INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals

Development of Hybrid Block Methods for Oscillatory Solutions of Second-Order Differential Equations

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ARTICLE INFO

Article history:
Received 10 January 2025
Received in revised form 03 March 2025
Accepted 19 March 2025

Keywords:
Oscillatory differential equations, hybrid block method, Numerical methods, Second-order initial value problems.

MSC 2020 Subject classification:
34A30, 34A34, 92B05, 92-10

ABSTRACT

This study explores the application of hybrid block method in solving oscillatory second-order initial value problems (IVPs), a category of differential equations relevant in modeling various real-life phenomena, such as mechanical and electrical oscillations. Traditional numerical methods often face challenges in accuracy and stability when applied to oscillatory problems, prompting a need for advanced methods like hybrid block method. This research developed a hybrid block method that offers improved error control, stability and convergence for oscillatory differential equations. Error analysis is conducted to assess the method's effectiveness, comparing it with existing approaches to highlight its robustness in handling oscillatory behavior. The proposed method demonstrates an expanded stability region, making it suitable for complex, real-world applications that require high precision. The study's findings emphasize the potential of block hybrid methods in advancing numerical solutions for differential equations, providing valuable insights for further research and application in science and engineering.

1. Introduction

Oscillatory differential equations are central to modeling phenomena where solutions exhibit wave-like or periodic characteristics, though their slow-changing amplitudes and phases pose unique challenges in applied mathematics. Traditional methods for tracking solutions over extended cycles often struggle with precision, especially when eigenvalues cluster near the imaginary axis, leading to oscillatory solutions. Researchers frequently approximate or compute quasi-envelopes to manage these complexities, aiming to maintain accuracy in both amplitude and phase angle over long intervals. This becomes essential in real-life applications, such as in mass-spring systems and other dynamic scenarios that demand oscillatory behavior modeling (Abdelrahim and Omar, 2016, Adee and Kumleng, 2022, Adee and Yunusa, 2022).

In practical applications, oscillatory differential equations describe various real-world systems like the cooling of a body, where factors like environmental fluctuations and medium conditions influence oscillatory temperature changes around an equilibrium. For instance, a mass-spring system illustrates how restoring forces generate oscillations around an equilibrium position, described by second-order equations. Additionally, adding second-order terms to Newton's Law of Cooling helps model oscillatory behavior due to periodic external influences, enhancing its ability to capture realistic temperature dynamics. This approach is significant as it broadens the understanding of cooling processes, moving beyond simple exponential decay to account for oscillatory factors in a more robust model (Skwame *et al.* 2017, Skwame *et al.,* 2020, Kyagya *et al.,* 2021, Ayinde *et al.,* 2023, Adewale and Sabo, 2024).

For this study, we aim to develop and apply a set of a numerical method for oscillatory solution of second-order oscillatory problems characterized by the following form:

$$y''(t) = f(t, y, y'), y(t_0) = y_0, y'(t_0) = y_0', \ \ t \in [t_0, t_n] \tag{1}$$

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https://doi.org/10.62054/ijdm/0201.03

where t_0 represents initial value/point, y_0 denotes the solution at time t_0, f remains continuous over the integration interval.

To solve these equations, reduction to first-order systems is often employed, though it comes with computational burdens, high error rates, and time constraints. These limitations have led to the development of new numerical methods, such as block hybrid methods, which offer more efficient, direct solutions to second-order initial value problems (Donald *et al.,* 2021, Donald *et al.,* 2022). By focusing on block hybrid methods, this study aims to provide a solution that balances computational efficiency and accuracy, allowing for more precise handling of oscillatory problems across disciplines from engineering to economics where modeling dynamic, repetitive behaviors around equilibrium points is critical (Areo and Rufai, 2016).

The block method, originally proposed by Milne in 1953, evolved from a tool for providing initial values in predictor-corrector algorithms to a fully independent numerical method (Donald *et al.,* 2021). Researchers like Mansor et al. (2023), Adewale and Sabo, (2024) have since advanced the method, focusing on improving computational efficiency by evaluating solutions at multiple points simultaneously. Several specific block methods have been developed to address various types of ordinary differential equations (ODEs). For example, Skwame et al. (2019) extended block methods with a double-step hybrid approach for second-order initial value problems, allowing larger step sizes and reduced computation. Additionally, Ayinde *et al.,* (2023) proposed a hybrid-block method specifically for nonlinear second-order ODEs, enhancing convergence and stability, which is vital in fields like fluid dynamics and biological systems. These contributions underscore the block hybrid method's versatility and significance in efficiently solving complex problems across scientific and engineering disciplines.

Definition 1.1: Consider the linear operator associated with the new method be defined as

$$L[y(x);h]=\sum_{j=0}^{k}\left\{\alpha_j y(x_n+jh)-\alpha_{vi}y(x_n+vih)-h^d\beta_j y^d(x_n+jh)-h^d\beta_{vi}y^d(x_n+vih)\right\} \tag{2}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,\ b]$. We expand $y(x_n+jh)$ and $y^d(x_n+jh)$ as a Taylor series about x_n and collecting like terms in h and y to obtain the expression;

$$\ell\{y(x);h\}=C_0y(x)+C_1y'(x)+\ldots+C_ph^py^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\ldots \tag{3}$$

Definition 1.2: The new method is said to be zero-stable if no roots Z_s, $s=1,\ldots,n$ of the first characteristic polynomial $\overline{\rho}(z)$ is define by $\overline{\rho}(z)=det\left[z\overline{A}-\overline{E}\right]$ has modulus greater than one, i.e $|Z_s|\le 1$ and every roots with $|Z_s|=1$ has multiplicity not exceeding two. That is if the roots $z_s,\ s=1,2,\ldots,n$ of the first characteristic polynomial $\overline{\rho}(z)$, defined by

$$\overline{\rho}(z)=\det\left[zA^{(0)}-E\right] \tag{4}$$

Definition 1.3: The new method is said be absolutely stable in the region of the complex plane if, for all $h(\lambda h)\in\Re$, all roots of the stability polynomial $\pi(r,h)$ associated with the method satisfy

$|r_s|<1,\ s=1,2,\cdots,k\ and\ |r_s|<|r_1|, s=2,3,\cdots,k$.

The following notations shall be used in Tables and figures below.

ES means Exact Solution

CS means Computed Solution

ENM means Error in New Method

EAR16 means Absolute Error in Areo and Rufai (2016)

ESBS17 means Absolute Error in Skwame et al. (2017)

EOO18 means Absolute Error in Omole and Ogunware (2018)

EOOA18 means Absolute Error in Olanegan, *et al.* (2018)

ESDKSB20 means Absolute Error in Skwame et al., (2020)

ESKV21 means Absolute Error in Sabo, et al., (2021)

2. Derivation of Computational Hybrid Block Method

In order to derive the new method, an approximate solution to power series polynomial of the form

$$y(t)=\sum_{j=0}^{p+q-1} a_j t^j \tag{5}$$

is considered as a basis function to approximate the solution of the initial value problems of general second order ordinary differential equation (1)

with the second derivative given by

$$y''(t)=\sum_{j=0}^{p+q-1} j(j-1)a_j t^{j-2} \tag{6}$$

where $t \in [a, b]$, the $a's$ are real unknown parameters to be determined and $p+q$ is the sum of the number of interpolation and collocation points. Let the solution of (1) be sought on the partition $\pi_N: a = t_0 < t_1 < t_2 < \cdots t_n < t_{n+1} < t_N = b$on the interval $[a, b]$ with a constant step size$h$, given by $h = t_n - t_{n-1}$, where $n = 0, 1, 2, \ldots, N$.

Then, substituting (6) in (1) gives

$$\sum_{j=0}^{p+q-1} j(j-1)a_j t^{j-2} = f(t, y, y') \tag{7}$$

Using (5) with $p = 2$ and $q = 7$, the polynomial of degree $p+q-1$ is as follows

$$y(t)=\sum_{j=0}^{8} a_j t^j \tag{8}$$

With its second derivative as

$$\sum_{j=0}^{8} j(j-1)a_j t^{j-2} = f(t, y, y') \tag{9}$$

now, interpolating (8) at the point g_{n+p}, $p=\frac{1}{2}$, 1 and collocating (9) at t_{n+q}, $q=0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1$ which lead to a system of equation in a matrix form as

$$\begin{bmatrix} 1 & t_{n+\frac{1}{2}} & t^2_{n+\frac{1}{2}} & t^3_{n+\frac{1}{2}} & t^4_{n+\frac{1}{2}} & t^5_{n+\frac{1}{2}} & t^6_{n+\frac{1}{2}} & t^7_{n+\frac{1}{2}} & t^8_{n+\frac{1}{2}} \\ 1 & t_{n+1} & t^2_{n+1} & t^3_{n+1} & t^4_{n+1} & t^5_{n+1} & t^6_{n+1} & t^7_{n+1} & t^8_{n+1} \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 & 56t_n^6 \\ 0 & 0 & 2 & 6t_{n+\frac{1}{4}} & 12t^2_{n+\frac{1}{4}} & 20t^3_{n+\frac{1}{4}} & 30t^4_{n+\frac{1}{4}} & 42t^5_{n+\frac{1}{4}} & 56t^6_{n+\frac{1}{4}} \\ 0 & 0 & 2 & 6t_{n+\frac{1}{3}} & 12t^2_{n+\frac{1}{3}} & 20t^3_{n+\frac{1}{3}} & 30t^4_{n+\frac{1}{3}} & 42t^5_{n+\frac{1}{3}} & 56t^6_{n+\frac{1}{3}} \\ 0 & 0 & 2 & 6t_{n+\frac{1}{2}} & 12t^2_{n+\frac{1}{2}} & 20t^3_{n+\frac{1}{2}} & 30t^4_{n+\frac{1}{2}} & 42t^5_{n+\frac{1}{2}} & 56t^6_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6t_{n+\frac{2}{3}} & 12t^2_{n+\frac{2}{3}} & 20t^3_{n+\frac{2}{3}} & 30t^4_{n+\frac{2}{3}} & 42t^5_{n+\frac{2}{3}} & 56t^6_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6t_{n+\frac{3}{4}} & 12t^2_{n+\frac{3}{4}} & 20t^3_{n+\frac{3}{4}} & 30t^4_{n+\frac{3}{4}} & 42t^5_{n+\frac{3}{4}} & 56t^6_{n+\frac{3}{4}} \\ 0 & 0 & 2 & 6t_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} & 30t^4_{n+1} & 42t^5_{n+1} & 56t^6_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \tag{10}$$

using Gaussian elimination method, (10) is solved for the a_j's. The values of the a_j's obtained are then substituted into (5), after some manipulations, this gives a continuous hybrid linear multistep method of the form;

$$y(t) = \alpha_{\frac{1}{2}}(t) y_{n+\frac{1}{2}} + \alpha_1(t) y_{n+1} + h^2 \left[\sum_{j=0}^{1} \beta_j(t) f_{n+j} + \beta_{v_i}(t) f_{n+v_i} \right], v_i = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \tag{11}$$

the coefficient $\alpha_{\frac{1}{2}}, \alpha_1, \beta_0, \beta_{\frac{1}{4}}, \beta_{\frac{1}{3}}, \beta_{\frac{1}{2}}, \beta_{\frac{2}{3}}, \beta_{\frac{3}{4}}, \beta_1$ are given by;

$$\alpha_{\frac{1}{2}} = 2 - 2t$$

$$\alpha_1 = 2t - 1$$

$$\beta_0 = \frac{41}{13440} - \frac{2539}{40320}t + \frac{1}{2}t^2 - \frac{77}{36}t^3 + \frac{49}{9}t^4 - \frac{203}{24}t^5 + \frac{707}{90}t^6 - 4t^7 + \frac{6}{7}t^8$$

$$\beta_{\frac{1}{4}} = \frac{2}{21} - \frac{74}{103}t + \frac{256}{15}t^3 - \frac{3392}{45}t^4 + \frac{768}{5}t^5 - \frac{7552}{45}t^6 + \frac{3328}{35}t^7 - \frac{768}{35}t^8$$

$$\beta_{\frac{1}{3}} = -\frac{243}{4480} + \frac{81}{128}t - \frac{243}{10}t^3 + \frac{4779}{40}t^4 - \frac{10449}{40}t^5 + \frac{2997}{10}t^6 - \frac{6156}{35}t^7 + \frac{1458}{35}t^8$$

$$\beta_{\frac{1}{2}} = \frac{17}{105} - \frac{23}{35}t + 16t^3 - \frac{260}{3}t^4 + \frac{1048}{5}t^5 - \frac{3928}{15}t^6 + \frac{1152}{7}t^7 - \frac{288}{7}t^8$$

$$\beta_{\frac{2}{3}} = -\frac{243}{4480} + \frac{1539}{4480}t - \frac{243}{20}t^3 + \frac{1377}{20}t^4 - \frac{7047}{40}t^5 + \frac{2349}{10}t^6 - \frac{5508}{35}t^7 + \frac{1458}{35}t^8$$

$$\beta_{\frac{3}{4}} = \frac{2}{21} - \frac{94}{315}t + \frac{256}{45}t^3 - \frac{1472}{45}t^4 + \frac{256}{3}t^5 - \frac{5248}{45}t^6 + \frac{2816}{35}t^7 - \frac{768}{35}t^8$$

$$\beta_1 = \frac{41}{13440} - \frac{41}{13440}t - \frac{1}{6}t^3 + \frac{71}{72}t^4 - \frac{107}{40}t^5 + \frac{347}{90}t^6 - \frac{20}{7}t^7 + \frac{6}{7}t^8$$

Evaluating (11) at non interpolating points to obtain the continuous form as,

$$\left.\begin{array}{l}
y_n - 2y_{n+\frac{1}{2}} + y_{n+1} = \frac{41}{13440}h^2 f_n + \frac{2}{21}h^2 f_{n+\frac{1}{4}} - \frac{243}{4480}h^2 f_{n+\frac{1}{3}} + \frac{17}{105}h^2 f_{n+\frac{1}{2}} - \frac{243}{4480}h^2 f_{n+\frac{2}{3}} + \frac{2}{21}h^2 f_{n+\frac{3}{4}} + \frac{41}{13440} f_{n+1} \\
y_{n+\frac{1}{4}} - \frac{3}{2}y_{n+\frac{1}{2}} + \frac{1}{2}y_{n+1} = -\frac{193}{1146880}h^2 f_n + \frac{31}{5376}h^2 f_{n+\frac{1}{4}} - \frac{243}{229376}h^2 f_{n+\frac{1}{3}} + \frac{4247}{71680}h^2 f_{n+\frac{1}{2}} - \frac{14823}{1146880}h^2 f_{n+\frac{2}{3}} - \frac{369}{8960}h^2 f_{n+\frac{3}{4}} + \frac{5861}{3440640}h^2 f_{n+1} \\
y_{n+\frac{1}{3}} - \frac{4}{3}y_{n+\frac{1}{2}} + \frac{1}{3}y_{n+1} = -\frac{1727}{14696640}h^2 f_n + \frac{884}{229635}h^2 f_{n+\frac{1}{4}} - \frac{379}{60480}h^2 f_{n+\frac{1}{3}} + \frac{2894}{76545}h^2 f_{n+\frac{1}{2}} - \frac{491}{60480}h^2 f_{n+\frac{2}{3}} + \frac{1252}{45927}h^2 f_{n+\frac{3}{4}} + \frac{16753}{14696640}h^2 f_{n+1} \\
y_n - \frac{2}{3}y_{n+\frac{1}{2}} - \frac{1}{3}y_{n+1} = \frac{3617}{29393280}h^2 f_n - \frac{206}{45927}h^2 f_{n+\frac{1}{4}} + \frac{241}{24192}h^2 f_{n+\frac{1}{3}} - \frac{1237}{76545}h^2 f_{n+\frac{1}{2}} + \frac{1429}{120960}h^2 f_{n+\frac{2}{3}} - \frac{6406}{229635}h^2 f_{n+\frac{3}{4}} + \frac{33343}{293393280} f_{n+1} \\
y_n - \frac{1}{2}y_{n+\frac{1}{2}} + \frac{1}{2}y_{n+1} = -\frac{613}{3440640}h^2 f_n - \frac{173}{26880}h^2 f_{n+\frac{1}{4}} + \frac{16281}{1146880}h^2 f_{n+\frac{1}{3}} + \frac{4667}{215040}h^2 f_{n+\frac{1}{2}} + \frac{29889}{1146880}h^2 f_{n+\frac{2}{3}} - \frac{75}{1792}h^2 f_{n+\frac{3}{4}} - \frac{5827}{3440640} f_{n+1}
\end{array}\right\} \tag{12}$$

Differentiating (12) once, yields

$$y'(t) = \alpha_{\frac{1}{2}}(t) y_{n+\frac{1}{2}} + \alpha_1(t) y_{n+1} + h^2\left[\sum_{j=0}^{1} \beta_j(t) f_{n+j} + \beta_{v_i}(t) f_{n+v_i}\right], v_i = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \tag{13}$$

the coefficient $\alpha'_{\frac{1}{2}}, \alpha'_1, \beta'_0, \beta'_{\frac{1}{4}}, \beta'_{\frac{1}{3}}, \beta'_{\frac{1}{2}}, \beta'_{\frac{2}{3}}, \beta'_{\frac{3}{4}}, \beta'_1$ are given by;

$$\alpha'_{\frac{1}{2}} = -2$$

$$\alpha'_1 = 2$$

$$\beta'_0 = -\frac{2539}{40320} + t - \frac{77}{12}t^2 + \frac{196}{9}t^3 - \frac{1015}{24}t^4 + \frac{707}{15}t^5 - 28t^6 + \frac{48}{7}t^7$$

$$\beta'_{\frac{1}{4}} = -\frac{74}{105} + \frac{256}{5}t^2 - \frac{13568}{45}t^3 + 768t^4 - \frac{15105}{15}t^5 + \frac{3328}{5}t^6 - \frac{6144}{35}t^7$$

$$\beta'_{\frac{1}{3}} = \frac{81}{128} - \frac{729}{10}t^2 + \frac{4779}{10}t^3 - \frac{10449}{8}t^4 + \frac{8991}{5}t^5 - \frac{6156}{5}t^6 + \frac{11664}{35}t^7$$

$$\beta'_{\frac{1}{2}} = -\frac{23}{35} + 48t^2 - \frac{1040}{3}t^3 + 1048t^4 - \frac{7856}{5}t^5 + 1152t^6 - \frac{2304}{7}t^7$$

$$\beta'_{\frac{2}{3}} = \frac{1539}{4480} - \frac{729}{20}t^2 + \frac{1377}{5}t^3 - \frac{7047}{8}t^4 + \frac{7047}{5}t^5 - \frac{5508}{5}t^6 + \frac{11664}{35}t^7$$

$$\beta'_{\frac{3}{4}} = -\frac{94}{315} + \frac{256}{15}t^2 - \frac{5888}{45}t^3 + \frac{1280}{3}t^4 - \frac{10496}{15}t^5 + \frac{2816}{5}t^6 - \frac{6144}{35}t^7$$

$$\beta'_1 = -\frac{41}{13440} - \frac{1}{2}t^2 + \frac{71}{18}t^3 - \frac{107}{8}t^4 + \frac{347}{15}t^5 - 20t^6 + \frac{48}{7}t^7$$

On evaluating (13) at all points, so that the following discrete methods are obtained as

$$
\left.\begin{aligned}
&hy'_n+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{2539}{40320}hf_n-\frac{74}{105}hf_{n+\frac{1}{4}}+\frac{81}{128}hf_{n+\frac{1}{3}}+\frac{23}{25}hf_{n+\frac{1}{2}}+\frac{1539}{4480}hf_{n+\frac{2}{3}}-\frac{94}{315}hf_{n+\frac{3}{4}}+\frac{41}{13440}hf_{n+1}\\
&hy'_{n+\frac{1}{4}}+2y_{n+\frac{1}{2}}-2y_{n+1}=\frac{29}{43008}hf_n-\frac{239}{5040}hf_{n+\frac{1}{4}}-\frac{5913}{71680}hf_{n+\frac{1}{3}}-\frac{851}{3360}hf_{n+\frac{1}{2}}+\frac{567}{10240}hf_{n+\frac{2}{3}}-\frac{93}{560}hf_{n+\frac{3}{4}}-\frac{4379}{645120}hf_{n+1}\\
&hy'_{n+\frac{1}{3}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{259}{466560}hf_n-\frac{106}{8505}hf_{n+\frac{1}{4}}-\frac{1157}{40320}hf_{n+\frac{1}{3}}+\frac{2221}{8505}hf_{n+\frac{1}{2}}+\frac{2411}{40320}hf_{n+\frac{2}{3}}-\frac{4286}{25515}hf_{n+\frac{3}{4}}+\frac{7337}{1088640}hf_{n+1}\\
&hy'_{n+\frac{1}{2}}+2y_{n+\frac{1}{2}}-2y_{n+1}=\frac{17}{20160}hf_n-\frac{2}{63}hf_{n+\frac{1}{4}}+\frac{81}{1120}hf_{n+\frac{1}{3}}-\frac{17}{105}hf_{n+\frac{1}{2}}+\frac{81}{2240}hf_{n+\frac{2}{3}}-\frac{10}{63}hf_{n+\frac{3}{4}}-\frac{1}{144}hf_{n+1}\\
&hy'_{n+\frac{2}{3}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{139}{217728}hf_n+\frac{82}{3645}hf_{n+\frac{1}{4}}+\frac{1963}{40320}hf_{n+\frac{1}{3}}-\frac{533}{8505}hf_{n+\frac{1}{2}}+\frac{5531}{40320}hf_{n+\frac{2}{3}}-\frac{1514}{8505}hf_{n+\frac{3}{4}}-\frac{21739}{3265920}hf_{n+1}\\
&hy'_{n+\frac{3}{4}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{443}{645120}hf_n\,\frac{41}{1680}hf_{n+\frac{1}{4}}+\frac{3807}{71680}hf_{n+\frac{1}{3}}-\frac{79}{1120}hf_{n+\frac{1}{2}}+\frac{13689}{71680}hf_{n+\frac{2}{3}}-\frac{103}{720}hf_{n+\frac{3}{4}}-\frac{1457}{215040}hf_{n+1}\\
&hy'_{n+1}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{41}{13440}hf_n+\frac{34}{315}hf_{n+\frac{1}{4}}-\frac{1053}{4480}hf_{n+\frac{1}{3}}+\frac{1}{3}hf_{n+\frac{1}{2}}-\frac{2349}{4480}hf_{n+\frac{2}{3}}+\frac{18}{35}hf_{n+\frac{3}{4}}+\frac{2293}{40320}hf_{n+1}
\end{aligned}\right\} \tag{14}
$$

Now, combining (12) and (14) to yield the computational hybrid block method, which can be written explicitly as

$$
\left.\begin{aligned}
&y_{n+\frac{1}{4}}=y_n+\frac{1}{4}hy'_n+\frac{129271}{10321920}h^2f_n+\frac{111}{1280}h^2f_{n+\frac{1}{4}}-\frac{120447}{1146880}h^2f_{n+\frac{1}{3}}+\frac{13253}{215040}h^2f_{n+\frac{1}{2}}-\frac{51111}{1146880}h^2f_{n+\frac{2}{3}}+\frac{1657}{80640}h^2f_{n+\frac{3}{4}}-\frac{2011}{3440640}h^2f_{n+1}\\
&y_{n+\frac{1}{3}}=y_n+\frac{1}{3}hy'_n+\frac{32741}{1837080}h^2f_n+\frac{6592}{45927}h^2f_{n+\frac{1}{4}}-\frac{22}{135}h^2f_{n+\frac{1}{3}}+\frac{7268}{76545}h^2f_{n+\frac{1}{2}}-\frac{517}{7560}h^2f_{n+\frac{2}{3}}+\frac{7232}{229635}h^2f_{n+\frac{3}{4}}-\frac{821}{918540}h^2f_{n+1}\\
&y_{n+\frac{1}{2}}=y_n+\frac{1}{2}hy'_n+\frac{2293}{80640}h^2f_n+\frac{9}{35}h^2f_{n+\frac{1}{4}}-\frac{2349}{8960}h^2f_{n+\frac{1}{3}}+\frac{1}{6}h^2f_{n+\frac{1}{2}}-\frac{1053}{8960}h^2f_{n+\frac{2}{3}}+\frac{17}{315}h^2f_{n+\frac{3}{4}}-\frac{41}{26880}h^2f_{n+1}\\
&y_{n+\frac{2}{3}}=y_n+\frac{2}{3}hy'_n+\frac{8968}{229635}h^2f_n+\frac{84992}{229635}h^2f_{n+\frac{1}{4}}-\frac{338}{945}h^2f_{n+\frac{1}{3}}+\frac{19904}{76545}h^2f_{n+\frac{1}{2}}-\frac{22}{135}h^2f_{n+\frac{2}{3}}+\frac{17408}{229635}h^2f_{n+\frac{3}{4}}-\frac{494}{229635}h^2f_{n+1}\\
&y_{n+\frac{3}{4}}=y_n+\frac{3}{4}hy'_n+\frac{50871}{1146880}h^2f_n+\frac{765}{1792}h^2f_{n+\frac{1}{4}}-\frac{465831}{1146880}h^2f_{n+\frac{1}{3}}+\frac{22167}{71680}h^2f_{n+\frac{1}{2}}-\frac{203391}{1146880}h^2f_{n+\frac{2}{3}}+\frac{111}{1280}h^2f_{n+\frac{3}{4}}-\frac{2817}{1146880}h^2f_{n+1}\\
&y_{n+1}=y_n+hy'_n+\frac{151}{2520}h^2f_n+\frac{64}{105}h^2f_{n+\frac{1}{4}}-\frac{81}{140}h^2f_{n+\frac{1}{3}}+\frac{52}{105}h^2f_{n+\frac{1}{2}}-\frac{81}{280}h^2f_{n+\frac{2}{3}}+\frac{64}{315}h^2f_{n+\frac{3}{4}}\\
&y'_{n+\frac{1}{4}}=y'_n+\frac{41059}{645120}hf_n+\frac{3313}{5040}hf_{n+\frac{1}{4}}-\frac{51273}{71680}hf_{n+\frac{1}{3}}+\frac{1357}{3360}hf_{n+\frac{1}{2}}-\frac{4131}{14336}hf_{n+\frac{2}{3}}+\frac{667}{5040}hf_{n+\frac{3}{4}}-\frac{2411}{645120}hf_{n+1}\\
&y'_{n+\frac{1}{3}}=y'_n+\frac{12967}{204120}hf_n+\frac{5888}{8505}hf_{n+\frac{1}{4}}-\frac{1667}{2520}hf_{n+\frac{1}{3}}+\frac{3368}{8505}hf_{n+\frac{1}{2}}-\frac{143}{504}hf_{n+\frac{2}{3}}+\frac{3328}{25515}hf_{n+\frac{3}{4}}-\frac{251}{68040}hf_{n+1}\\
&y'_{n+\frac{1}{2}}=y'_n+\frac{2573}{40320}hf_n+\frac{212}{315}hf_{n+\frac{1}{4}}-\frac{2511}{4480}hf_{n+\frac{1}{3}}+\frac{52}{105}hf_{n+\frac{1}{2}}-\frac{1377}{4480}hf_{n+\frac{2}{3}}+\frac{44}{315}hf_{n+\frac{3}{4}}-\frac{157}{40320}hf_{n+1}\\
&y'_{n+\frac{2}{3}}=y'_n+\frac{541}{8505}hf_n+\frac{17408}{25515}hf_{n+\frac{1}{4}}-\frac{184}{315}hf_{n+\frac{1}{3}}+\frac{5056}{8505}hf_{n+\frac{1}{2}}-\frac{13}{63}hf_{n+\frac{2}{3}}+\frac{1024}{8505}hf_{n+\frac{3}{4}}-\frac{92}{25515}hf_{n+1}\\
&y'_{n+\frac{3}{4}}=y'_n+\frac{4563}{71680}hf_n+\frac{381}{560}hf_{n+\frac{1}{4}}-\frac{41553}{71680}hf_{n+\frac{1}{3}}+\frac{657}{1120}hf_{n+\frac{1}{2}}-\frac{2187}{14336}hf_{n+\frac{2}{3}}+\frac{87}{560}hf_{n+\frac{3}{4}}-\frac{267}{71680}hf_{n+1}\\
&y'_{n+1}=y'_n+\frac{151}{2520}hf_n+\frac{256}{315}hf_{n+\frac{1}{4}}-\frac{243}{280}hf_{n+\frac{1}{3}}+\frac{104}{105}hf_{n+\frac{1}{2}}-\frac{280}{243}hf_{n+\frac{2}{3}}+\frac{256}{315}hf_{n+\frac{3}{4}}-\frac{151}{2520}hf_{n+1}
\end{aligned}\right\} \tag{15}
$$

3. Analysis of Basic Properties of the Method

These properties include; order, error constant, consistency and zero-stability, which reveal the nature of convergence of the method. The region of absolute stability of the method has also been obtained in this section.

3.1 Order and Error Constant of the new method

The new method and the associated linear difference operators are said to have order p if$C_0 = C_1 = \ldots = C_p = C_{p+1} = C_{p+2} = 0, C_{p+3} \neq 0$. The order is also defined as the largest positive real number p that quantifies the rate of convergence of a numerical approximation of a differential equation to that of the exact solution, the term $C_{p+3}$is called the error constant and implies that the local truncation error for the new method is given by,

$$T_{n+k} = C_{p+3}h^{p+3}y^{p+3}(t) + O(h^{p+4}) \tag{16}$$

By definition (1), we compute the new method as

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)^j}{j!} - y_n - \frac{1}{4}hy'_n - \frac{129271}{10321920}hy''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{111}{1280}\left(\frac{1}{4}\right) - \frac{120447}{1146880}\left(\frac{1}{3}\right) + \frac{13253}{215040}\left(\frac{1}{2}\right) - \frac{51111}{1146880}\left(\frac{2}{3}\right) + \frac{1657}{80640}\left(\frac{3}{4}\right) - \frac{2011}{3440640}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^j}{j!} - y_n - \frac{1}{3}hy'_n - \frac{32741}{1837080}hy''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{6592}{45927}\left(\frac{1}{4}\right) - \frac{22}{135}\left(\frac{1}{3}\right) + \frac{7268}{76545}\left(\frac{1}{2}\right) - \frac{517}{7560}\left(\frac{2}{3}\right) + \frac{7232}{229635}\left(\frac{3}{4}\right) - \frac{821}{918540}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j}{j!} - y_n - \frac{1}{2}hy'_n - \frac{2293}{80640}hy''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{9}{35}\left(\frac{1}{4}\right) - \frac{2349}{8960}\left(\frac{1}{3}\right) + \frac{1}{6}\left(\frac{1}{2}\right) - \frac{1053}{8960}\left(\frac{2}{3}\right) + \frac{17}{315}\left(\frac{3}{4}\right) - \frac{41}{26880}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^j}{j!} - y_n - \frac{2}{3}hy'_n - \frac{8968}{229635}hy''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{84992}{229635}\left(\frac{1}{4}\right) - \frac{338}{945}\left(\frac{1}{3}\right) + \frac{19904}{76545}\left(\frac{1}{2}\right) - \frac{22}{135}\left(\frac{2}{3}\right) + \frac{17408}{229635}\left(\frac{3}{4}\right) - \frac{494}{229635}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)^j}{j!} - y_n - \frac{3}{4}hy'_n - \frac{50871}{1146880}hy''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{765}{1792}\left(\frac{1}{4}\right) - \frac{465831}{1146880}\left(\frac{1}{3}\right) + \frac{22167}{71680}\left(\frac{1}{2}\right) - \frac{203391}{1146880}\left(\frac{2}{3}\right) + \frac{111}{1280}\left(\frac{3}{4}\right) - \frac{2817}{1146880}(1)\right] \\ \sum_{j=0}^{\infty}\frac{(1)^j}{j!} - y_n - hy'_n - \frac{151}{2520}hy''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{64}{105}\left(\frac{1}{4}\right) - \frac{81}{140}\left(\frac{1}{3}\right) + \frac{52}{105}\left(\frac{1}{2}\right) - \frac{81}{280}\left(\frac{2}{3}\right) + \frac{64}{315}\left(\frac{3}{4}\right) + 0(1)\right] \end{bmatrix} = 0 \tag{17}$$

The new method is of uniform order 6 with error constant given by

$$C_{p+2} = \left[1.4002\times10^{-08} \quad 1.3861\times10^{-08} \quad -9.1571\times10^{-07} \quad 1.3861\times10^{-08} \quad 1.4002\times10^{-08} \quad 1.3861\times10^{-08}\right].$$

3.2 Consistency of the new method

The new method is said to be consistent since it has order $p \geq 1$.

3.3 Zero-Stability the new method

Applying definition 1.2, the first characteristic polynomial is given by,

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} z & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & -1 \\ 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & z-1 \end{bmatrix} \right| = z^4(z-1)$$

Thus, solving for Z in

$$z^4(z-1) \tag{18}$$

gives $Z = 0,\ 0, 0,\ 0,\ 1$. Hence, the new method is zero-stable.

3.4 Convergence of the New Method

The new method is convergent since it is consistent and zero-stable.

3.5 Region of Absolute Stability of the New Method

Applying the Definition 1.3 on the new method using the boundary locus method, we obtain the stability polynomial of the new method as

$$\bar{h}(w)=\left(-\frac{1}{241920}w^5-\frac{1}{241920}w^6\right)h^{12}+\left(-\frac{11}{103680}w^5-\frac{11}{103680}w^6\right)h^{10}+\left(-\frac{19}{10368}w^5+\frac{7}{4320}w^6\right)h^8$$
$$+\left(-\frac{29}{1728}w^5-\frac{29}{1728}w^6\right)h^6+\left(-\frac{65}{432}w^4+\frac{101}{864}w^5\right)h^4+\left(-\frac{1}{2}w^5-\frac{1}{2}w^6\right)h^2-2w^4+w^5 \quad (19)$$

Using the stability polynomial (19), the region of absolute stability of the new method is shown in Figure 3.1 as

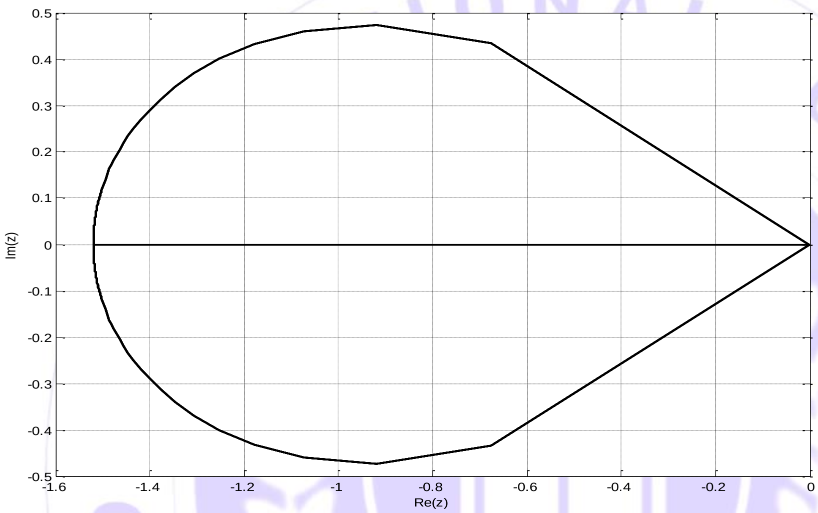

Figure 3.1: Stability region of the new method.

The stability region obtained in Figure 3.1 is $A-stable$.

4. Numerical Examples

Problem 4.1: Consider the Second Order Oscillatory Real-Life Problem (Dynamic Problem)

A 10 kilogram mass is attached to a spring having a spring constant of $140\ N/M$. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(t) = 5\,sin\,t$. Find the subsequent motion of the mass $(t{:}\,0.10 \le t \le 1.00)$ if the force due to air resistance is $90\left(\frac{dy}{dt}\right)N$.

Applying the procedure, where $m = 10,\ k = 140,\ a = 90\ and\ F(t) = 5\,sin\,t$

Problem 4.1 reduces to

$$\frac{d^2y}{dt^2}+9\frac{dy}{dt}+14y(t)=\frac{1}{2}\sin(t),\ y(0)=0,\ y'(0)=-1 \quad (20)$$

with the exact solution of (20) is given by,

$$y(t)=\frac{1}{500}\left(-90e^{-2t}+99e^{-7t}+13\sin t-9\cos t\right) \tag{21}$$

Source [Skwame et al. (2017), Skwame et al., (2020), Sabo, et al., (2021)].

Problem 4.2: Consider the Second Order Oscillatory Real-Life Problem (Cooling of a Body)

The temperature t degrees of a body, x minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2}+\frac{dy}{dt}=0$. By using the substitution $z=\frac{dy}{dt}$ or otherwise, find y in terms of x given that $y=60$ when $t=0$ and $y=35$ when$t=6\,ln\,4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

Formulation of the problem,

$$y''(t)=\frac{-y'}{3},\ y(0)=60,\ y'(0)=-\frac{80}{9},\ h=0.1 \tag{22}$$

with analytic solution

$$y(t)=\frac{80}{3}e^{-\left(\frac{1}{3}\right)t}+\frac{100}{3} \tag{23}$$

Source [Omole and Ogunware (2018), Olanegan, *et al.* (2018)].

Problem 4.3: Consider the Second Order Oscillatory Real-Life Problem (Mass Spring System)

A $128lb$ weight is attached to a spring having a spring constant of $64lb/ft$. The weight is started in motion with no initial velocity by displacing it $6inches$ above the equilibrium position and by simultaneously applying to the weight an external force $F_4(t)=8\,sin\,4\,t$. Assuming no air resistance, compute the subsequent motion of the weight at t: $0.01\le t\le 0.10$.

Now, we model this problem into a mathematical model and then apply our method to compute the motion on the weight attached to the spring. Here,

$$m=4,\ k=64,\ b=0,\ and\ F_4(t)=8\sin 4t$$

Thus, problem 4.3 boils down to

$$\frac{d^2y}{dt^2}+16y=2\sin 4t,\ y(0)=-\frac{1}{2},\ y'(0)=0 \tag{24}$$

with the exact solution of (24) is given by,

$$y(t)=-\frac{1}{2}\cos 4t+\frac{1}{16}\sin 4t-\frac{1}{4}t\cos 4t \tag{25}$$

Source: [Skwame et al., (2017) Sabo, et al., (2021)].

Problem 4.4: Consider the Second Order Oscillatory Real-Life Problem (Simple Harmonic Motion)

An object stretches a spring 6 inches in equilibrium.

i. Set up the equation of motion and find its general solution.

ii. Find the displacement of the object for $t > 0$, if it's initially displaced 18 inches above equilibrium and given a downward velocity of $3\frac{ft}{s}$.

From Newton's second law of motion, we have

$$my''(t)+cy'(t)+ky(t)=F \tag{26}$$

By setting $c = 0$ and $F = 0$, we get

$$my''(t)+ky=0 \Rightarrow y''(t)+\frac{k}{m}y(t)=0 \tag{27}$$

The equation of the weight of the object is given as follow:

$$mg=k\Delta l \Rightarrow \frac{k}{m}=\frac{t}{\Delta l} \tag{28}$$

Substituting $t = 32\frac{ft}{s^2}$, $\Delta l = \frac{6}{12}ft$ into (28) we obtain

$$\frac{k}{m}=\frac{32}{\frac{6}{12}}=64 \tag{29}$$

Substituting equation (29) into the equation (27) we get

$$y''(t)+64y=0 \tag{30}$$

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus, $y(0) = \frac{3}{2}$, $y'(0) = -3$ and $h = 0.1$. We make use of (4.11) as

$$dsolver\left(\left\{y''(t)+64y(t)=0,\ y(0)=\frac{3}{2},\ y'(0)=-3\right\}\right) \tag{31}$$

We obtain the exact solution (31) as

$$y(t)=-\frac{3}{8}\sin(8t)+\frac{3}{2}\cos(8t) \tag{32}$$

Source [Areo and Rufai (2016)].

4.2 Numerical Results

Table 4.1: Showing the absolute error for problem 4.1 using the new method with that of Skwame et al. (2017), Skwame et al., (2020) Sabo, et al., (2021)

t	Exact Solution	Computed Solution	ENM	ESBS17	ESKV21	ESDKSB20
0.1	-0.06436205154552458248	-0.06436205150464058085	4.0884e-11	1.2744e-08	2.0453e-10	4.4268e-09
0.2	-0.08430720522644774945	-0.08430720519437203889	3.2076e-11	3.0442e-08	4.8485e-10	2.2383e-08
0.3	-0.08405225313390041905	-0.08405225311910860351	1.4792e-11	4.1501e-08	6.6174e-10	3.5865e-08
0.4	-0.07529304213333374810	-0.07529304213226985165	1.0639e-12	4.5385e-08	7.2649e-10	4.2157e-08
0.5	-0.06357063960355798563	-0.06357063961082972478	7.2717e-12	4.4298e-08	7.1295e-10	4.2895e-08
0.6	-0.05142117069384508163	-0.05142117070516176084	1.1317e-11	4.0466e-08	6.5550e-10	4.0288e-08
0.7	-0.03993052956438697070	-0.03993052957697073103	1.2584e-11	3.5475e-08	5.7884e-10	3.6051e-08
0.8	-0.02949865862803573900	-0.02949865864031169561	1.2276e-11	3.0285e-08	4.9808e-10	3.1287e-08
0.9	-0.02021269131259124546	-0.02021269132378338304	1.1192e-11	2.5408e-08	4.2140e-10	2.6618e-08
1.0	-0.01202699425403169607	-0.01202699426384207333	9.8104e-12	2.1071e-08	3.5257e-10	2.2352e-08

Table 4.2: Showing the absolute error for problem 4.2 using the new method with that of Omole and Ogunware, (2018), Olanegan et al. (2018)

t	ES	CS	ENM	EOO18	EOOA18
0.1	59.12576267952015738700	59.12576267952015738700	0.0000e00	3.5500e-11	7.4764e-06
0.2	58.28018626750980633900	58.28018626750980633900	0.0000e00	4.5800e-11	2.9394e-05
0.3	57.46233114762558861700	57.46233114762558861700	0.0000e00	7.0000e-11	6.4802e-05
0.4	56.67128850781193210600	56.67128850781193210600	0.0000e00	6.5000e-12	1.1279e-05
0.5	55.90617933041637530700	55.90617933041637530700	0.0000e00	3.3300e-11	1.7250e-04
0.6	55.16615341541284956400	55.16615341541284956400	0.0000e00	4.2000e-11	2.4310e-04
0.7	54.45038843564751105000	54.45038843564751105000	0.0000e00	4.3800e-11	3.2383e-04
0.8	53.75808902305729847200	53.75808902305729847200	0.0000e00	1.0700e-10	4.1393e-04
0.9	53.08848588484580976200	53.08848588484580976200	0.0000e00	6.5800e-11	5.1271e-04
1.0	52.44083494863438001100	52.44083494863438001100	0.0000e00	1.6900e-10	6.1951e-04

Table 4.3: Showing the absolute error for problem 4.3 using the new method with that of Skwame et al., (2017) Sabo, et al., (2021)

t	Exact Solution	Computed Solution	NM	ESBS17	ESKV21
0.01	- 0.49959872021047678004	-0.49959872021047678004	0.0000e00	1.6621e-09	1.0000e-19
0.02	- 0.49839019330974949646	-0.49839019330974949647	1.0000e-20	1.1586e-08	4.1000e-19
0.03	- 0.49636836974027966301	-0.49636836974027966304	3.0000e-20	2.9743e-08	9.1000e-19
0.04	- 0.49352852660817937130	-0.49352852660817937133	3.0000e-20	5.6076e-08	1.6600e-18
0.05	- 0.48986728796894500998	-0.48986728796894501001	3.0000e-20	9.0504e-08	2.6200e-18
0.06	- 0.48538264289709933476	-0.48538264289709933479	3.0000e-20	1.3291e-07	3.8000e-18
0.07	- 0.48007396129056685722	-0.48007396129056685725	3.0000e-20	1.8317e-07	5.2000e-18
0.08	- 0.47394200736436189072	-0.47394200736436189075	3.0000e-20	2.4110e-07	6.8500e-18
0.09	- 0.46698895079202783994	-0.46698895079202783997	3.0000e-20	3.0653e-07	8.7500e-18
0.1	- 0.45921837545722401274	-0.45921837545722401277	3.0000e-20	1.6621e-09	1.0850e-17

Table 4.4: Showing the absolute error for problem 4.4 using the new methods with that of Areo and Rufai, (2016)

t	Exact Solution	Computed Solution	ENM	EAR16
0.1	0.77605152993342709579	0.77605153113454792281	1.2011e-09	3.3496e-07
0.2	- 0.41863938459249752594	-0.41863938218202686939	2.4105e-09	1.6371e-06
0.3	- 1.3593892660185498469	-1.3593892642583606346	1.7602e-09	3.2716e-06
0.4	- 1.4755518599067871611	-1.4755518611552423129	1.2485e-09	3.5979e-06
0.5	- 0.69666449555494477770	-0.69666450045569309273	4.9008e-09	1.3589e-06
0.6	0.50481020347261010590	0.50481019723085707534	6.2416e-09	2.9143e-06
0.7	1.4000738069674951883	1.4000738036501870356	3.3173e-09	6.7226e-06
0.8	1.4460714263183540043	1.4460714292670085498	2.9487e-09	7.0589e-06
0.9	0.61490152285494961183	0.61490153165390767581	8.7989e-09	2.6543e-06
1.0	- 0.58925939319668845548	-0.58925938330094591132	9.8957e-09	4.6056e-06

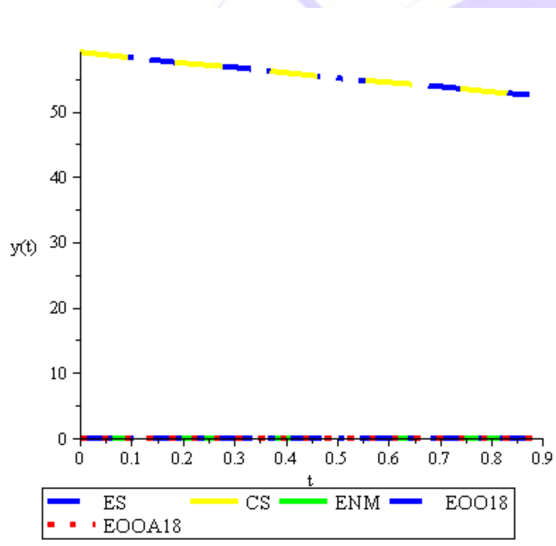

Figure 4.1: Graphical curve of table 4.1

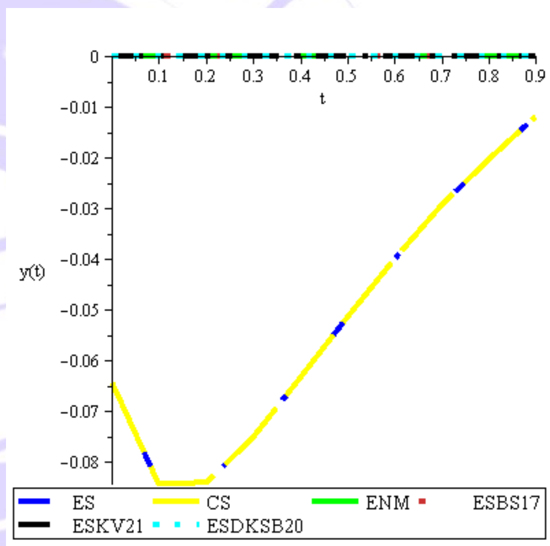

Figure 4.2: Graphical curve of table 4.2

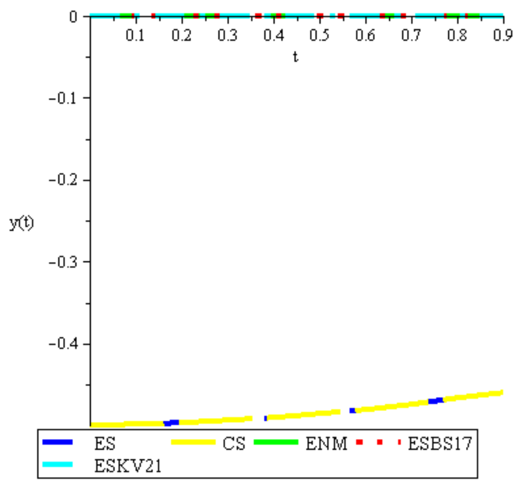

Figure 4.3: Graphical curve of Table 4.3

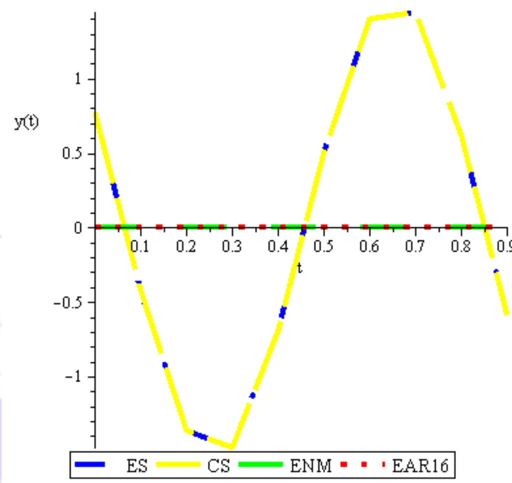

Figure 4.4: Graphical curve of Table 4.4

4.3 Discussion of Results

Table 4.1 presents the computed solutions alongside the exact solutions for Problem 4.1, showing minimal absolute errors for the new method. The errors are consistently smaller compared to those from Skwame et al. (2017), Skwame et al. (2020), and Sabo et al. (2021). The data indicates the new method's superior precision, even for small time increments. Figure 4.1 illustrating the errors further demonstrate that the new method's error curve is nearly flat, signifying a stable solution across the time interval.

Table 4.2 highlights the analytical and computed solutions for Problem 4.2, where the new method achieves zero absolute error (ENM) at every time step. This level of accuracy is unmatched by the methods of Omole and Ogunware (2018) and Olanegan et al. (2018), which show increasing errors. The graphical representation of the temperature decay confirms the new method's precise tracking of the exact solution, emphasizing its capability to handle real-life cooling problems effectively (see figure 4.2).

For Problem 4.3, Table 4.3 shows the absolute errors in the computed solutions using the new method compared with Skwame et al. (2017) and Sabo et al. (2021). The errors from the new method are orders of magnitude smaller, demonstrating its high accuracy. The graphs in figure 4.3 reveal the oscillatory behavior of the mass-spring system, with the new method's solution closely overlapping the exact solution, while competing methods deviate slightly at higher time intervals.

In Problem 4.4, the absolute errors presented in the table confirm the new method's consistency and accuracy. The displacement and velocity calculations align with the exact solution, outperforming Areo and Rufai (2016). The figure 4.4 is the graph of displacement over time exhibit smooth harmonic oscillations, validating the new method's capability to solve problems involving periodic motion.

The tables and graphs collectively demonstrate the new method's efficiency in minimizing computational errors across all problems. The method's ability to produce negligible errors, even for complex oscillatory and dynamic systems, highlights its robustness. The graphical comparisons between exact and computed solutions, along with error curves, provide visual evidence of the method's precision and stability. This analysis underscores the potential of the new block hybrid method as a reliable tool for solving second-order real-life initial value problems in ordinary differential equations.

5. Conclusion

This research focuses on enhancing numerical solutions for oscillatory second-order initial value problems (IVPs) using block hybrid methods. Oscillatory IVPs are challenging to solve accurately due to their unique characteristics, which often lead to numerical instability when traditional methods are applied. The block hybrid method proposed in this study addresses these limitations by offering a solution that combines accuracy, stability and a broader convergence region. Through error analysis and comparative performance evaluations, the method has been shown to outperform existing techniques, providing a reliable approach to solving oscillatory problems commonly encountered in fields like physics, engineering and applied sciences. The findings of this study underscore the advantages of the block hybrid method for tackling oscillatory second-order IVPs, as it maintains stability and accuracy where traditional methods may falter. By expanding the stability region and improving error control, this method demonstrates considerable potential for real-world applications requiring precision in oscillatory contexts. The research provides a strong foundation for future exploration into block hybrid methods, suggesting that further refinement could make these techniques even more effective and widely applicable across scientific disciplines.

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