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Effective Fuzzy Parameterized Soft Set with Application in Decision Making

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ABSTRACT

In this paper, we introduce the concept of effective fuzzy parameterized soft set in the framework of effective set; where the decision on the selection of any element in the power set of the universal set is influenced by the effects the elements in the effective set has over each parameter. In this context, we define some basic set operations and consider some algebraic properties by proving the De-morgan's laws and the distributive laws. Lastly, we develop a new algorithm and apply the concept to efficiently solve a hypothetical case involving evaluation process.

1. Introduction

Zadeh in 1965 did a remarkable job by developing a very interesting concept (fuzzy set) which seems more accurate in dealing with uncertainties. As opposed to the bivalent nature of classical set, fuzzy set assign degree of belonging to each element of a set within the unit interval. As time passes by, Molodtsov (1999) introduced soft set in order to take care of the inadequacy of parameterization tool associated with fuzzy set and other non-classical set theories. Molodtsov also outlined several areas of application of soft set theory such as: Riemann Integration, Game theory, Operations Research, Probability theory, etc. As a way forward in trying to handle uncertainties, fuzzy set was introduced into the study of soft set theory by Maji et al. (2001), who were the first to define fuzzy soft set. In 2002, Maji and Roy defined several fundamental operations, such as subset, superset, null set, absolute set, union, intersection, AND, and OR operations of soft set theory, to correlate with Molodtsov's soft set. The concept of fuzzy soft set served as inspiration for these definitions. Additionally, a few findings in the soft set context were confirmed and the theory was applied to resolve a decision-making problem. In their study, Roy and Maji (2007) explored several applications of the theory, thereby introducing a novel approach to object recognition from imprecise multi-observer data, which entails building a comparison table from fuzzy soft sets in a parametric sense for decision-making. However, this technique has significant drawbacks including the possibility of information loss during the process of creating the final fuzzy soft set from the multi observer data. In order to compare choices, Kong et al. (2008) updated Roy's approach (Roy and Maji, 2007: A Fuzzy Soft Set Theoretic Approach to Decision Making Problems) and provided a heuristic algorithm of normal parameter reduction. Also, the problems of sub-optimal choice and added parameter set of soft set are analysed. The same year, in addition; Yao et al. (2008) introduced the idea of a soft fuzzy set, went over some related qualities, and provided examples of how fuzzy soft sets and soft

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fuzzy sets connect to one another.

In sequel, Ali et al. in 2009 defined some new operations like restricted union, extended intersection, restricted intersection and restricted difference of two soft sets and also proposed several assertions by Maji and Roy (2002) on soft set theory are not true by given counter examples. Furthermore, in 2010, Feng et al. proposed that the concept of choice value (Maji and Roy, 2002: An application of soft set in a decision-making problem) designed for crisp soft sets is unfit to solve decision making problems involving fuzzy soft sets. And at the same time, proposed an adjustable approach which can be more suitable to solve real life problems in an imprecise environment. Also, Kharal and Ahmad (2010) investigated the notion of a mapping on soft classes and study several properties of images and inverse images of soft sets supported by examples and counter examples; the notion was applied to the problems of medical diagnosis in medical expert system. Additionally, Cagman et al. (2010a) explored the use of soft set theory to commutative ideals in BCK-algebras. Also, in the same year, same authors introduce fuzzy parameterized fuzzy soft set and its applications; where the fuzzification touches both the universal set and the set of parameters (Cagman et al., 2010b).

In line with the fuzzy soft set definition, Cagman et al. (2011a) provided the fuzzy soft set theory definition along with some related properties. They also developed the fuzzy soft aggregation operator, which enables the creation of more effective decision-making techniques. The same authors (Cagman et al. 2011b) also tried to fuzzify a set of parameters in order to solve more complex problems, and they developed the idea of fuzzy parameterized soft set theory and they also provide properties associated with this theory and suggest a method of decision-making. Furthermore, Adam and Hassan (2014a) introduced the notion of Q fuzzy soft sets and studied its operations (Adam and Hassan, 2014b). Also, they introduced multi-Q fuzzy parameterized soft set and its application (Adam and Hassan, 2014c). In 2016, Rodriguez et al. proposed the comparison score-based approach to solving fuzzy soft set-based decision-making problems. Liu et al. (2017) pointed out that the choice value algorithm which was proposed by Maji and Roy (2002) is not strongly fit for decision making because all attributes are good descriptions, and the greater the value, the better. However, in practice, attributes may be good, bad or constrained. That is why the algorithm needs to be further improved according to actual problem. So, the authors dedicated to the analysis of the choice based algorithm and the comparison score algorithm and proposed a new decision making algorithm which focus on the application of fuzzy soft set and ideal solutions in decision making problem by laying emphasis on the analysis of decision objectives, attributes analysis and explicit decision function; for which they found that the core of the decision process is the design phase, which is to formulate a model for an identified decision. Again, in recent years, in addition to applying soft set and fuzzy soft sets to decision making, many authors like Nasef et al. (2018a) presented another application of soft sets in a decision-making problem for real estate marketing with the help of rough mathematics, and provided an algorithm to select the optimal choice of an object. Also, in the same year (Nasef et al., 2018b) applied the notion of fuzzy soft sets in Sanchez's method (1976, 1979) of decision making. Furthermore, Alcantud and Torrecillas (2018) developed a way to solve intertemporal choice problems like; savings, investments, spendings, etc., for fuzzy soft sets. Also in 2020, Dey et al. combined hesitant fuzzy set and multi fuzzy soft set to develop hesitant multi fuzzy soft set and also introduced a new type of level soft set and called it RMSS-level soft set, and present an algorithm and a novel approach to hesitant multi fuzzy soft set-based decision-making problems by using the RMSS-level soft set. In 2020, Renukadevi and Sangeetha introduced a new notion pertaining to certainty and coverage of a parameter, which is associated with the soft set and they proposed a novel approach that leverages parameter certainty to address decision-making challenges. In trying to address the parameterized difficulties of instability, Somen in 2021 proposed a fuzzy hyper soft set as a hybrid of a fuzzy set and a hyper soft set. Also, Phaengtan et al. (2021) defined partial averages of fuzzy soft sets and provided a new approach for handling a few partial average-based decision-making problems.

Interestingly, Alkhazaleh in 2022, devised the idea of effective set, thereby introducing the notion of effective fuzzy soft set. The author outlined the fundamentals and gave several examples of how the novel idea may be used to solve practical decision-making and medical diagnostic challenges. In this direction of research, we introduce the concept of effective fuzzy parameterized soft set in the framework of effective set; where the decision on the selection of any element in the power set ($P(U)$) of the universal set U is influenced by the effect the elements in the effective set has over each parameter. Thereby introducing a new concept upon which some results are verified. In the last part of the paper, the new concept is applied to solve decision making problems by developing new algorithm.

2. Preliminaries

In this section, we provide some basic definitions following from: Zadeh, 1965; Molodtsov, 1999; Maji et al., 2001; Cagman et al., 2011a; Cagman et al., 2011b and Alkhazaleh, 2022.

Definition 2.1. (**Fuzzy Set**): A pair (μ, U) is called a fuzzy set, where U is a non-empty universe set and μ is a function from U into the unit interval $[0,1]$, (i.e, $\mu: U \to [0,1]$) and for all $x \in U$, $\mu(x)$ is referred to as the degree of membership of $x \in U$.

Definition 2.2. (**Soft Set**): Let U be a universe set, and E be the set of parameters. A pair (F, E) is called a soft set over U if and only if F is a mapping from E into the set of all subsets of the universe set U, i.e., $F: E \to P(U)$, where $P(U)$ is the power set of U.
In other words, soft set over U is a parameterized family of subsets of U.
Every set $F(e)$, for every $e \in E$, from this family may be considered as the set of e-elements of the soft set (F, E) or considered as the set of e-approximate elements of the soft set. Accordingly, we can view a soft set (F, E) as a collection of approximations: $(F, E) = \{F(e): e \in E\}$.

Definition 2.3. (**Fuzzy Soft Set**): Let I^U be the set of all fuzzy sets of U. Then a pair (f, A) is called a fuzzy soft set over U, where A is a subset of the set of parameters E, and f is a mapping from A into I^U. That is, $f: A \to I^U$, and for each $a \in A$, $f(a) = f_a: U \to I^U$, is a fuzzy set on U.

Definition 2.4. (**Fuzzy Parameterized Soft Set**): Let U be an initial universe, E be the set of parameters and A be the fuzzy set over E. A fuzzy parameterized soft set (FP-soft set) $\digamma_A$ on the universe U is defined by the set of ordered pairs:

$$\digamma_A = \{(\mu_A(x)/x, \gamma_A(x)): x \in E, \gamma_A(x) \in P(U), \mu_A(x) \in [0,1]\},$$

Where $P(U)$ is the power set of U and the function $\gamma_A: E \to P(U)$ is called approximate function such that $\gamma_A(x) = \varnothing$ if $\mu_A(x) = 0$, characterized by the membership function $\mu_A: E \to [0,1]$
The value $\mu_A(x)$ of an element x of the parameters represents its degree of importance. And it is solely based on the desirability of the decision maker.
Hence, this means that the approximate function is defined from fuzzy subset of E to the crisp subset of the Universe set U.

Definition 2.5. (**Effective Set**): An effective set is a fuzzy set Λ in a universe of discourse A where $\Lambda: A \to [0,1]$. A is the set of effective parameters that may change the membership values by making positive effect (or no effect) on values of memberships after applying it and defined as: $\Lambda = \{\langle a, \delta_\Lambda(a)\rangle: a \in A\}$

Definition 2.6. (**Effective Fuzzy Soft Set**): Let U be an initial universal set, E be the set of parameters, A be a set of effective parameters, and Λ be the effective set over A. Let I^U denote all fuzzy subsets of U; a pair $(F, E)_\Lambda$ is called an effective fuzzy soft set (EFSS in short) over U where F is a mapping given by $F: E \to I^U$, defined as follows:

$$F(e_i)_\Lambda = \left\{ \frac{x_j}{\mu(x_j)_\Lambda} : x_j \in U, e_i \in E \right\}$$

where $\forall a_k \in A$,

$$\mu_U(x_j)_\Lambda = \begin{cases} \mu_U(x_j) + \left(\frac{\left(1 - \mu_U(x_j)\right) \sum_k \delta_{\Lambda_{x_j}}(a_k)}{|A|} \right), & if \mu_U(x_j) \in (0,1), \\ \mu_U(x_j), & O.W. \end{cases}$$

Where $|A|$ stands for cardinality of A, and $O.W$ stands for "otherwise".

3. Results

In this section, we introduce the new concept "Effective Fuzzy Parameterized Soft Set".
Note that throughout the rest of the work, the set of all Effective fuzzy parameterized soft set over U will be denoted as $EFPS(U)$

3.1 Effective Fuzzy Parameterized Soft Set

Before defining the new concept, we need to consider some basic definitions.

Definition 3.1.1. An effective set is a fuzzy set ω in a universe of discourse U, where ω is a function given by $\omega: A \to [0,1]$, and A is the set of effective parameters that may change the membership values by making positive (or no) effect on them; defined as follows: $\omega = \{\langle a, \delta_\omega(a) \rangle : a \in A\}$

Definition 3.1.2. A fuzzy parameterized set is a fuzzy set γ in a universe of discourse U, where γ is a function given by $\gamma: E \to [0,1]$, and E is the set of parameters from Molodstov's soft set; defined as follows: $\gamma = \{\langle e, \Gamma_\gamma(e) \rangle : e \in E\}$
Now, we define the new concept.

Definition 3.1.3. **(Effective Fuzzy Parameterized Soft Set):** Let U be an initial universe, E be the set of parameters and γ be the fuzzy parameters set over E, let A be the set of effective parameters and ω be the fuzzy effective set over A. Then, a pair (F_{E^*}, U) is called an **effective fuzzy parameterized soft set (EFPS-set)**, where F is a mapping given by $F: E^* \to U$, defined as follows:

$$F_{E^*} = \left\{ \left(\frac{e_i^*}{\mu_{E^*}(e_i^*)}, f_{E^*}(e_i^*) \right) : e^* \in E^*, f_{E^*}(e_i^*) \in P(U), \mu_{E^*}(e_i^*) \in [0,1] \right\} \quad \dots (1)$$

where E^* is the set of reformed parameters obtained after the influence of the effective set on the set of parameters E, f_{E^*} is called the approximate function and $\mu_{E^*}: E^* \to [0,1]$ is called the membership function of the EFPS-set F_{E^*} such that for all $a_i \in A$ and all $e_i \in E$

$$\mu_{E^*}(e_i^*) = \Gamma_{\gamma_E}(e_i) + \left(\frac{\left(1 - \Gamma_{\gamma_E}(e_i)\right) . \sum_i \delta_{\omega_A}(a_i)}{|A|} \right) \quad \dots (2)$$

Example 3.1.4. Let $U = \{u_1, u_2, u_3\}$ be an initial universe, let $E = \{e_1, e_2, e_3\}$ be the set of parameters, let $A = \{a_1, a_2, a_3\}$ be the set of effective parameters and $E^* = \{e_1^*, e_2^*, e_3^*\}$ be the set of reformed parameters.

Suppose the fuzzy parameters set over E is given by:

$$\gamma(u_1) = \left\{\frac{e_1}{0.5}, \frac{e_2}{0.6}, \frac{e_3}{0.7}\right\}, \gamma(u_2) = \left\{\frac{e_1}{0}, \frac{e_2}{0.3}, \frac{e_3}{0.7}\right\}, \gamma(u_3) = \left\{\frac{e_1}{1}, \frac{e_2}{0.8}, \frac{e_3}{0.5}\right\}.$$

Suppose the fuzzy effective set A is given by:

$$\omega(u_1) = \left\{\frac{a_1}{0.6}, \frac{a_2}{0.5}, \frac{a_3}{0.9}\right\}, \omega(u_2) = \left\{\frac{a_1}{0.3}, \frac{a_2}{1}, \frac{a_3}{0.8}\right\}, \omega(u_3) = \left\{\frac{a_1}{1}, \frac{a_2}{0.5}, \frac{a_3}{0.8}\right\}.$$

So that after applying equation (2) using the data above, we have;

<u>For u_1:</u>

$$F_{E^*} = \left\{\frac{e_1^*}{0.5 + \left(\frac{(1-0.5)(0.6+0.5+0.9)}{3}\right)}, \frac{e_2^*}{0.6 + \left(\frac{(1-0.6)(0.6+0.5+0.9)}{3}\right)}, \frac{e_3^*}{0.7 + \left(\frac{(1-0.7)(0.6+0.5+0.9)}{3}\right)}, \{u_1\}\right\}$$

$$\Rightarrow F_{E^*} = \left\{\frac{e_1^*}{0.84}, \frac{e_2^*}{0.87}, \frac{e_3^*}{0.9}, \{u_1\}\right\}$$

Here, the above shows how we obtained the set of reformed parameters. It can be seen that the first element e_1^* is obtained after the influence of the fuzzy values from the effective parameters set on the parameter e_1. The rest follows.

So, by using the same method for u_2 and u_3, we arrived at the following effective fuzzy parameterized soft set (EFPS-set):

$$(F_{E^*}, U) = \left\{\left(\frac{e_1^*}{0.84}, \frac{e_2^*}{0.87}, \frac{e_3^*}{0.9}, \{u_1\}\right), \left(\frac{e_1^*}{0.7}, \frac{e_2^*}{0.79}, \frac{e_3^*}{0.91}, \{u_2\}\right), \left(\frac{e_1^*}{1}, \frac{e_2^*}{0.94}, \frac{e_3^*}{0.89}, \{u_3\}\right)\right\}$$

In a tabular form, the above EFPS-set can be shown below:

Table 1: ***EFPS-set Table***

U/E^*	u_1	u_2	u_3
e_1^*	0.84	0.7	1
e_2^*	0.87	0.79	0.94
e_3^*	0.9	0.91	0.89

Interpretation: The table above shows the fuzzy values of the reformed parameters with respect to the elements of the universal set. For the element u_1, the fuzzy values of e_1^*, e_2^* and e_3^* are 0.84, 0.87 and 0.9 respectively.

Next, we consider some of the basic operations of the concept of EFPS-set.

3.2 Basic Operations of EFPS-set

Here, we basically give the definitions of complement, union and intersection under the EFPS-set.

Definition 3.2.1. Suppose $F_{E^*} = \left\{\left(\frac{e_i^*}{\mu(e_i^*)}, f_{E^*}(e_i^*)\right) : e_i^* \in E_I^*, f_{E^*}(e_i^*) \in P(U)\right\}$, then the complement of F_{E^*} denoted as $F_{(E^*)^c}$ is defined by:

$$F_{(E^*)^c} = \left\{\left(\frac{e_i^*}{\mu(e_i^*)^c}, f_{E^*}(e_i^*)\right) : f_{E^*}(e_i^*) \in P(U)\right\}$$, where $\mu(e_i^*)^c = 1 - \mu(e^*)$.

Example 3.2.2. Let $F_{E^*} = \left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}, \{u_i\}\right\}$. Then, the complement of F_{E^*} is given by:

$$F_{(E^*)^c} = \left\{\frac{e_1^*}{0.1}, \frac{e_2^*}{0.15}, \frac{e_3^*}{0.2}, \{u_i\}\right\}$$

Definition 3.2.3. Suppose $F_{E_1^*} = \left\{\left(\frac{e_i^*}{\mu(e_i^*)}, f_{E_1^*}(e_i^*)\right) : e_i^* \in E_1^*, f_{E_1^*}(e_i^*) \in P(U)\right\}$ and $G_{E_2^*} = \left\{\left(\frac{e_j^*}{\mu(e_j^*)}, g_{E_2^*}(e_j^*)\right) : e_j^* \in E_2^*, g_{E_2^*}(e_j^*) \in P(U)\right\}$. Then, the union of $F_{E_1^*}$ and $G_{E_2^*}$ is a set H_{E^*} defined by:

$$H_{E^*} = \left\{\left(\frac{e_h^*}{\mu(e_h^*)}, h_{E^*}(e_h^*)\right) : e_h^* \in E^*, h_{E^*}(e_h^*) \in P(U)\right\}$$

where $\mu(e_h^*) = max\{\mu(e_i^*), \mu(e_j^*)\}$ and $h_{E^*}(e_h^*) = f_{E_1^*}(e_i^*) \cup g_{E_2^*}(e_j^*)$

Example 3.2.4. Let $F_{E_1^*} = \left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}, \{u_i\}\right\}$ and $G_{E_2^*} = \left\{\frac{e_1^*}{0.92}, \frac{e_2^*}{0.88}, \frac{e_3^*}{0.92}, \{u_j\}\right\}$. Then, the union of $F_{E_1^*}$ and $G_{E_2^*}$ is given by: $H_{(E^*)} = \left\{\frac{e_1^*}{0.92}, \frac{e_2^*}{0.88}, \frac{e_3^*}{0.92} \{u_h\}\right\}$.

Definition 3.2.5. Suppose $F_{E_1^*} = \left\{\left(\frac{e_i^*}{\mu(e_i^*)}, f_{E_1^*}(e_i^*)\right) : e_i^* \in E_1^*, f_{E_1^*}(e_i^*) \in P(U)\right\}$ and $G_{E_2^*} = \left\{\left(\frac{e_j^*}{\mu(e_j^*)}, g_{E_2^*}(e_j^*)\right) : e_j^* \in E_2^*, g_{E_2^*}(e_j^*) \in P(U)\right\}$. Then, the intersection of $F_{E_1^*}$ and $G_{E_2^*}$ is a set K_{E^*} defined by:

$$K_{E^*} = \left\{\left(\frac{e_k^*}{\mu(e_k^*)}, k_{E^*}(e_k^*)\right) : e_k^* \in E^*, k_{E^*}(e_k^*) \in P(U)\right\}$$

where $\mu(e_k^*) = min\{\mu(e_i^*), \mu(e_j^*)\}$ and $k_{E^*}(e_k^*) = f_{E_1^*}(e_i^*) \cup g_{E_2^*}(e_j^*)$.

Example 3.2.6. Let $F_{E_1^*} = \left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}, \{u_i\}\right\}$ and $G_{E_2^*} = \left\{\frac{e_1^*}{0.92}, \frac{e_2^*}{0.88}, \frac{e_3^*}{0.92}, \{u_j\}\right\}$. Then, the intersection of $F_{E_1^*}$ and $G_{E_2^*}$ is given by: $K_{(E^*)} = \left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}, \{u_k\}\right\}$.

3.3 Extended Definitions of the Basic Operations

Below, we re-defined the basic operations of EFPS-set based on the idea gotten from the work of Alkhazaleh (2022).

Definition 3.3.1. The $\omega_{complement}$ of the EFPS-set (F_{E^*}, U) is the set ω^c, where ω^c is the fuzzy compliment defined by: $\omega^c = \{\langle a, 1 - \delta_\omega(a)\rangle : a \in A\}$

Example 3.3.2. Suppose $\omega(u_1) = \left\{\frac{a_1}{0.6}, \frac{a_2}{0.5}, \frac{a_3}{0.9}\right\}$, then the $\omega_{complement}$ is given as;

$$\omega^c(u_1) = \left\{\frac{a_1}{0.4}, \frac{a_2}{0.5}, \frac{a_3}{0.1}\right\}$$

Definition 3.3.3. The $\gamma_{complement}$ of the EFPS-set (F_{E^*}, U) is the set γ^c, where γ^c is the fuzzy compliment defined by: $\gamma^c = \{\langle e, 1 - \Gamma_\gamma(e)\rangle : e \in E\}$

Example 3.3.4. Suppose $\gamma(u_1) = \left\{\frac{e_1}{0.5}, \frac{e_2}{0.6}, \frac{e_3}{0.7}\right\}$, then the $\gamma_{complement}$ is given as;

$$\gamma^c(u_1) = \left\{\frac{e_1}{0.5}, \frac{e_2}{0.4}, \frac{e_3}{0.3}\right\}$$

Definition 3.3.5. The $Total_{complement}$ of the EFPS-set (F_{E^*}, U) is the EFPS-set $\left(F_{(E^*)^c}, U\right)$ where $(E^*)^c$ is the set of complements of the reformed parameters defined by:

$$F_{(E^*)^c} = \left\{\left(\frac{e_i^*}{\mu_{E^*}(e_i^*)^c}\right), f_{E^*}(e_i^*)\right\}, \text{ such that;}$$

$$\mu_{E^*}(e_i^*)^c = \Gamma_{\gamma_E^c}(e_i) + \left(\frac{\left(1 - \Gamma_{\gamma_E^c}(e_i)\right).\sum_i \delta_{\omega_A^c}(a_i)}{|A|}\right) \qquad \ldots (3)$$

Example 3.3.6. Using the data from 3.1.4, the $\omega_{complement}$ (ω^c) is given by:

$$\omega^c(u_1) = \left\{\frac{a_1}{0.4}, \frac{a_2}{0.5}, \frac{a_3}{0.1}\right\}, \omega^c(u_2) = \left\{\frac{a_1}{0.7}, \frac{a_2}{0}, \frac{a_3}{0.2}\right\}, \omega^c(u_3) = \left\{\frac{a_1}{0}, \frac{a_2}{0.5}, \frac{a_3}{0.2}\right\}.$$

Also, from 3.1.4, the $\gamma_{complement}$ (γ^c) is given by:

$$\gamma^c(u_1) = \left\{\frac{e_1}{0.5}, \frac{e_2}{0.4}, \frac{e_3}{0.3}\right\}, \gamma^c(u_2) = \left\{\frac{e_1}{0.1}, \frac{e_2}{0.7}, \frac{e_3}{0.3}\right\}, \gamma^c(u_3) = \left\{\frac{e_1}{0}, \frac{e_2}{0.2}, \frac{e_3}{0.5}\right\}.$$

Hence, by applying equation (3) using the above data, we have;

For u_1:

$$F_{(E^*)^c} = \left\{\frac{e_1^*}{0.5 + \left(\frac{(1-0.5)(0.4+0.5+0.1)}{3}\right)}, \frac{e_2^*}{0.4 + \left(\frac{(1-0.4)(0.4+0.5+0.1)}{3}\right)}, \frac{e_3^*}{0.3 + \left(\frac{(1-0.3)(0.4+0.5+0.1)}{3}\right)}, \{u_1\}\right\}$$

$$\Rightarrow F_{(E^*)^c} = \left\{\frac{e_1^*}{0.67}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.53}, \{u_1\}\right\}$$

By using the same method for both u_2 and u_3, we have;

$$\left(F_{(E^*)^c}, U\right) = \left\{\left(\frac{e_1^*}{0.67}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.53}, \{u_1\}\right), \left(\frac{e_1^*}{1}, \frac{e_2^*}{0.79}, \frac{e_3^*}{0.51}, \{u_2\}\right), \left(\frac{e_1^*}{0.23}, \frac{e_2^*}{0.38}, \frac{e_3^*}{0.62}, \{u_3\}\right)\right\}$$

In a tabular form, the above EFPS-set can be shown below:

Table 2: *EFPS-complement Table*

$U/(E^*)^c$	u_1	u_2	u_3
e_1^*	0.67	1	0.23
e_2^*	0.6	0.79	0.38
e_3^*	0.53	0.51	0.62

Definition 3.3.7. The union of two EFPS-set $\left(F_{E_1^*}, U\right)$ and $\left(G_{E_2^*}, U\right)$ is the EFPS-set (H_{E^*}, U) where $E^* = E_1^* \cup E_2^*$ and H_{E^*} is defined by;

$H_{E^*} = \left\{\left(e_h^*/\mu(e_h^*), h_{E^*}(e_h^*)\right): e_h^* \in E^*, h_{E^*}(e_h^*) \in P(U)\right\}$, such that $h_{E^*}(e_h^*) = f_{E_1^*}(e_i^*) \cup g_{E_2^*}\left(e_j^*\right)$.

The idea of this union is to create new fuzzy parameter set ω resulting from the union of ω' and ω'', and a new fuzzy effective set γ resulting from the union of γ' and γ''.

i.e, $\omega(u_1) = max\{\omega'(u_1), \omega''(u_1)\}$ and $\gamma(u_1) = max\{\gamma'(u_1), \gamma''(u_1)\}$

Hence, H_{E^*} is obtained by applying equation (2) using the data provided from ω and γ.

Example 3.3.8. Suppose for $\left(F_{E_1^*}, U\right)$, we have;

$\omega'(u_1) = \left\{\frac{a_1}{0.3}, \frac{a_2}{0.6}, \frac{a_3}{0.5}\right\}$, $\omega'(u_2) = \left\{\frac{a_1}{0.2}, \frac{a_2}{0.7}, \frac{a_3}{0}\right\}$, $\omega'(u_3) = \left\{\frac{a_1}{0.8}, \frac{a_2}{0.1}, \frac{a_3}{0.5}\right\}$, and

$\gamma'(u_1) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.7}, \frac{e_3}{0.6}\right\}$, $\gamma'(u_2) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.5}, \frac{e_3}{0.4}\right\}$, $\gamma'(u_3) = \left\{\frac{e_1}{0.3}, \frac{e_2}{0}, \frac{e_3}{0.2}\right\}$,

Also, suppose for $\left(G_{E_2^*}, U\right)$, we have;

$\omega''(u_1) = \left\{\frac{a_1}{0.6}, \frac{a_2}{0.5}, \frac{a_3}{0.7}\right\}$, $\omega''(u_2) = \left\{\frac{a_1}{0.8}, \frac{a_2}{0.3}, \frac{a_3}{0.5}\right\}$, $\omega''(u_3) = \left\{\frac{a_1}{1}, \frac{a_2}{0.6}, \frac{a_3}{0.2}\right\}$, and

$\gamma''(u_1) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.7}, \frac{e_3}{0.8}\right\}$, $\gamma''(u_2) = \left\{\frac{e_1}{0.2}, \frac{e_2}{0.6}, \frac{e_3}{0.3}\right\}$, $\gamma''(u_3) = \left\{\frac{e_1}{0.5}, \frac{e_2}{0.4}, \frac{e_3}{0.1}\right\}$,

Then, for (H_{E^*}, U), we have;

$\omega(u_1) = \left\{\frac{a_1}{0.6}, \frac{a_2}{0.6}, \frac{a_3}{0.7}\right\}$, $\omega(u_2) = \left\{\frac{a_1}{0.8}, \frac{a_2}{0.7}, \frac{a_3}{0.5}\right\}$, $\omega(u_3) = \left\{\frac{a_1}{1}, \frac{a_2}{0.6}, \frac{a_3}{0.5}\right\}$, and

$\gamma(u_1) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.7}, \frac{e_3}{0.8}\right\}$, $\gamma(u_2) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.6}, \frac{e_3}{0.4}\right\}$, $\gamma(u_3) = \left\{\frac{e_1}{0.5}, \frac{e_2}{0.4}, \frac{e_3}{0.2}\right\}$,

Now, to get H_{E^*}, we apply equation (2) using the above data.

<u>For u_1:</u>

$$H_{E^*} = \left\{\frac{e_1^*}{0.8 + \left(\frac{(1-0.8)(0.6+0.6+0.7)}{3}\right)}, \frac{e_2^*}{0.7 + \left(\frac{(1-0.7)(0.6+0.6+0.7)}{3}\right)}, \frac{e_3^*}{0.8 + \left(\frac{(1-0.8)(0.6+0.6+0.7)}{3}\right)}, \{u_1\}\right\}$$

$$\Rightarrow H_{E^*} = \left\{\frac{e_1^*}{0.93}, \frac{e_2^*}{0.89}, \frac{e_3^*}{0.93}, \{u_1\}\right\}$$

By using the same method for u_2 and u_3, we arrived at the following effective fuzzy parameterized soft set (EFPS-set):

$$(H_{E^*}, U) = \left\{\left(\frac{e_1^*}{0.93}, \frac{e_2^*}{0.89}, \frac{e_3^*}{0.93}, \{u_1\}\right), \left(\frac{e_1^*}{0.93}, \frac{e_2^*}{0.87}, \frac{e_3^*}{0.8}, \{u_2\}\right), \left(\frac{e_1^*}{0.85}, \frac{e_2^*}{0.82}, \frac{e_3^*}{0.76}, \{u_3\}\right)\right\}$$

Definition 3.3.9. The intersection of two EFPS-set $\left(F_{E_1^*}, U\right)$ and $\left(G_{E_2^*}, U\right)$ is the EFPS-set (K_{E^*}, U) where $E^* = E_1^* \cap E_2^*$ and K_{E^*} is defined by;

$K_{E^*} = \{(e_k^*/\mu(e_k^*), k_{E^*}(e_k^*)) : e_k^* \in E^*, k_{E^*}(e_k^*) \in P(U)\}$, such that $k_{E^*}(e_k^*) = f_{E_1^*}(e_i^*) \cap g_{E_2^*}(e_j^*)$.

The idea of this intersection is to create new fuzzy parameter set ω resulting from the intersection of ω' and ω'', and a new fuzzy effective set γ resulting from the from the intersection of γ' and γ''.

i.e, $\omega(u_1) = min\{\omega'(u_1), \omega''(u_1)\}$ and $\gamma(u_1) = min\{\gamma'(u_1), \gamma''(u_1)\}$

Hence, K_{E^*} is obtained by applying equation (2) using the data provided from ω and γ.

Example 3.3.10. Suppose for $\left(F_{E_1^*}, U\right)$, we have;

$\omega'(u_1) = \left\{\frac{a_1}{0.3}, \frac{a_2}{0.6}, \frac{a_3}{0.5}\right\}$, $\omega'(u_2) = \left\{\frac{a_1}{0.2}, \frac{a_2}{0.7}, \frac{a_3}{0}\right\}$, $\omega'(u_3) = \left\{\frac{a_1}{0.8}, \frac{a_2}{0.1}, \frac{a_3}{0.5}\right\}$, and

$\gamma'(u_1) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.7}, \frac{e_3}{0.6}\right\}$, $\gamma'(u_2) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.5}, \frac{e_3}{0.4}\right\}$, $\gamma'(u_3) = \left\{\frac{e_1}{0.3}, \frac{e_2}{0}, \frac{e_3}{0.2}\right\}$,

Also, suppose for $\left(G_{E_2^*}, U\right)$, we have;

$\omega''(u_1) = \left\{\frac{a_1}{0.6}, \frac{a_2}{0.5}, \frac{a_3}{0.7}\right\}$, $\omega''(u_2) = \left\{\frac{a_1}{0.8}, \frac{a_2}{0.3}, \frac{a_3}{0.5}\right\}$, $\omega''(u_3) = \left\{\frac{a_1}{1}, \frac{a_2}{0.6}, \frac{a_3}{0.2}\right\}$, and

$\gamma''(u_1) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.7}, \frac{e_3}{0.8}\right\}$, $\gamma''(u_2) = \left\{\frac{e_1}{0.2}, \frac{e_2}{0.6}, \frac{e_3}{0.3}\right\}$, $\gamma''(u_3) = \left\{\frac{e_1}{0.5}, \frac{e_2}{0.4}, \frac{e_3}{0.1}\right\}$,

Then, for (K_{E^*}, U), we have;
$\omega(u_1) = \left\{\frac{a_1}{0.3}, \frac{a_2}{0.5}, \frac{a_3}{0.5}\right\}$, $\omega(u_2) = \left\{\frac{a_1}{0.2}, \frac{a_2}{0.3}, \frac{a_3}{0}\right\}$, $\omega(u_3) = \left\{\frac{a_1}{0.8}, \frac{a_2}{0.1}, \frac{a_3}{0.2}\right\}$, and
$\gamma(u_1) = \left\{\frac{e_1}{0.8}, \frac{e_2}{0.7}, \frac{e_3}{0.6}\right\}$, $\gamma(u_2) = \left\{\frac{e_1}{0.2}, \frac{e_2}{0.5}, \frac{e_3}{0.3}\right\}$, $\gamma(u_3) = \left\{\frac{e_1}{0.3}, \frac{e_2}{0}, \frac{e_3}{0.1}\right\}$,

Now, to get K_{E^*}, we apply equation (2) using the above data.
For u_1:

$$K_{E^*} = \left\{\frac{e_1^*}{0.8 + \left(\frac{(1-0.8)(0.3+0.5+0.5)}{3}\right)}, \frac{e_2^*}{0.7 + \left(\frac{(1-0.7)(0.3+0.5+0.5)}{3}\right)}, \frac{e_3^*}{0.6 + \left(\frac{(1-0.6)(0.3+0.5+0.5)}{3}\right)}, \{u_1\}\right\}$$

$$\Rightarrow K_{E^*} = \left\{\frac{e_1^*}{0.89}, \frac{e_2^*}{0.83}, \frac{e_3^*}{0.77}, \{u_1\}\right\}$$

By using the same method for u_2 and u_3, we arrived at the following effective fuzzy parameterized soft set (EFPS-set):

$$(K_{E^*}, U) = \left\{\left(\frac{e_1^*}{0.89}, \frac{e_2^*}{0.83}, \frac{e_3^*}{0.77}, \{u_1\}\right), \left(\frac{e_1^*}{0.34}, \frac{e_2^*}{0.59}, \frac{e_3^*}{0.42}, \{u_2\}\right), \left(\frac{e_1^*}{0.56}, \frac{e_2^*}{0.37}, \frac{e_3^*}{0.43}, \{u_3\}\right)\right\}$$

3.4 Algebraic Properties of EFPS-set Operations

Here, we consider some of the algebraic properties of the basic operations of the new concept (EFPS-set).

Proposition 3.4.1. ***(De Morgan's Laws):*** *Let $F_{E_I^*}, G_{E_J^*} \in EFPS(U)$. Then De Morgan's laws are valid:*

1. $\left(F_{E_I^*} \cup G_{E_J^*}\right)^c = F_{(E_I^*)^c} \cap G_{E_J^*)^c}$
2. $\left(F_{E_I^*} \cap G_{E_J^*}\right)^c = F_{(E_I^*)^c} \cup G_{E_J^*)^c}$

Proof. From definition 3.1.3, we can see that;

$F_{E_I^*} \Rightarrow \mu(e_i^*)$, $G_{E_J^*} \Rightarrow \mu(e_j^*)$, $H_{E_H^*} \Rightarrow \mu(e_h^*)$ and $K_{E_K^*} \Rightarrow \mu(e_k^*)$

1. $\left(F_{E_I^*} \cup G_{E_J^*}\right)^c = F_{(E_I^*)^c} \cap G_{E_J^*)^c}$

   From the left-hand side; By definition 3.2.3;

$$\left(F_{E_I^*} \cup G_{E_J^*}\right)^c = 1 - max\{\mu(e_i^*), \mu(e_j^*)\}$$
$$= 1 - \mu(e_h^*)$$
$$= \mu(e_h^*)^c$$

Also, by definition 3.2.5;

$$F_{(E_I^*)^c} \cap G_{(E_J^*)^c} = min\left\{\left(1 - \mu(e_i^*)\right), \left(1 - \mu(e_j^*)\right)\right\}$$
$$= 1 - max\{\mu(e_i^*), \mu(e_j^*)\}$$
$$= 1 - \mu(e_h^*)$$
$$= \mu(e_h^*)^c$$

which implies the left-hand side and the right-hand side are equal, hence the proof.

$$\left(F_{E_I^*} \cap G_{E_J^*}\right)^c = F_{(E_I^*)^c} \cup G_{(E_J^*)^c}$$

By definition 3.2.5;

$$\left(F_{E_I^*} \cap G_{E_J^*}\right)^c = 1 - min\{\mu(e_i^*), \mu(e_j^*)\}$$
$$= 1 - \mu(e_k^*)$$
$$= \mu(e_k^*)^c$$

Also, by definition 3.2.3;

$$F_{(E_I^*)^c} \cup G_{(E_J^*)^c} = max\left\{\left(1 - \mu(e_i^*)\right), \left(1 - \mu(e_j^*)\right)\right\}$$
$$= 1 - min\{\mu(e_i^*), \mu(e_j^*)\}$$
$$= 1 - \mu(e_k^*)$$
$$= \mu(e_k^*)^c$$

which implies the left-hand side and the right-hand side are equal, hence the proof. □

We now give a numerical proof of the above proposition.

Example 3.4.2. Suppose: $F_{E_I^*} = \left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}, \{u_i\}\right\}$ and $G_{E_J^*} = \left\{\frac{e_1^*}{0.92}, \frac{e_2^*}{0.88}, \frac{e_3^*}{0.92}, \{u_j\}\right\}$,

The union $\Rightarrow H_{E_H^*} = \left\{\frac{e_1^*}{0.92}, \frac{e_2^*}{0.88}, \frac{e_3^*}{0.92}, \{u_h\}\right\}$

The intersection $\Rightarrow K_{E_K^*} = \left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}, \{u_k\}\right\}$

Also, the complements:

$F_{(E_I^*)^c} = \left\{\frac{e_1^*}{0.1}, \frac{e_2^*}{0.15}, \frac{e_3^*}{0.2}\{u_i\}\right\}$ and $G_{(E_J^*)^c} = \left\{\frac{e_1^*}{0.08}, \frac{e_2^*}{0.12}, \frac{e_3^*}{0.08}, \{u_j\}\right\}$,

The complement of the union $\Rightarrow H_{(E_H^*)^c} = \left\{\frac{e_1^*}{0.08}, \frac{e_2^*}{0.12}, \frac{e_3^*}{0.08}, \{u_h\}\right\}$

The complement of the intersection $\Rightarrow K_{(E_K^*)^c} = \left\{\frac{e_1^*}{0.1}, \frac{e_2^*}{0.15}, \frac{e_3^*}{0.2}, \{u_k\}\right\}$

Now, $\left(F_{E_I^*} \cup G_{E_J^*}\right)^c = F_{(E_I^*)^c} \cap G_{E_J^*)^c}$

For the left-hand side:

$$\left(F_{E_I^*} \cup G_{E_J^*}\right)^c = \left(\left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}\{u_i\}\right\} \cup \left\{\frac{e_1^*}{0.92}, \frac{e_2^*}{0.88}, \frac{e_3^*}{0.92}\{u_j\}\right\}\right)^c$$
$$= \left(\left\{\frac{e_1^*}{0.92}, \frac{e_2^*}{0.88}, \frac{e_3^*}{0.92}\{u_i, u_j\}\right\}\right)^c = \left\{\frac{e_1^*}{0.08}, \frac{e_2^*}{0.12}, \frac{e_3^*}{0.08}, \{u_i, u_j\}\right\}$$

And for the right-hand side:

$$F_{(E_I^*)^c} \cap G_{E_J^*)^c} = \left\{\frac{e_1^*}{0.1}, \frac{e_2^*}{0.15}, \frac{e_3^*}{0.2}, \{u_i\}\right\} \cap \left\{\frac{e_1^*}{0.08}, \frac{e_2^*}{0.12}, \frac{e_3^*}{0.08}, \{u_j\}\right\}$$
$$= \left\{\frac{e_1^*}{0.08}, \frac{e_2^*}{0.12}, \frac{e_3^*}{0.08}, \{u_i, u_j\}\right\}$$

which implies the left-hand side is equal to the right-hand side.

$$\left(F_{E_I^*} \cap G_{E_J^*}\right)^c = F_{(E_I^*)^c} \cup G_{(E_J^*)^c}$$

For the left-hand side:

$$\left(F_{E_I^*} \cap G_{E_J^*}\right)^c = \left(\left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}\{u_i\}\right\} \cap \left\{\frac{e_1^*}{0.92}, \frac{e_2^*}{0.88}, \frac{e_3^*}{0.92}\{u_j\}\right\}\right)^c$$
$$= \left(\left\{\frac{e_1^*}{0.9}, \frac{e_2^*}{0.85}, \frac{e_3^*}{0.8}\{u_i, u_j\}\right\}\right)^c = \left\{\frac{e_1^*}{0.1}, \frac{e_2^*}{0.15}, \frac{e_3^*}{0.2}, \{u_i, u_j\}\right\}$$

And for the right-hand side:

$$F_{(E_I^*)^c} \cup G_{E_J^*)^c} = \left\{\frac{e_1^*}{0.1}, \frac{e_2^*}{0.15}, \frac{e_3^*}{0.2}, \{u_i\}\right\} \cup \left\{\frac{e_1^*}{0.08}, \frac{e_2^*}{0.12}, \frac{e_3^*}{0.08}, \{u_j\}\right\}$$
$$= \left\{\frac{e_1^*}{0.1}, \frac{e_2^*}{0.15}, \frac{e_3^*}{0.2}, \{u_i, u_j\}\right\}$$

Also implies the left-hand side is equal to the right-hand side. Hence, the proof.

Proposition 3.4.3. ***(Distributivity):*** *Let* $A_{E_A^*}, B_{E_B^*}$ *and* $C_{E_C^*} \in EFPS(U)$*, then;*

1. $A_{E_A^*} \cap \left(B_{E_B^*} \cup C_{E_C^*}\right) = \left(A_{E_A^*} \cap B_{E_B^*}\right) \cup \left(A_{E_A^*} \cap C_{E_C^*}\right)$
2. $A_{E_A^*} \cup \left(B_{E_B^*} \cap C_{E_C^*}\right) = \left(A_{E_A^*} \cup B_{E_B^*}\right) \cap \left(A_{E_A^*} \cup C_{E_C^*}\right)$

Proof. Suppose by definition;

$A_{E_A^*} \Rightarrow \mu(e_a^*)$, $B_{E_B^*} \Rightarrow \mu(e_b^*)$, $C_{E_C^*} \Rightarrow \mu(e_c^*)$, $D_{E_D^*} \Rightarrow \mu(e_d^*)$, $F_{E_F^*} \Rightarrow \mu(e_f^*)$, $G_{E_G^*} \Rightarrow \mu(e_g^*)$, $H_{E_H^*} \Rightarrow \mu(e_h^*)$ *and* $K_{E_K^*} \Rightarrow \mu(e_k^*)$
such that:

1. $B_{E_B^*} \cup C_{E_C^*} = D_{E_D^*}$, $A_{E_A^*} \cap B_{E_B^*} = F_{E_F^*}$, $A_{E_A^*} \cap C_{E_C^*} = G_{E_G^*}$, $A_{E_A^*} \cap D_{E_D^*} = H_{E_H^*}$, $F_{E_F^*} \cup G_{E_G^*} = K_{E_K^*}$

   Therefore,

   $$A_{E_A^*} \cap \left(B_{E_B^*} \cup C_{E_C^*}\right) \Rightarrow min\{\mu(e_a^*), max\{\mu(e_b^*), \mu(e_c^*)\}\}$$
   $$= max\{min\{\mu(e_a^*), \mu(e_b^*)\}, min\{\mu(e_a^*), \mu(e_c^*)\}\}$$
   $$= max\{\mu(e_f^*), \mu(e_g^*)\}$$

   Since $= max\{\mu(e_f^*), \mu(e_g^*)\} \Rightarrow F_{E_F^*} \cup G_{E_G^*}$,
   this implies left hand side is equal to the right-hand side.

2. *The proof follows easily if we assume the following:*

$B_{E_B^*} \cap C_{E_C^*} = D_{E_D^*}$, $A_{E_A^*} \cup B_{E_B^*} = F_{E_F^*}$, $A_{E_A^*} \cup C_{E_C^*} = G_{E_G^*}$, $A_{E_A^*} \cup D_{E_D^*} = H_{E_H^*}$, $F_{E_F^*} \cap G_{E_G^*} = K_{E_K^*}$

□

In what follows, we give a numerical example to show the proof of the above proposition.

Example 3.4.4. Suppose:

$A_{E_A^*} = \left\{\frac{e_1^*}{0.5}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.7}, \{u_a\}\right\}$, $B_{E_B^*} = \left\{\frac{e_1^*}{0.3}, \frac{e_2^*}{0.7}, \frac{e_3^*}{0.4}, \{u_b\}\right\}$ and $C_{E_C^*} = \left\{\frac{e_1^*}{0.8}, \frac{e_2^*}{0.9}, \frac{e_3^*}{0.6}, \{u_c\}\right\}$

1. For the left-hand side; $A_{E_A^*} \cap \left(B_{E_B^*} \cup C_{E_C^*}\right)$

   $$\left\{\frac{e_1^*}{0.5}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.7}, \{u_a\}\right\} \cap \left(\left\{\frac{e_1^*}{0.3}, \frac{e_2^*}{0.7}, \frac{e_3^*}{0.4}, \{u_b\}\right\} \cup \left\{\frac{e_1^*}{0.8}, \frac{e_2^*}{0.9}, \frac{e_3^*}{0.6}, \{u_c\}\right\}\right)$$

   $$= \left\{\frac{e_1^*}{0.5}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.7}, \{u_a\}\right\} \cap \left\{\frac{e_1^*}{0.8}, \frac{e_2^*}{0.9}, \frac{e_3^*}{0.6}, \{u_d\}\right\}$$

   $$= \left\{\frac{e_1^*}{0.5}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.6}, \{u_h\}\right\} \Rightarrow H_{E_H^*}$$

   For the right-hand side; $\left(A_{E_A^*} \cap B_{E_B^*}\right) \cup \left(A_{E_A^*} \cap C_{E_C^*}\right)$

$$= \left(\left\{\frac{e_1^*}{0.5}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.7}, \{u_a\}\right\} \cap \left\{\frac{e_1^*}{0.3}, \frac{e_2^*}{0.7}, \frac{e_3^*}{0.4}, \{u_b\}\right\}\right) \cup \left(\left\{\frac{e_1^*}{0.5}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.7}, \{u_a\}\right\} \cap \left\{\frac{e_1^*}{0.8}, \frac{e_2^*}{0.9}, \frac{e_3^*}{0.6}, \{u_c\}\right\}\right)$$

$$= \left(\left\{\frac{e_1^*}{0.3}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.4}, \{u_f\}\right\} \cup \left\{\frac{e_1^*}{0.5}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.6}, \{u_g\}\right\}\right)$$

$$= \left\{\frac{e_1^*}{0.5}, \frac{e_2^*}{0.6}, \frac{e_3^*}{0.6}, \{u_k\}\right\} \Rightarrow K_{E_K^*}$$

Since $\mu(e_h^*)$ and $\mu(e_k^*)$ are equal, it implies the left-hand side and the right-hand side are equal.

2. The solution follows easily.

4. EFPS-set in Decision-Making

As an illustration, we used the data presented in the application part in Alkhazaleh (2022). The idea here, is to outline new algorithm that is easier and more effective than that of Roy & Maji (2007) and that of Alkhazaleh (2022).

Algorithm 4.1. *Here, we give a new algorithm as a modification of Roy & Maji's and Alkhazaleh's.*

1. *In two separate tables, input the fuzzy effective set ω and the fuzzy parameters set* γ.
2. *Input the set of the reformed parameters* E^* *over* U.
3. *Compute the corresponding resultant EFPS-set* F_{E^*} *and place it in a tabular form.*
4. *Compute the row sums* r_i *and the average row sum* c_i $\forall i$ *in order to deduce the fuzzy score of each candidate and place it in a tabular form.* ($Where\ r_i = \sum\mu(e_i^*)$ *and* $c_i = \frac{\sum\mu(e_i^*)}{|E^*|}$).
5. *Compute the ranking table.*

Note that the ranking table contains different categories deduced based on the scores obtained in item 4 of the algorithm. Also, there is no need for the comparison table because we are not dealing with multi-observer data.

So, for Roy & Maji's algorithm, we used (H, E) which is clearly the basic union of (F, E_1) and (G, E_2) as follows:

Table 3: ***Tabular Representation of*** (H, E)

U/E	e_1	e_2	e_3	e_4	e_5	e_6
x_1	0.8	0.4	0.6	0.6	0.8	0.6
x_2	0.6	0.5	0.8	0.9	0.8	0.6
x_3	0.9	0.7	0.9	0.8	0.9	0.9
x_4	0.4	0.4	0.6	0.7	0.7	0.7
x_5	0.8	0.5	0.7	0.4	0.5	0.6
x_6	0.4	0.5	0.8	0.4	0.6	0.5

Now, for the row-sum and the average-row sum table of the above data, we have;

Table 4: ***Tabular Representation of the row sums and the average row sums of*** (H, E)

U	row-sum (r_i)	average row sum (c_i)
x_1	3.8	0.63
x_2	4.2	0.70
x_3	5.1	0.85
x_4	3.5	0.58
x_5	3.5	0.58
x_6	3.2	0.53

Clearly, we can see that after computing the average row-sums, x_3 has the highest fuzzy value 0.85. It's therefore the best to select.
Also, from Alkhazaleh (2022);

Table 5: ***Tabular Representation of*** (H_{λ_s}, E)

U/E	e_1	e_2	e_3	e_4	e_5	e_6
x_1	0.91	0.73	0.82	0.82	0.91	0.82
x_2	0.88	0.85	0.94	0.97	0.94	0.88
x_3	0.93	0.78	0.93	0.86	0.93	0.93
x_4	0.73	0.73	0.82	0.87	0.82	0.87
x_5	0.94	0.85	0.91	0.82	0.85	0.88
x_6	0.90	0.92	0.97	0.90	0.93	0.92

Now, for the row-sum and the average row-sum table of the above data, we have;

Table 6: ***Tabular Representation of the row sums and the average row sums of*** (H_{λ_s}, E)

U	row-sum (r_i)	average row sum (c_i)
x_1	5.01	0.84
x_2	5.46	0.91
x_3	5.36	0.89
x_4	4.84	0.81
x_5	5.25	0.88
x_6	5.54	0.92

From the above table, after computing the average row-sums, it is very clear that the decision is to select x_6 since it has the highest fuzzy value 0.92. This clearly indicates that the introduction of the effective set made changes in the decision-making process as prescribed before by Roy and Maji (2007).

4.2 Application of EFPS-set in Efficiency Test

Example 4.2.1. Suppose two law firms merged together and are trying to run an efficiency test among the junior staffers in order to determine whom to let go. Things to consider for the test are categorize into two; the individual key performance indicator and professional skills.

Suppose we have the following information for the test:

Let $U = \{u_1, u_2, \ldots, u_{20}\}$ represents the members of staff in consideration- by which the new firm will be capable of accommodating 14 out of the 20 members. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be the set of parameters to consider in determining the individual key performance, where e_1=*client list*, e_2=*billable hours*, e_3=*percentage of cases won*, e_4=*monthly expenses*, e_5=*number of task completed*, e_6=*number of matters opened* and e_7=*originating revenue referred to the firm*. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of effective parameters which will be used to determine the professional skills of the staffers, where a_1=*analytical skills*, a_2=*communication skills*, a_3=*teamwork*, a_4=*commercial awareness* and a_5=*creative problem solving*.

Suppose after collating the information of every staff, we have the following fuzzy values for each parameter in relation to all staffers:

Table 7: ***Tabular Representation of the fuzzy parameters set***

$\gamma(U)/E$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\gamma(u_1)$	0.4	0.5	0.4	0.6	0.6	0.7	0.6
$\gamma(u_2)$	0.6	0.7	0.7	0.6	0.5	0.8	0.9
$\gamma(u_3)$	0.6	0.6	0.5	0.7	0.5	0.6	0.5
$\gamma(u_4)$	0.5	0.6	0.5	0.7	0.6	0.5	0.5
$\gamma(u_5)$	0.7	0.7	0.8	0.6	0.7	0.6	0.8
$\gamma(u_6)$	0.3	0.4	0.8	0.6	0.9	0.8	0.4
$\gamma(u_7)$	0.4	0.4	0.6	0.5	0.6	0.7	0.7
$\gamma(u_8)$	0.5	0.6	0.5	0.7	0.8	0.8	0.8
$\gamma(u_9)$	0.3	0.4	0.4	0.5	0.4	0.6	0.5
$\gamma(u_{10})$	0.8	0.8	0.8	0.7	0.8	0.5	0.9
$\gamma(u_{11})$	0.7	0.8	0.7	0.7	0.6	0.7	0.8
$\gamma(u_{12})$	0.5	0.5	0.9	0.5	0.9	0.6	0.4
$\gamma(u_{13})$	0.1	0.2	0.5	0.4	0.3	0.2	0.3
$\gamma(u_{14})$	0.8	0.7	0.8	0.6	0.7	0.8	0.8
$\gamma(u_{15})$	0.3	0.4	0.4	0.3	0.5	0.5	0.5
$\gamma(u_{16})$	0.3	0.3	0.5	0.4	0.5	0.4	0.6
$\gamma(u_{17})$	0.9	0.9	0.7	0.8	0.7	0.6	0.9
$\gamma(u_{18})$	0.5	0.4	0.5	0.6	0.6	0.5	0.4
$\gamma(u_{19})$	0.8	0.7	0.8	0.6	0.6	0.7	0.7
$\gamma(u_{20})$	0.6	0.6	0.6	0.7	0.6	0.5	0.7

Also, we have the following fuzzy values for each effective parameter in relation to all staffers:

Table 8: ***Tabular Representation of the fuzzy effective set***

$\omega(U)/A$	a_1	a_2	a_3	a_4	a_5
$\omega(u_1)$	0.5	0.7	0.2	0.7	0.5
$\omega(u_2)$	0.7	0.7	0.8	0.9	0.6
$\omega(u_3)$	0.8	0.4	0.8	0.6	0.7
$\omega(u_4)$	0.3	0.8	0.8	0.7	0.4
$\omega(u_5)$	0.7	0.7	0.4	0.6	0.7
$\omega(u_6)$	0.9	0.8	0.9	0.5	0.8
$\omega(u_7)$	0.3	0.5	0.6	0.5	0.4
$\omega(u_8)$	0.6	0.5	0.7	0.7	0.6
$\omega(u_9)$	0.4	0.5	0.4	0.4	0.5
$\omega(u_{10})$	0.9	0.7	0.6	0.7	0.9
$\omega(u_{11})$	0.5	0.3	0.6	0.6	0.4
$\omega(u_{12})$	0.7	0.7	0.4	0.8	0.6
$\omega(u_{13})$	0.7	0.4	0.4	0.6	0.7
$\omega(u_{14})$	0.6	0.8	0.8	0.7	0.5
$\omega(u_{15})$	0.2	0.5	0.4	0.4	0.1
$\omega(u_{16})$	0.4	0.4	0.3	0.6	0.5
$\omega(u_{17})$	0.8	0.7	0.7	0.6	0.8
$\omega(u_{18})$	0.4	0.7	0.7	0.6	0.6
$\omega(u_{19})$	0.5	0.5	0.3	0.9	0.6
$\omega(u_{20})$	0.1	0.2	0.2	0.3	0.2

So that after applying equation (2) using the data above, we have;

For u_1:

$$F_{E^*} = \left\{ \begin{array}{c} \frac{e_1^*}{0.4 + \left(\frac{(1-0.4)(0.5+0.7+0.2+0.7+0.5)}{5}\right)}, \frac{e_2^*}{0.5 + \left(\frac{(1-0.5)(0.5+0.7+0.2+0.7+0.5)}{5}\right)}, \\ \frac{e_3^*}{0.4 + \left(\frac{(1-0.4)(0.5+0.7+0.2+0.7+0.5)}{5}\right)}, \frac{e_4^*}{0.6 + \left(\frac{(1-0.6)(0.5+0.7+0.2+0.7+0.5)}{5}\right)}, \\ \frac{e_5^*}{0.6 + \left(\frac{(1-0.6)(0.5+0.7+0.2+0.7+0.5)}{5}\right)}, \frac{e_6^*}{0.7 + \left(\frac{(1-0.7)(0.5+0.7+0.2+0.7+0.5)}{5}\right)}, \\ \frac{e_7^*}{0.6 + \left(\frac{(1-0.6)(0.5+0.7+0.2+0.7+0.5)}{5}\right)}, \{u_1\} \end{array} \right\}$$

$$\Rightarrow F_{E^*} = \left\{\frac{e_1^*}{0.71}, \frac{e_2^*}{0.76}, \frac{e_3^*}{0.71}, \frac{e_4^*}{0.81}, \frac{e_5^*}{0.81}, \frac{e_6^*}{0.86}, \frac{e_7^*}{0.81}, \{u_1\}\right\}$$

Here, the above shows how we obtained the set of reformed parameters for the first staff u_1. It can be seen that the first element e_1^* is obtained after the influence of the fuzzy values from the effective parameters set on the parameter e_1. The rest follows.

So, by using the same method for the rest of the staffers (i.e., from u_2 to u_{20}), we have the following result:

Below is the tabular representation of the resultant EFPS-set (F_{E^*}, U).

Table 9: ***Tabular Representation of the EFPS-set*** (F_{E^*}, U)

U/E^*	e_1^*	e_2^*	e_3^*	e_4^*	e_5^*	e_6^*	e_7^*
u_1	0.71	0.76	0.71	0.81	0.81	0.86	0.81
u_2	0.9	0.92	0.92	0.9	0.87	0.95	0.97
u_3	0.86	0.86	0.83	0.9	0.83	0.86	0.83
u_4	0.8	0.84	0.8	0.88	0.84	0.8	0.8
u_5	0.89	0.89	0.92	0.85	0.89	0.85	0.92
u_6	0.86	0.87	0.96	0.91	0.98	0.96	0.87
u_7	0.68	0.68	0.78	0.73	0.78	0.84	0.84
u_8	0.81	0.85	0.81	0.89	0.92	0.92	0.92
u_9	0.61	0.66	0.66	0.72	0.66	0.78	0.72
u_{10}	0.95	0.95	0.95	0.93	0.95	0.88	0.98
u_{11}	0.84	0.9	0.84	0.84	0.79	0.84	0.9
u_{12}	0.82	0.82	0.96	0.82	0.96	0.86	0.78
u_{13}	0.6	0.65	0.78	0.74	0.69	0.65	0.69
u_{14}	0.94	0.9	0.94	0.87	0.9	0.94	0.94
u_{15}	0.52	0.59	0.59	0.52	0.66	0.66	0.66
u_{16}	0.48	0.48	0.72	0.66	0.72	0.66	0.78
u_{17}	0.97	0.97	0.92	0.94	0.92	0.89	0.97
u_{18}	0.8	0.76	0.8	0.84	0.84	0.8	0.76
u_{19}	0.91	0.87	0.91	0.82	0.82	0.87	0.87
u_{20}	0.68	0.68	0.68	0.76	0.68	0.6	0.76

Table 10: ***Tabular Representation of the row-sums*** (r_i) ***and the average row-sums*** (c_i) ***of the*** (F_{E^*}, U)

U	row-sum (r_i)	average row-sum (c_i)
u_1	5.47	0.78
u_2	6.43	0.92
u_3	5.97	0.85
u_4	5.76	0.82
u_5	6.21	0.89
u_6	6.41	0.92
u_7	5.33	0.76
u_8	6.12	0.87
u_9	4.81	0.69
u_{10}	6.59	0.94
u_{11}	5.95	0.85
u_{12}	6.02	0.86
u_{13}	4.8	0.69
u_{14}	6.43	0.92
u_{15}	4.2	0.6
u_{16}	4.5	0.64
u_{17}	6.58	0.94
u_{18}	5.6	0.8
u_{19}	6.07	0.87
u_{20}	4.84	0.69

Now, we used the scores obtained from the above table in order to deduce staff ranking as our final step:

Table 11: ***The Ranking Table***

Ranking	Members of Staff (U)	Score (c_i)
1^{st}	u_{10} and u_{17}	0.94
2^{nd}	u_2, u_6 and u_{14}	0.92
3^{rd}	u_5	0.89
4^{th}	u_8 and u_{19}	0.87
5^{th}	u_{12}	0.86
6^{th}	u_3 and u_{11}	0.85
7^{th}	u_4	0.82
8^{th}	u_{18}	0.80
9^{th}	u_1	0.78

Ranking	Members of Staff (U)	Score (c_i)
10^{th}	u_7, u_9, u_{13} and u_{20}	0.69
11^{th}	u_{16}	0.64
12^{th}	u_{15}	0.60

We can see that from the above table, we have 12 categories, and that staff u_{10} and u_{17} has the highest scores, while staff u_{15} is with the lowest score. Since the firm wants to maintain 14 out of the 20 members of staff in consideration, therefore staff $u_{15}, u_{16}, u_{20}, u_{13}, u_9$ and u_7 are the ones to let go.

5 Conclusion

As a way forward in dealing with uncertainties, we have introduced a new concept (EFPS-set theory) within the frame work of effective set; where the fuzzification goes to the elements in the set of parameters, which proves to be more perfect and efficient in decision-making. We study some of the basic set operations such as: union, intersection and complement; and we used these operations to consider some algebraic properties by proving results like the De Morgan's laws and the distributive laws. Additionally, we developed a new algorithm by modifying the algorithms given by Roy & Maji and Alkhazaleh; thereby giving a comparison test which shows that the new algorithm can be adopted in solving choice making problems. Also, the new concept has been applied to solve a case involving efficiency evaluation test; where a sample of 20 candidates are successfully examined. In sequel, further research can be conducted on the application of the EFPS-set theory in areas like: decision-making, data analysis and optimization problems. Also, experts in fuzzy logic and soft set theory may refine the framework and expand its potential to handle uncertainties.

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References

Adam F. and Hassan N. (2014a). Q-Fuzzy Soft Set. *Applied Mathematical Sciences.* 174(8), 8689-8695.

Adam F. and Hassan N. (2014b). Operations on Q-Fuzzy Soft set. *Applied Mathematical Sciences.* 175(8), 8697-8701.

Adam F. and Hassan N. (2014c). Multi Q-Fuzzy Parameterized Soft Set and its Application. *Journal of Intelligent and Fuzzy Systems,* 27(1), 419-424.

Alcantud, J. C. R. and Torrecillas, M. J. M. (2018). Intertemporal Choice of Fuzzy Soft Sets. *Symmetry.*10(9), 371. DOI:10.3390/Sym10090371.

Alkhazaleh, S. (2022). Effective Fuzzy Soft Set Theory and its Applications. *Applied Computational Intelligence and Soft Computing.* vol. 2022.

Ali, M. I., Feng, F., Liu, X. and Min, W. K. (2009). On Some New Operations in Soft Set Theory. *Computers and Mathematics with Applications,* 57(9), 1547-1553.

Cagman, N. and Enginoglu, S. (2010a). Soft matrix theory and its decision making. *Computer and Mathematics with Applications.* 59(10), 3308. DOI:10.1016/j.camwa.2010.03.015.

Cagman, N., Citak, F. and Enginoglu, S. (2010b). Fuzzy Parameterized Fuzzy Soft Set Theory and its Applications. *Turkish Journal of Fuzzy Systems.* 1(1), 21-35.

Cagman, N., Enginoglu, S. and Citak, F. (2011a). Fuzzy Soft Set Theory and its Applications. *Iranian Journal of Fuzzy System.* 8(3), 137-147.

Cagman, N., Citak, F. and Enginoglu, S. (2011b). Fuzzy Parameterized Soft Set Theory And its Applications. *Annals of Fuzzy Mathematics and Informatics,* 2(2), 219-226.

Dey, A., Senapati, T., Pal, M. and Chen, G. (2020). A Novel Approach to Hesitant Multi-Fuzzy Soft Set Based Decision-Making. *AIMS Mathematic,* 5(3), 1985-2008. DOI:10.3934/math.2020132.

Feng, F., Jun, Y., Liu, X. and Li, L. (2010). An Adjustable Approach to Fuzzy Soft Set Based Decision Making. *Journal of Computational and Applied Mathematics,* 234, 10-20.

Kharal, A. and Ahmad, B. (2010). Mappings on Soft Classes. *New Mathematics and Natural Computation.* vol. 07, no. 03. DOI:10.1142/S1793005711002025.

Kong, Z., Gao, L., Wang, L. and Li, S. (2008). The Normal Parameter Reduction of Soft Sets and its Algorithm. *Computers and Mathematics with Applications.* 56, 3029-3037.

Liu, Z., Qin, K. and Pei, Z. (2017). A Method for Fuzzy Soft Set in Decision-Making Based on an Ideal Solution. *Symmetry.* vol. 9, pp. 246. DOI:10.3390/sym9100246.

Maji, P. K., Biswas, R. and Roy, A. R. (2001). Fuzzy Soft Sets. *Journal of Fuzzy Mathematics, 9*(3), 677-691.

Maji, P. K. and Roy, A. R. (2002). An Application of Soft Sets in a Decision-Making Problem. *Computers and Mathematics with Applications,* 44, 1077-1083.

Molodtsov, D. A. (1999). Soft Set Theory-First Results. *Computers and Mathematics with Applications,* 37, 19-31.

Nasef, A. A., El-Maghrabi, A. I., Elfeky, A. M. and Jafari, S. (2018). Soft Set Theory and Its Applications. *International Conference on Topology and its Application.* pp. 211-222.

Nasef, A. A., Elfeky, A. M., El-Maghrabi, A. I. and Jafari, S. (2018). Another Application of Fuzzy Soft Sets in Real Life Problems. *International Conference on Topology and its Application.* pp. 202-210.

Phaengtan, K., Aimcharoen, N. and Suebsan, P. (2021). Partial averages of fuzzy soft sets in decision-making problems. *Journal of interdisciplinary Mathematics*, 24(4), 1035-1052.

Renukadevi, V. and Sangeetha, G. (2020). A new decision-making approach for winning strategy based on multi soft set logic *Journal of Fuzzy Extension and Applications,* 1(2), 112–121.

Rodriguez, R. M., Labella, A. and Martinez, L. (2016). An overview on fuzzy modeling of complex linguistic preferences in decision making. *International Journal Computer Intelligence System,* 9, 81-94.

Roy, A. R. and Maji, P. K. (2007). A Fuzzy Soft Set Theoretic Approach to Decision Making Problems. *Journal of Computational and Applied Mathematics,* 203, 412-418.

Sanchez, E. (1976). Resolution of composite fuzzy relation equations. *Information Control,* 30(1), 38-48.

Sanchez, E. (1979). Inverses of fuzzy relations, application to possibility distributions and medical diagnosis. *Fuzzy Sets System,* 2(1), 75-86.

Somen, D. (2021). Fuzzy hypersoft sets and its weightage operator for decision-making. *Journal of fuzzy extension and Applications,* 2(2), 163-170.

Yao, B. X., Liu, J. and Yan, R. X. (2008). Fuzzy Soft Set and Soft Fuzzy Set. *IEEE Fourth International Conference on Natural Computation,* pp. 252-255.

Zadeh, L. A. (1965). Fuzzy Sets. *Information and Control,* 8, 338-353.