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{257}{1400}\left(\frac{1}{9}\right)^j+\frac{17}{700}\left(\frac{2}{9}\right)^j-\frac{199}{630}\left(\frac{1}{3}\right)^j+\frac{83}{700}\left(\frac{4}{9}\right)^j-\frac{211}{700}\left(\frac{5}{9}\right)^j+\frac{299}{6300}\left(\frac{2}{3}\right)^j-\frac{11}{350}\left(\frac{7}{9}\right)^j+\frac{1}{140}\left(\frac{8}{9}\right)^j-\frac{1}{1400}(1)^j\right]$$
$$\sum_{j=0}^{\infty}\frac{\left(\frac{7}{9}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{341383}{1866240}\left(\frac{1}{9}\right)^j+\frac{24647}{1166400}\left(\frac{2}{9}\right)^j-\frac{178703}{583200}\left(\frac{1}{3}\right)^j+\frac{460649}{4665600}\left(\frac{4}{9}\right)^j-\frac{1251607}{4665600}\left(\frac{5}{9}\right)^j+\frac{4459}{116640}\left(\frac{2}{3}\right)^j-\frac{92617}{1166400}\left(\frac{7}{9}\right)^j+\frac{90013}{9331200}\left(\frac{8}{9}\right)^j-\frac{8183}{9331200}(1)^j\right]$$
$$\sum_{j=0}^{\infty}\frac{\left(\frac{8}{9}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{23552}{127575}\left(\frac{1}{9}\right)^j+\frac{3712}{127575}\left(\frac{2}{9}\right)^j-\frac{41984}{127575}\left(\frac{1}{3}\right)^j+\frac{3632}{25515}\left(\frac{4}{9}\right)^j-\frac{41984}{127575}\left(\frac{5}{9}\right)^j+\frac{3712}{127575}\left(\frac{2}{3}\right)^j-\frac{23552}{127575}\left(\frac{7}{9}\right)^j-\frac{3956}{127575}(1)^j\right]$$
$$\sum_{j=0}^{\infty}\frac{(1)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[-\frac{15741}{89600}\left(\frac{1}{9}\right)^j-\frac{27}{2240}\left(\frac{2}{9}\right)^j-\frac{1209}{5600}\left(\frac{1}{3}\right)^j-\frac{2889}{4800}\left(\frac{4}{9}\right)^j+\frac{2889}{4800}\left(\frac{5}{9}\right)^j-\frac{1209}{5600}\left(\frac{2}{3}\right)^j-\frac{27}{2240}\left(\frac{7}{9}\right)^j-\frac{15741}{89600}\left(\frac{8}{9}\right)^j-\frac{2857}{89600}(1)^j\right]$$

Thus, $\overline{C}_0=0,\overline{C}_1=0,\overline{C}_2=0,\overline{C}_3=0,\ldots=0,\overline{C}_p=0$; implying that the order of the new method is $p=[9\quad 9\quad 9\quad 9\quad 9\quad 9\quad 9\quad 9]^T$. That is, the new method is of order nine, with the error constant is given by

$$C_{10}=[7.3505\times10^{-12}\quad 5.9871\times10^{-12}\quad 6.4964\times10^{-12}\quad 6.1760\times10^{-12}\quad 6.4964\times10^{-12}\quad 5.9871\times10^{-12}\quad 7.3505\times10^{-12}\quad -5.8932\times10^{-13}]^T$$

3.2 Consistency

According to Adewale and Sabo (2024), the new method is consistent, since the order is greater than or equal to one $(p\geq1)$.

3.3 Zero Stability

Definition 3.1

A block hybrid method is said to be zero-stable if the roots $z_s,\ s=1,2,\ldots,n$ of the first characteristic polynomial $\overline{\rho}(z)$, defined by

$$\overline{\rho}(z)=\det\left[zA^{(0)}-E\right] \tag{20}$$

Satisfies $|z_s|\leq1$ and every root with $|z_s|=1$ has multiplicity not exceeding the order of the differential equation as $h\to0$. Moreover, as $h\to0$, $\rho(z)=z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ *and* E. The primary implication of zero stability is to regulate the propagation of error throughout the integration process.

The new method is said to be zero-stable if as $h\to0$, the root $z_j,\ j=1(1)k$ of the first characteristic polynomial $\rho(z)=0$ that is $\rho(z)=\det\left[\sum_{j=0}^{k}A^{(j)}z^{k-j}\right]=0$ Satisfies $|z_j|\leq1$ and for those roots with $|z_j|=1$, multiplicity must not exceed two. The new method is expressed in the form

$$\rho(z)=\left|z\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\right|=z^7(z-1)$$

Thus, solving for z in

$$\rho(z)=z^7(z-1)=0 \tag{21}$$

Gives $z_1=0,\ z_2=0,\ z_3=0,\ z_4=0,\ z_5=0,\ z_6=0,\ z_7=0$ and $z_8=1$

Therefore, the new method is zero-stable.

3.4 Convergence

Theorem 3.1:

The requisite and complete prerequisites for a linear multistep method to achieve convergence entail both consistency and zero-stability.

Based on Dahlquist's Theorem 3.1 as stated by Raymond and Sabo (2023), the new method demonstrates convergence due to its consistency and zero-stability.

3.5 Absolute Stability Region

Definition 3.2

Region of absolute stability is a region in the complex z plane, where $z=\lambda h$. It is defined as those values of z such that the numerical solutions of $y'=-\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition (Adewale and Sabo, 2024). In order to ascertain the regions of absolute stability for the block hybrid method, a technique was adopted that circumvents the need for computing polynomial roots or solving simultaneous inequalities. This method is referred to as the Boundary Locus Method (BLM). Based on Definition 3.2, the boundary locus method is been applied on the new method to obtain the stability polynomial as

$$\bar{h}(w)=h^8\left(\frac{417744659}{76710263390208000}w^7+\frac{485207}{71915871928320}w^8\right)-h^7\left(\frac{49763165419}{2013644413992960000}w^7-\frac{27522907}{94389581905920}w^8\right)+$$
$$h^6\left(-\frac{6724482907}{12585277587456000}w^7-\frac{705170539}{94389581905920}w^8\right)+h^5\left(\frac{5204419680941}{107873807892480000}w^7-\frac{344466127}{2528292372480}w^8\right)+h^4\left(\frac{862373358001}{1123685498880000}w^7+\frac{76898191}{42138206208}w^8\right)+$$
$$h^3\left(\frac{6040298487439}{561842749440000}w^7-\frac{46017679}{2633637888}w^8\right)+h^2\left(-\frac{5931801838133}{70230343680000}w^7+\frac{1601063}{13436928}w^8\right)+h\left(\frac{3179713}{7257600}w^7-\frac{1}{2}w^8\right)-w^7+w^8 \tag{22}$$

The stability polynomial is used to plot the absolute stability region of the new method as

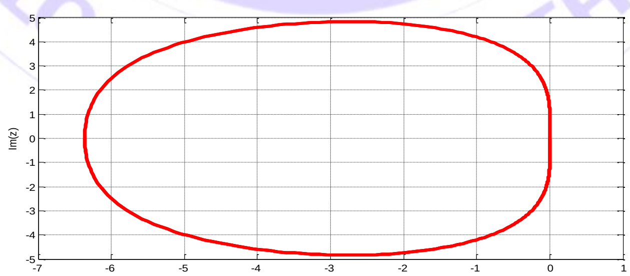

Figure 3.1: Stability region of the new method

4 Results and Discussion

In this section, the numerical results will be presented in both tabular and graphical formats to evaluate the computational reliability of the newly derived block method using Maple 18 software package. The absolute errors of the computed solutions are examined and compared with those from existing methods. Additionally, the outcomes from both computational approaches are analyzed and discussed.

4.1 Numerical Experiments

The newly developed block methods will be tested on several oscillatory problems of the form (1), as shown below.

The following notations shall be used in Tables 4.1 to 4.5 below.

Notations	Meaning
X	Point of Evaluation
EOA16	Error in Omar and Adeyeye, (2016)
ESYA16	Error in Sunday, Yusuf and Andest, (2016)
ES13	Error in Sunday *et al.* (2013)
ES11	Error in Sunday, (2011)
EJ13	Error in James *et al.* (2013)

Problem 1 (SIR Model):

The susceptible infected recovered disease is an epidemic disease which computes the total of number of people that are sick with a communicable infection in a population over a period of time. Such models were developed from the fact that they contain coupled equations about the number of individuals who are susceptible to disease is appropriately large within the host $S(t)$, number of individual person whose level of parasite $I(t)$ and the whole individuals who have recovered $R(t)$. The model is coupled as:

$$\frac{dS}{dt} = \mu - \beta SI - \mu S \tag{23}$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I \tag{24}$$

$$\frac{dR}{dt} = \gamma I - \mu R \tag{25}$$

for $\mu, \gamma \; and \; \beta$ are positive parameters. y is given as,

$$y = S + I + R \tag{26}$$

Summing (4.1), (4.2) and (4.3) to get

$$y' = \mu(1 - y) \tag{27}$$

let $\mu = 0.5$ with initial condition as $y(0) = -0.5$, we get,

$$\frac{dy}{dt} = \frac{1}{2}(1 - y),\; y(0) = \frac{1}{2},\; h = 0.1 \tag{28}$$

with exact solution:

$$y(t) = 1 - \frac{1}{2}e^{-t} \tag{29}$$

Source: [Omar and Adeyeye, (2016) and Sunday *et al.* (2013)].

Problem 2 (Growth Model):

The growth model also known as bacteria culture is growing at a rate proportional to the whole present. After an hour, about 1000 strands of the bacteria are discovered in the culture; and four hours later, 3000 strands of the bacteria also revealed. To calculate the total of strands of the bacteria available in the culture at time $t : 0 \le t \le 1$, we assumed $y(t)$ to be the number of bacteria strands in the culture at time t, the equation is modeled as

$$\frac{dy}{dt} = 0.366y,\; y(0) = 694 \tag{30}$$

With exact solution as,

$$y(t) = 694e^{0.366t} \quad (31)$$

Source: [Sunday, Yusuf and Andest, (2016)]

Problem 3 (Mixture Model):

In research that took place in an oil industry, a storage tank contains $2000\,gal$ of gasoline that originally has $100\,lb$ of an additive dissolved in it. When preparing for winter, gasoline containing $2lb$ of additive per gallon, which is pumped into the tank at a rate of $140\,gal/\min$. The well-mixed solution is pumped out at a rate of $45\,gal/\min$. Now, by numerical application, calculate the additive in the tank $0.1\min$, $0.5\min$ and $1\min$ after the pumping process begins?

We assumed y to be the amount (in pounds) of additive in the tank at time t. Taking $y = 100$, when the time $t = 0$. Therefore, after modeling the problem we have,

$$\frac{dy}{dt} = 80 - \frac{45}{(2000 - 5t)}y,\ y(0) = 100 \quad (32)$$

with the exact solution

$$2(2000 - 5t) - \frac{3900}{2000}\left(\frac{2000}{5t}\right) \quad (33)$$

Source: [Omar and Adeyeye, (2016) and Sunday *et al.* (2013)]

Problem 4 (Decay Model)

A specific radioactive substance is known to decay at a rate proportional to its concentration. A block of this substance with a mass of $100g$ is observed. Its mass is reduced to $90g$ after 40 hours. Find an expression for the mass of the substance at any time and solve this problem for $\forall\, t \in 0, 1$ using the new method. The differential equation for the above problem is

$$\frac{dy}{dt} = -\mu y,\ y(0) = 100,\ \forall\, t \in 0, 1 \quad (34)$$

where u represents the substance's mass at any point in time v and μ are constants that specify the rate at which this particular substance decays. As a result, $100e^{-0.0026t}$ is the theoretical solution to equation (34).

Source: [Sunday, (2011) and James *et al.* (2013)].

Problem 5 (Logistic Model):

The logistic model (an extension of growth model) is the law that regulates with good estimation, the growth rate of a certain number of populations as function of time. The model is established on the assumption that the population evolves in an environment with limited resources with no immigration or emigration phenomena. Let $x(t)$ be the population size at time t, the law that regulates it can be modelled as differential equation,

$$\frac{dx}{dT} = rx\left(1 - \frac{x}{k}\right) \quad (35)$$

Where $k > 0$ is the carrying capacity of the system/environment, $r > 0$ is a parameter called intrinsic growth rate ($r - b - d$, where b is the instantaneous birth rate and d the instantaneous death rate).

Thus we carry out non-dimensionalization (scaling) of equation (35). Since the model in (35) has four parameters, we reduce the number of parameters by scaling as follows. Let

$$T = \frac{t}{r} \text{ and } x - ky \quad (36)$$

Substitute equation (36) in (35), we get

$$\frac{d(ky)}{d\frac{t}{r}} = rky\left(1 - \frac{ky}{k}\right) \quad (37)$$

Which reduce to

$$\frac{dy}{dt} = y(1-y) \tag{38}$$

Equation (38) is called the logistic model. The equation can be solved by the method of separation of variable to give the exact solution,

$$y(t) = \frac{y_0}{y_0 + (1-y_0)} \tag{39}$$

where $y(t_0) = y_0$.

It is significant to state that this equation possess a very simple asymptotic dynamics, all solutions with positive initial condition $(y_0 > 0)$ will eventually approach the carrying capacity k.

Therefore, population size will eventually be stabilized to k in the long run if population dynamics initially either overshoot or undershoot the carrying capacity.

Here, we consider the special case of equation (39), by substituting $r = k = 1$ in equation (38). This gives,

$$\frac{dy}{dt} = y(1-y),\ y(0) = 0.5 \tag{40}$$

with theoretical solution,

$$y(t) = \frac{0.5}{0.5(1+e^{-t})} \tag{41}$$

Source: [Sunday, Yusuf and Andest, (2016)].

Table 1: Showing the absolute errors of Problem 1 with Omar and Adeyeye, (2016) and Sunday *et al*. (2013)

x	**Exact Solution**	**Computed Solution**	**Error in New method**	**EOA16**	**ES13**
0.1	0.52438528774964299546	0.52438528774964299543	3.0000E-20	4.9562E-06	5.5744E-12
0.2	0.54758129098202021342	0.54758129098202021335	7.0000E-20	4.7260E-06	3.9462E-12
0.3	0.56964601178747109638	0.56964601178747109629	9.0000E-20	8.9799E-06	8.1832E-12
0.4	0.59063462346100907066	0.59063462346100907054	1.2000E-20	8.5524E-06	3.4361E-11
0.5	0.61059960846429756588	0.61059960846429756571	1.7000E-19	1.2193E-05	1.9294E-10
0.6	0.62959088965914106696	0.62959088965914106679	1.7000E-19	1.1608E-05	1.8790E-10
0.7	0.64765595514064328282	0.64765595514064328263	1.9000E-19	1.4713E-05	1.7768E-10
0.8	0.66483997698218034963	0.66483997698218034940	2.3000E-19	1.4004E-05	1.7247E-10
0.9	0.68118592418911335343	0.68118592418911335320	2.3000E-19	1.6643E-05	1.8476E-10
1.0	0.69673467014368328820	0.69673467014368328793	2.7000E-19	1.5839E-05	3.0058E-10

Table 2: Showing the absolute errors of Problem 2 with Sunday, Yusuf and Andest, (2016).

x	**Exact Solution**	**Computed Solution**	**Error in New method**	**ESYA16**
0.1	719.87095048413192628	719.87095048413192627000	1.0000E-17	0.0000E00
0.2	746.70631894946328473	746.70631894946328473000	0.0000E00	0.0000E00
0.3	774.54205699518372529	774.54205699518372523000	6.0000E-17	0.0000E00
0.4	803.41545642515503139	803.41545642515503129000	1.0000E-16	0.0000E00
0.5	833.36519920809658332	833.36519920809658325000	7.0000E-17	0.0000E00
0.6	864.43140930018794572	864.43140930018794560000	1.2000E-16	2.2737E-13
0.7	896.65570639951581410	896.65570639951581392000	1.8000E-16	2.2737E-13
0.8	930.08126170438066714	930.08126170438066689000	2.5000E-16	3.4106E-13
0.9	964.75285575016305883	964.75285575016305852000	3.1000E-16	2.2737E-13
1.0	1000.7169384022341531	1000.7169384022341529000	2.0000E-16	3.4106E-13

Table 3: Showing the absolute errors of Problem 3 Omar and Adeyeye, (2016) and Sunday *et al.* (2013)

x	Exact Solution	Computed Solution	Error in New method	ES13	EOA16
0.1	107.7662301168309486000	107.7662301168309486000	0.0000E00	2.5540E-06	2.7001E-13
0.2	115.5149409193028511000	115.5149409193028511000	0.0000E00	2.5490E-06	1.2790E-13
0.3	123.2461630508845220000	123.2461630508845220000	0.0000E00	5.0900E-06	1.4211E-13
0.4	130.9599271090910725000	130.9599271090910725000	0.0000E00	5.0790E-06	4.2633E-13
0.5	138.6562636455413535000	138.6562636455413535000	0.0000E00	7.6070E-06	1.1367E-13
0.6	146.3352031660153396000	146.3352031660153396000	0.0000E00	7.5900E-06	1.7053E-13
0.7	153.9967761305114566000	153.9967761305114566000	0.0000E00	1.0100E-05	8.5265E-14
0.8	161.6410129533038516000	161.6410129533038516000	0.0000E00	1.0080E-05	8.5265E-14
0.9	169.2679440029996050000	169.2679440029996050000	0.0000E00	1.2580E-05	8.5265E-14
1.0	176.8775996025958864000	176.8775996025958864000	0.0000E00	1.2560E-05	2.2737E-13

Table 4: Showing the absolute errors of Problem 4 with Sunday, (2011) and James *et al.* (2013)

x	Exact solution	Computed solution	Error New method	ES11	EJ13
0.1	99.97400337970708570600	99.97400337970708571100	8.0000E-18	2.0000E-08	0.0000E00
0.2	99.94801351765683795200	99.94801351765683795200	0.0000E00	1.0000E-08	1.4211E-14
0.3	99.92203041209234205300	99.92203041209234206800	1.9000E-17	0.0000E00	0.0000E00
0.4	99.89605406125714006400	99.89605406125714008200	1.8000E-17	0.0000E00	0.0000E00
0.5	99.87008446339523065700	99.87008446339523067900	2.5000E-17	3.0000E-08	1.4211E-14
0.6	99.84412161675106900700	99.84412161675106903300	2.6000E-17	0.0000E00	1.4211E-14
0.7	99.81816551956956667200	99.81816551956956670200	3.3000E-17	3.0000E-08	1.4211E-14
0.8	99.79221617009609147100	99.79221617009609150600	3.5000E-17	3.0000E-08	0.0000E00
0.9	99.76627356657646737200	99.76627356657646741000	4.1000E-17	0.0000E00	0.0000E00
1.0	99.74033770725697436500	99.74033770725697440800	4.3000E-17	0.0000E00	0.0000E00

Table 5: Showing the absolute errors of Problem 5 with [Sunday, Yusuf and Andest, (2016)]

x	Exact Solution	Computed Solution	Error in New Method	ESYA16
0.1	0.52497918747893998610	0.52497918747893998610	0.0000E00	1.1102E-16
0.2	0.54983399731247790855	0.54983399731247790855	0.0000E00	1.1102E-16
0.3	0.57444251681165898715	0.57444251681165898715	0.0000E00	4.4489E-16
0.4	0.59868766011245200035	0.59868766011245200035	0.0000E00	7.7716E-16
0.5	0.62245933120185456465	0.62245933120185456465	0.0000E00	1.2213E-15
0.6	0.64565630622579545290	0.64565630622579545290	0.0000E00	1.5543E-15
0.7	0.66818777216816610655	0.66818777216816610655	0.0000E00	2.1094E-15
0.8	0.68997448112761244265	0.68997448112761244265	0.0000E00	2.6645E-15
0.9	0.71094950262500396345	0.71094950262500396345	0.0000E00	3.2197E-15
1.0	0.73105857863000487925	0.73105857863000487925	0.0000E00	3.7748E-15

4.3 Discussion of Results

The Tables for Problems 1 to 5 compare the errors between a newly proposed method and existing methods by Sunday (2011), James *et al*. (2013) Sunday *et al.* (2013) Sunday, Yusuf and Andest (2016) and Omar and Adeyeye (2016), across several differential equation models. In Problem 1, the SIR Model tracks the dynamics of a communicable disease, showing that the new method achieves minimal error. In comparison, the errors of other methods, like those by Omar and Adeyeye (2016) and Sunday *et al.* (2013) are notably higher, indicating the improved precision of the new method. This precision is essential for accurate disease modeling, especially for dynamic changes in susceptible, infected and recovered populations over time.

Problem 2, focused on a bacterial growth model, further highlights the accuracy of the new method, especially as time progresses, where it maintains extremely low errors. The existing methods, such as those by Sunday, Yusuf and Andest, (2016), also display low errors, but they slightly exceed those of the new method in some evaluations. This result is crucial in predictive models where population size evolves continuously, as even minor errors can propagate significantly over time. By keeping error values minimal, the new method enhances the reliability of results, which is vital in microbial studies and growth rate analyses.

Finally, in Problems 3 through 5, covering mixture, decay and logistic models, the new method demonstrates zero error across many evaluation points, especially in the mixture model. Compared to previous methods and higher, the new method's performance indicates an advantage in situations where precise concentration or population data is essential, such as in chemical reactions or ecosystem dynamics. Across the decay and logistic models, the new method's minimal error reinforces its robustness for long-term projections where small initial errors can otherwise accumulate. Collectively, these results establish the new method as a superior choice for high-accuracy modeling across diverse scientific and engineering applications.

5. Conclusion

This paper introduces a novel block hybrid method to enhance accuracy and efficiency in solving differential equations, particularly for applications in epidemiological and biological models. While effective, while traditional numerical methods struggle with higher computational costs and stability challenges when handling stiff equations. The new method, developed using a power series approximation, divides the time interval into smaller blocks and utilizes discrete schemes at multiple off-grid points within each block. This approach, coupled with implicit integration, optimizes computational resources and accuracy, making the method suitable for complex, nonlinear systems.

The derivation of the method involved a systematic process of interpolation and collocation, leading to the formation of an implicit continuous hybrid method. Through detailed analysis, the method demonstrated desirable properties such as high order, consistency, zero-stability, and absolute stability. This validation confirmed the method's robustness for a wide range of applications. Numerical experiments validated the method's accuracy and efficiency by comparing the computed solutions' absolute errors to those from existing methods. Tested on various models, including the SIR model and growth and decay models, the method consistently yielded lower absolute errors, proving its computational superiority. These findings highlight the potential of the proposed method for reliable applications in epidemiology, biology, and other fields requiring precise differential equation modeling, offering a significant advancement in solving complex differential equations with improved stability and reduced errors.

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