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Two Higher-Order Single-Step Embedded Block Hybrid Integrators for First-Order Ordinary Differential Equations

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ABSTRACT

In this research work, the extended Backward Differentiation Formula (EBDF) is used to develop an embedded block hybrid method (EBHM) for step numbers $k = 5$ and 6 for the solution of first-order initial-value problems (IVPs) in ordinary differential equations (ODEs). The methods are obtained by continuous approximation using multi-step collocation techniques. The new embedded block hybrid method for $k = 5$ $(EBHM5)$ and $k = 6$ $(EBHM6)$ consist of five and six discrete formulae, respectively, which are simultaneously used as integrators. Analyses of the EBHM5 and EBHM6 indicate that the methods are respectively of orders six and seven, hence consistent. In addition, both methods are zero-stable and therefore convergent. Linear stability analyses indicate that the methods are A-stable, hence suitable for stiff problems. The new methods are used to solve some numerical examples, and the absolute errors show a better rate of convergence when compared with existing methods in the literature.

1. Introduction

The wide applications of ordinary differential equations (ODEs) in the natural sciences, technology, physics, economics, engineering, mechanics, astronomy, biology, the computation of radio technical circuits or satellite trajectories, stability of a plane flight, and the theory of oscillations (Atabo & Adee, 2021) make ODEs active area for research especially as researchers continue to seek for more accurate solutions to such problems.

However, in most cases, analytical methods of solution to ODEs are difficult to come by or are applicable to a limited class of equations. Many times, ODEs appearing in real life do not fall under the 'limited class of equations', therefore, one is obliged to use numerical methods. Numerical methods are of great advantages and importance, due to the introduction of computing machines which makes numerical work much easier and faster.

We seek an approximate solution to first-order ODE as an initial value problem (IVP) of the form

$$y' = f(x, y(x)), y(x_0) = y_0 \quad (1)$$

on the interval $[x_0, x_N]$. Several scholars proposed numerical methods for solving (1). They include Aboiyar & Luga (2015), Abdelrahim *et al*. (2016), Adesanya *et al.* (2012), Adeyefa & Fadaka (2017), Abolarin *et al.* (2020), Akinfenwa *et al.* (2011), Bothayna & Muhammed (2019), Chu & Hamilton (1987), Fotta *et al.* (2015) and Akinfenwa & Jator (2015) among others. Among these, Akinfenwa & Jator (2015) developed an extended continuous block backward differentiation formula of step number $k = 4$ (ECBDF4) for the solution of (1), which was now embedded into a single step (i.e. $k = 1$) by Adee *et al*. (2022) for the EHBM which yielded smaller absolute errors, hence better performance.

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Extending the work of Adee *et al.* (2022) and in the context of Akinfenwa and Jator (2015) for $k = 4$, this study seeks to develop and implement higher single-step methods by constructing similar one-step embedded block methods for step numbers $k = 5$ and 6 respectively.

2. Derivation of the Proposed New Methods

For $k = 5$ (EBHM5)

Adee *et al.* (2022) defined the general one-step embedded block hybrid method as

$$y(t) = \sum_{j=0}^{\frac{k-1}{k}} \alpha_{\frac{j}{k}}(t) y_{n+\frac{j}{k}} + h\left[\beta_{\frac{k-1}{k}}(t) f_{n+\frac{k-1}{k}} + \beta_1(t) f_{n+1}\right] \tag{2}$$

where $\alpha_j(t), j = 0, \dots, \frac{k-1}{k}, \beta_{\frac{k-1}{k}}$ and β_1 are continuous coefficients to be determined.

They also considered an approximate polynomial $p(x)$ of the form

$$p(x) = \sum_{j=0}^{t+m} \alpha_j x^j \tag{3}$$

as basis function, where t and m are the interpolation and collocation points respectively and $\alpha_j \in R$, are coefficients to be determined. The idea is to approximate the exact solution $y(x)$ of (1) in the interval $[a, b]$ with partitions at a constant step size of $h = x_i - x_{i-1}, i = 0,1, \dots, N$. Taking the first derivative of (3) yields

$$p'(x) = \sum_{j=0}^{t+m} j\alpha_j x^{j-1} \tag{4}$$

Furthermore, they interpolated (3) at $x_{n+t}, t = 0, \lambda_i$ and collocated (4) at $x_{n+m}, m = 0, \lambda_i, k$, where i is the off-grid points and k is the step number.

Now, fix $k = 5$ in (2), evaluating (3) at the points $x_{n+\frac{i}{5}}, i = 0(1)4$ and (4) at the points $x_{n+\frac{i}{5}}, i = 4,5$, we generate a system of seven linear algebraic equations expressed in matrix form as

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+\frac{1}{5}} & x^2_{n+\frac{1}{5}} & x^3_{n+\frac{1}{5}} & x^4_{n+\frac{1}{5}} & x^5_{n+\frac{1}{5}} & x^6_{n+\frac{1}{5}} \\ 1 & x_{n+\frac{2}{5}} & x^2_{n+\frac{2}{5}} & x^3_{n+\frac{2}{5}} & x^4_{n+\frac{2}{5}} & x^5_{n+\frac{2}{5}} & x^6_{n+\frac{2}{5}} \\ 1 & x_{n+\frac{3}{5}} & x^2_{n+\frac{3}{5}} & x^3_{n+\frac{3}{5}} & x^4_{n+\frac{3}{5}} & x^5_{n+\frac{3}{5}} & x^6_{n+\frac{3}{5}} \\ 1 & x_{n+\frac{4}{5}} & x^2_{n+\frac{4}{5}} & x^3_{n+\frac{4}{5}} & x^4_{n+\frac{4}{5}} & x^5_{n+\frac{4}{5}} & x^6_{n+\frac{4}{5}} \\ 0 & 1 & 2x_{n+\frac{4}{5}} & 3x^2_{n+\frac{4}{5}} & 4x^3_{n+\frac{4}{5}} & 5x^4_{n+\frac{4}{5}} & 6x^5_{n+\frac{4}{5}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} \tag{5}$$

Solving for the unknowns, $a_j, j = 0(1)6$ in (5) and after some simplifications, we obtain the continuous method of the form

$$p(t)=\sum_{j=0}^{4}\alpha_{\frac{j}{5}}(t)y_{n+\frac{j}{5}}+\sum_{j=4}^{5}\beta_{\frac{j}{5}}(t)f_{n+\frac{j}{5}} \tag{6}$$

where $t=x-x_n, 0\le t\le 1$,

$$\alpha_0(t)=-\frac{7450}{591h}+\frac{571475t}{4728h^2}-\frac{2725625t^2}{6304h^3}+\frac{3458125t^3}{4728h^4}-\frac{11171875t^4}{18912h^5}+\frac{578125t^5}{3152h^6}$$

$$\alpha_{\frac{1}{5}}(t)=\frac{19400}{591h}-\frac{885200t}{1773h^2}+\frac{869125t^2}{394h^3}-\frac{7466875t^3}{1773h^4}+\frac{2171875t^4}{591h^5}-\frac{1421875t^5}{1182h^6}$$

$$\alpha_{\frac{2}{5}}(t)=-\frac{9450}{197h}+\frac{190950t}{197h^2}-\frac{7971375t^2}{1576h^3}+\frac{4243125t^3}{394h^4}+\frac{15984375t^4}{1576h^5}+\frac{27656625t^5}{788h^6}$$

$$\alpha_{\frac{3}{5}}(t)=\frac{35800}{591h}-\frac{782800t}{591h^2}+\frac{3017375t^2}{394h^3}-\frac{10536875t^3}{591h^4}+\frac{10671875t^4}{591h^5}-\frac{2609375t^5}{394h^6}$$

$$\alpha_{\frac{4}{5}}(t)=-\frac{19400}{591h}+\frac{10405975t}{14184h^2}-\frac{27572875t^2}{6304h^3}+\frac{149493125t^3}{14184h^4}-\frac{208015625t^4}{18912h^5}+\frac{39078125t^5}{9456h^6}$$

$$\beta_{\frac{4}{5}}(t)=\frac{745}{197}-\frac{101345t}{1182}+\frac{823125t^2}{1576h^2}-\frac{1530625t^3}{1182h^3}+\frac{2203125t^4}{1576h^4}-\frac{428125t^5}{788h^5}$$

$$\beta_1(t)=-\frac{48}{197}+\frac{1120t}{197h}-\frac{7125t^2}{197h^2}+\frac{18750t^3}{197h^3}-\frac{21875t^4}{197h^4}+\frac{9375t^5}{197h^5}$$

Evaluate (6) at $t=h$, its derivative, $p'(t)$ at $t=\frac{h}{5},\frac{2h}{5},\frac{3h}{5},0$ and after some rearrangement, we obtain the five discrete formulae:

$$y_{n+1}=-\frac{3}{197}y_n+\frac{25}{197}y_{n+\frac{1}{5}}-\frac{100}{197}y_{n+\frac{2}{5}}+\frac{300}{197}y_{n+\frac{3}{5}}-\frac{25}{197}y_{n+\frac{4}{5}}+\frac{60}{197}hf_{n+\frac{4}{5}}+\frac{12}{197}hf_{n+1}$$

$$y_{n+\frac{1}{5}}=-\frac{228}{2065}hf_{n+\frac{4}{5}}+\frac{27}{4130}hf_{n+1}+y_{n+\frac{4}{5}}-\frac{117}{59}y_{n+\frac{3}{5}}+\frac{864}{413}y_{n+\frac{2}{5}}-\frac{45}{413}y_n-\frac{591}{4130}hf_{n+\frac{1}{5}}$$

$$y_{n+\frac{2}{5}}=\frac{157}{1575}hf_{n+\frac{4}{5}}-\frac{8}{1575}hf_{n+1}-\frac{529}{540}y_{n+\frac{4}{5}}+\frac{748}{315}y_{n+\frac{3}{5}}-\frac{404}{945}y_{n+\frac{1}{5}}+\frac{41}{1260}y_n-\frac{394}{1575}hf_{n+\frac{2}{5}}$$

$$y_{n+\frac{3}{5}}=-\frac{501}{3490}hf_{n+\frac{4}{5}}+\frac{9}{1745}hf_{n+1}+\frac{4895}{2792}y_{n+\frac{4}{5}}-\frac{621}{698}y_{n+\frac{2}{5}}+\frac{53}{349}y_{n+\frac{1}{5}}-\frac{43}{2792}y_n-\frac{591}{1745}hf_{n+\frac{3}{5}}$$

$$y_{n+\frac{4}{5}}=\frac{447}{3880}hf_{n+\frac{4}{5}}-\frac{18}{2425}hf_{n+1}+\frac{179}{97}y_{n+\frac{3}{5}}-\frac{567}{388}y_{n+\frac{2}{5}}+y_{n+\frac{1}{5}}-\frac{149}{388}y_n-\frac{591}{19400}hf_n \tag{7}$$

The discrete methods in (7) constitute the one-step embedded hybrid block method for $k=5$ (EBHM5) for the solution of (1).

For $k = 6$ (EBHM6)

Similarly, fix $k = 6$ in (2), evaluating (3) at the points $x_{n+\frac{i}{6}}, i = 0(1)5$ and (4) at the points $x_{n+\frac{i}{6}}, i = 5,6$ generates a system of seven algebraic equations of the form

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+\frac{1}{6}} & x^2_{n+\frac{1}{6}} & x^3_{n+\frac{1}{6}} & x^4_{n+\frac{1}{6}} & x^5_{n+\frac{1}{6}} & x^6_{n+\frac{1}{6}} & x^7_{n+\frac{1}{6}} \\ 1 & x_{n+\frac{1}{3}} & x^2_{n+\frac{1}{3}} & x^3_{n+\frac{1}{3}} & x^4_{n+\frac{1}{3}} & x^5_{n+\frac{1}{3}} & x^6_{n+\frac{1}{3}} & x^7_{n+\frac{1}{3}} \\ 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} & x^7_{n+\frac{1}{2}} \\ 1 & x_{n+\frac{2}{3}} & x^2_{n+\frac{2}{3}} & x^3_{n+\frac{2}{3}} & x^4_{n+\frac{2}{3}} & x^5_{n+\frac{2}{3}} & x^6_{n+\frac{2}{3}} & x^7_{n+\frac{2}{3}} \\ 1 & x_{n+\frac{5}{6}} & x^2_{n+\frac{5}{6}} & x^3_{n+\frac{5}{6}} & x^4_{n+\frac{5}{6}} & x^5_{n+\frac{5}{6}} & x^6_{n+\frac{5}{6}} & x^7_{n+\frac{5}{6}} \\ 0 & 1 & 2x_{n+\frac{5}{6}} & 3x^2_{n+\frac{5}{6}} & 4x^3_{n+\frac{5}{6}} & 5x^4_{n+\frac{5}{6}} & 6x^5_{n+\frac{5}{6}} & 7x^6_{n+\frac{5}{6}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{2}{3}} \\ y_{n+\frac{5}{6}} \\ f_{n+\frac{5}{6}} \\ f_{n+1} \end{pmatrix} \tag{8}$$

Solving for the unknowns $a_j, j = 0(1)7$ in (8) leads to the continuous interpolant of the form

$$p(t) = \sum_{j=0}^{5} \alpha_{\frac{j}{6}}(t) y_{n+\frac{j}{6}} + \sum_{j=5}^{6} \beta_{\frac{j}{6}}(t) f_{n+\frac{j}{6}} \tag{9}$$

where $t = x - x_n, 0 \le t \le 1,$

$$\alpha_0(t) = -\frac{32813}{2070h} + \frac{1015544t}{5175h^2} - \frac{1082903t^2}{1150h^3} + \frac{261416t^3b}{115h^4} - \frac{67632t^4}{23h^5} + \frac{1119312t^5}{575h^6} - \frac{297864t^6}{575h^7}$$

$$\alpha_{\frac{1}{6}}(t) = \frac{1}{23}\left(\frac{1025}{h} - \frac{20195t}{h^2} + \frac{119943t^2}{h^3} - \frac{327222t^3}{h^4} + \frac{457785t^4}{h^5} - \frac{319788t^5}{h^6} + \frac{88452t^6}{h^7}\right)$$

$$\alpha_{\frac{1}{3}}(t) = \frac{1}{23}\left(\frac{5050}{3h} + \frac{129790t}{3h^2} - \frac{300732t^2}{h^3} + \frac{909888t^3}{h^4} - \frac{1368720t^4}{h^5} + \frac{1008288t^5}{h^6} - \frac{290304t^6}{h^7}\right)$$

$$\alpha_{\frac{1}{2}}(t) = \frac{1}{23}\left(\frac{19700}{9h} - \frac{545660t}{9h^2} + \frac{462196t^2}{h^3} - \frac{1513912t^3}{h^4} + \frac{2428620t^4}{h^5} - \frac{1883952t^5}{h^6} + \frac{565488t^6}{h^7}\right)$$

$$\alpha_{\frac{2}{3}}(t) = \frac{1}{23}\left(-\frac{4675}{2h} + \frac{67070t}{h^2} - \frac{1073241t^2}{2h^3} + \frac{1857528t^3}{h^4} - \frac{3144960t^4}{h^5} + \frac{2562192t^5}{h^6} - \frac{802872t^6}{h^7}\right)$$

$$\alpha_{\frac{5}{6}}(t) = \frac{17573}{345h} - \frac{2551723t}{1725h^2} + \frac{6921789t^2}{575h^3} - \frac{4892826t^3}{115h^4} + \frac{1694907t^4}{23h^5} - \frac{35287812t^5}{575h^6} + \frac{11278764t^6}{575h^7}$$

$$\beta_{\frac{5}{6}}(t) = -\frac{3145}{69} + \frac{45958t}{345h} - \frac{126114t^2}{115h^2} + \frac{90516t^3}{23h^3} - \frac{159750t^4}{23h^4} + \frac{679752t^5}{115h^5} - \frac{222264t^6}{115h^6}$$

$$\beta_1(t) = \frac{50}{207} - \frac{1490t}{207h} + \frac{1399t^2}{23h^2} - \frac{5200t^3}{23h^3} + \frac{9600t^4}{23h^4} - \frac{8640t^5}{23h^5} + \frac{3024t^6}{23h^6}$$

Evaluate (9) at $t = h$, its derivative, $p'(t)$ at $t = \frac{h}{6}, \frac{h}{3}, \frac{h}{2}, \frac{2h}{3}, 0$ and after some rearrangements yield six discrete formulae:

$$y_{n+1} = \frac{1}{23}\left(\frac{2}{9}y_n - 2y_{n+\frac{1}{6}} + \frac{25}{3}y_{n+\frac{1}{3}} - \frac{200}{9}y_{n+\frac{1}{2}} + 50y_{n+\frac{2}{3}} - \frac{34}{3}y_{n+\frac{5}{6}} + \frac{20}{3}hf_{n+\frac{5}{6}} + \frac{10}{9}hf_{n+1}\right)$$

$$y_{n+\frac{1}{6}} = \frac{53}{630}hf_{n+\frac{5}{6}} - \frac{4}{945}hf_{n+1} - \frac{217}{225}y_{n+\frac{5}{6}} + 2y_{n+\frac{2}{3}} + \frac{136}{63}y_{n+\frac{1}{3}} - \frac{398}{189}y_{n+\frac{1}{2}} - \frac{418}{4725}y_n - \frac{23}{210}hf_{n+\frac{1}{6}}$$

$$y_{n+\frac{1}{3}} = -\frac{9}{155}hf_{n+\frac{5}{6}} + \frac{1}{372}hf_{n+1} + \frac{1073}{1550}y_{n+\frac{5}{6}} - \frac{48}{31}y_{n+\frac{2}{3}} - \frac{21}{62}y_{n+\frac{1}{6}} + \frac{202}{93}y_{n+\frac{1}{2}} + \frac{53}{2325}y_n - \frac{23}{124}hf_{n+\frac{1}{3}}$$

$$y_{n+\frac{1}{2}} = \frac{501}{5810}hf_{n+\frac{5}{6}} - \frac{2}{581}hf_{n+1} - \frac{9201}{8300}y_{n+\frac{5}{6}} + \frac{1773}{581}y_{n+\frac{2}{3}} + \frac{387}{2324}y_{n+\frac{1}{6}} - \frac{636}{581}y_{n+\frac{1}{3}} - \frac{31}{2075}y_n - \frac{207}{581}hf_{n+\frac{1}{2}}$$

$$y_{n+\frac{2}{3}} = -\frac{236}{1425}hf_{n+\frac{5}{6}} + \frac{4}{855}hf_{n+1} + \frac{18566}{7125}y_{n+\frac{5}{6}} - \frac{2}{19}y_{n+\frac{1}{6}} + \frac{148}{285}y_{n+\frac{1}{3}} - \frac{1736}{855}y_{n+\frac{1}{2}} + \frac{227}{21375}y_n - \frac{46}{95}hf_{n+\frac{2}{3}}$$

$$y_{n+\frac{5}{6}} = \frac{1570}{17573}hf_{n+\frac{5}{6}} - \frac{250}{52719}hf_{n+1} + \frac{70125}{35146}y_{n+\frac{2}{3}} - \frac{15375}{17573}y_{n+\frac{1}{6}} + \frac{25250}{17573}y_{n+\frac{1}{3}} - \frac{98500}{52719}y_{n+\frac{1}{2}} + \frac{32813}{105438}y_n + \frac{345}{17573}hf_n \quad (10)$$

Again, the discrete methods in (10) constitute the one-step embedded hybrid block method for $k = 6$ (EBHM6) for the solution of (1).

3. Analysis of The New Block Methods

Error constant and order

The local truncation error of (2) is the linear operator L, given as

$$L[y(x);h] = \sum_{j=0}^{\frac{k-1}{k}} \alpha_{\frac{j}{k}} y\left(x + \frac{j}{k}h\right) - h\beta_{\frac{k-1}{k}} y'\left(x + \frac{k-1}{k}h\right) - h\beta_1 y'(x+h) \quad (11)$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. By expanding $y\left(x + \frac{j}{k}h\right)$ and the derivatives $y'\left(x + \frac{k-1}{k}h\right)$ and $y'(x+h)$ using Taylor series about x and collecting the like terms in (11) we obtain

$$L[y(x);h] = \sum_{j=0}^{k} c_j h^j y^j(x) = c_0 y(x) + c_1 h y'(x) + \cdots + c_q h^q y^x(x) + \cdots$$

where

$$c_0 = \sum_{j=0}^{k} \alpha_{\frac{j}{k}}$$

$$c_1 = \sum_{j=0}^{k} \left(\frac{j}{k} \alpha_{\frac{j}{k}} - \beta_{\frac{j}{k}} \right)$$

$$\vdots$$

$$c_q = \sum_{j=0}^{k} \left(\frac{\left(\frac{j}{k}\right)^q}{q!} \alpha_{\frac{j}{k}} - \frac{\left(\frac{j}{k}\right)^{(q-1)}}{(q-1)!} \beta_{\frac{j}{k}} \right), q = 2,3,\ldots \tag{12}$$

Therefore, the method (1) is said to be of order p if $c_0 = c_1 = c_2 = \cdots = c_p = 0$ but $c_{p+1} \neq 0$ and c_{p+1} is called the error constant (Adee *et al.* 2022). Following from (11) and (12), the error constants and order for EBHM5 and EBHM6 are given in Table 1 and Table 2 respectively.

Table 1: Error constants and order for EBHM5

Method	Order	Error constant
y_{n+1}	6	$-\frac{2}{21556875}$
$y_{n+\frac{1}{5}}$	6	$-\frac{159}{1129296875}$
$y_{n+\frac{2}{5}}$	6	$\frac{31}{369140625}$
$y_{n+\frac{3}{5}}$	6	$-\frac{227}{3817187500}$
$y_{n+\frac{4}{5}}$	6	$\frac{209}{1060937500}$

Table 2: Error constants and order for EBHM6

Method	Order	Error constant
y_{n+1}	7	$-\frac{5}{1622509056}$
$y_{n+\frac{1}{6}}$	7	$\frac{73}{14814213120}$
$y_{n+\frac{1}{3}}$	7	$-\frac{37}{14579066880}$
$y_{n+\frac{1}{2}}$	7	$\frac{227}{91079976960}$
$y_{n+\frac{2}{3}}$	7	$-\frac{79}{33508339200}$
$y_{n+\frac{5}{6}}$	7	$\frac{775}{118063567872}$

Tables 1 and 2 indicate that EBHM5 and EBHM6 are consistent respectively, having orders 6 and 7 accordingly.

3.1 Stability Analysis

The stability analysis of the methods EBHM5 and EBHM6 are carried out by applying the test condition $y' = \lambda y$ on each method. The stability polynomial of EBHM5 yields

$$\begin{aligned}\rho_5(r,z) = &-\frac{68808357}{3058406968750}r^4z^5 + \frac{206425071}{195738046}r^5z - \frac{1169742069}{4893451150}r^5z^2 + \frac{619275213}{19573804600}r^5z^3 \\ &- \frac{3142248303}{1223362787500}r^5z^4 + \frac{68808357}{611681393750}r^5z^5 + \frac{619275213}{978690230}r^4z + \frac{68808357}{978690230}r^4z^2 \\ &+ \frac{206425071}{97869023000}r^4z^3 - \frac{298169547}{12233627875}r^4z^4 - \frac{206425071}{97869023}r^5 + \frac{206425071}{97869023}r^4\end{aligned} \tag{13}$$

Similarly, the stability of EBHM6 gives

$$\begin{aligned}\rho_6(r,z) = &\frac{839523}{42095471299}r^6z^6 - \frac{2518569}{4295456255}r^6z^5 + \frac{279841}{84190942598}r^5z^6 + \frac{292154004}{30068193785}r^6z^4 \\ &+ \frac{3078251}{60136387570}r^{5z^5} - \frac{90668484}{859091251}r^{6z^3} - \frac{2518569}{8590912510}r^{5z^4} + \frac{4533424200}{6013638757}r^6z^2 \\ &- \frac{120891312}{6013638757}r^5z^3 - \frac{19584392544}{6013638757}r^6z - \frac{1813369680}{6013638757}r^5z^2 + \frac{39168785088}{6013638757}r^6 \\ &- \frac{13056261696}{6013638757}r^5z - \frac{39168785088}{6013638757}r^5\end{aligned} \tag{14}$$

3.1.1 Zero stability

A linear multistep method (LMM) is said to be zero stable if no roots of the characteristic polynomial have modulus greater than one, $|z_r| \leq 1$, *and every root of modulus one is simple and distinct* (Endre & David, 2003).

Therefore, setting $z = 0$ in (13) yields

$$\rho_5(r,0) = -\frac{206425071}{97869023}r^4(r-1) \tag{15}$$

Finding the zeros of (15) and their moduli leads to $|r_1| = 1, |r_{2,3,4,5}| = 0$ which confirms that EBHM5 is zero-stable. Likewise, by setting $z = 0$ in (14) gives

$$\rho_6(r,0) = \frac{39168785088}{6013638757}r^5(r-1) \tag{16}$$

The zeros of (16) and their moduli yield $|r_1| = 1, |r_{2,3,4,5,6}| = 0$ which confirms that EBHM6 is also zero-stable.

3.1.2 Convergence

The necessary and sufficient condition for LMM to be convergent is that it must be consistent and zero-stable (Endre & David, 2003).

Hence, both methods (7) and (10) are convergent since they are all consistent and zero-stable.

3.1.3 Region of Absolute stability

The region of absolute stability of EBHM5 and EBHM6 are obtained by plotting stability polynomials (13) and (14) respectively with its derivatives in written MATLAB codes. These are jointly shown in shown in Figure 1.

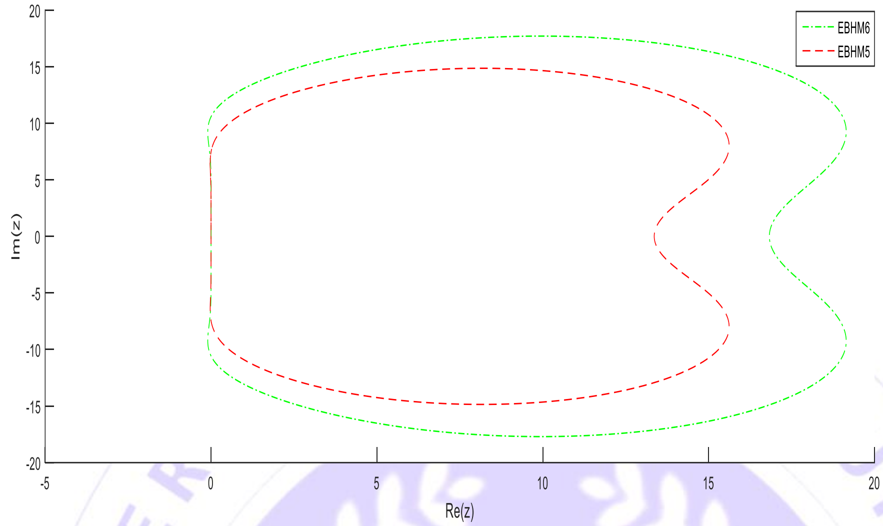

Figure 1: Regions of absolute stability for EBHM5 and EBHM6.

By definition, *a numerical method is said to be absolutely stable if its region of absolute stability contains the whole of the left-hand plane* $(Re\lambda h < 0)$ (Dahlquist, 1963).

Figure 1 clearly shows that EBHM5 and EBHM6 of eq. (7) and (10) respectively are absolutely stable with EBHM5 possessing a larger region of absolute stability than EBHM6.

4. Numerical Examples

To test the efficiency of the methods EBHM5 (7) and EBHM6 (10), we implemented each of them as a block integrator that simultaneously yields approximate solutions $y_{n+1-i\mu}$ to the exact solution $y(x_{n+1-i\mu})$ for $i = 0(1)5, \mu = \frac{1}{5}$ in the case of EBHM5 and $i = 0(1)6, \mu = \frac{1}{6}$ for EBHM6 on

$$\pi_N: a = x_0 < x_1 < x_2 \dots < x_n < x_{n+1} < \cdots < x_N = b$$

$$h = x_{n+1} - x_n, n = 0(1)N - 1$$

where h is the constant step-size of the partition π_N of $[a, b]$ by solving some numerical examples in the literature. The results obtained using EBHM5 and EBHM6 were compared in terms of the absolute errors $|y(x_n) - y_n|$ or the maximum absolute errors $|y(x_n) - y_n|$ in the specified interval with some existing problems in the literature as follows:

EHBM = Method by Adee *et al.* (2022) for $k = 1, p = 5$
7SBM= 7-step block method by Abolarin *et al.* (2020) for $k = 7, p = 8$.

EBHM5 = Method (7) for $k = 1, p = 6$.

EBHM6 = Method (10) for $k = 1, p = 7$.
All computations are carried out in MATLAB codes.

Problem 1

We consider a stiff system:

$$y_1' = -y_1 - 30y_2 + 30e^{-x} \qquad y_1(0) = 1$$

$$y_2' = 30y_1 - y_2 - 30e^{-x} \qquad y_2(0) = 1,\ 0 \le x \le 20, h = 0.01$$

The stiffness ratio of the system is 1:200, and its exact solution is $y_1(x) = e^{-x}, y_2(x) = e^{-x}$.

Source: Adee *et al.* (2022). The result is measured in terms of the absolute errors are shown in Table 3.

Table 3: Absolute errors $|y(x_n) - y_n|$ for problem 1 for $h = 0.01$.

x	y_i	EHBM $(k = 1, p = 5)$	EBHM5 $(k = 1, p = 6)$	EBHM6 $(k = 1, p = 7)$
1.0	y_1	1.11×10^{-15}	0	5×10^{-15}
	y_2	8.88×10^{-16}	2.0×10^{-15}	4.0×10^{-15}
10.0	y_1	1.22×10^{-19}	0	0
	y_2	3.86×10^{-19}	0	0
20.0	y_1	5.33×10^{-23}	0	0
	y_2	2.77×10^{-23}	0	0

Problem 2

We consider another stiff system of initial value problem solve by Adee *et al* (2022) for different step sizes;

$$y'(x) = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y(x), \qquad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, 0 \le x \le 20$$

The three components of the theoretical solution of the problem are given as

$$y(x) = \frac{1}{2}\begin{pmatrix} e^{-2x} + e^{-40x}(cos(40x) + sin(40x)) \\ e^{-2x} - e^{-40x}(cos(40x) + sin(40x)) \\ 2e^{-40x}(sin(40x) - cos(40x)) \end{pmatrix}$$

The main aim is to show the order and accuracy of the EBHM5 and EBHM6 for different choices of the constant step size h. The results are shown in Table 4.

Table 4: Maximum absolute errors $|y(x_n) - y_n|$ on $[0,20]$ for problem 2

$\boldsymbol{h}$	EHBM ($\boldsymbol{k=1,p=5}$)	EBHM5 ($\boldsymbol{k=1,p=6}$)	EBHM6 ($\boldsymbol{k=1,p=7}$)
$\mathbf{0.01}$	2.52×10^{-8}	3.27×10^{-8}	1.52×10^{-10}
$\mathbf{0.005}$	2.54×10^{-10}	1.03×10^{-9}	3.59×10^{-12}
$\mathbf{0.0025}$	6.74×10^{-12}	3.24×10^{-11}	1.23×10^{-11}
$\mathbf{0.00125}$	1.07×10^{-13}	3.57×10^{-11}	4.82×10^{-11}
$\mathbf{0.00625}$	1.61×10^{-14}	1.22×10^{-10}	1.91×10^{-10}

Problem 3

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people $S(t)$, number of people infected $I(t)$, and the number of people who have recovered $R(t)$. This is a good and simple model for many infectious diseases including measles, mumps and rubella. It is given by the following three coupled equations:

$$\begin{bmatrix} \frac{dS}{dt} = \mu(1-S) - \beta IS \\ \frac{dI}{dt} = -\mu I - \gamma I + \beta IS \\ \frac{dR}{dt} = -\mu R + \gamma I \end{bmatrix}$$

where μ , γ , β , are positive parameters to be determined. Define by y, to be $y = S + I + R$

and adding this coupled equation, we obtain the following evolution equation for y,

$$y' = \mu(1-y)$$

Taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population), we obtain,

$$y' = 0.5(1-y), y(0) = 0.5$$

whose exact solution is

$$y(t) = 1 - 0.5e^{-0.5t}$$

Source: Abolarin *et al.* (2020).

The goal here is to compare the absolute errors and the results are shown in Table 5.

Table 5: Absolute errors $|y(x_n) - y_n|$ for problem 3

x	7SBM ($k = 7, p = 8$)	EBHM5 ($k = 1, p = 6$)	EBHM6 ($k = 1, p = 7$)
0.1	9.103829×10^{-15}	0	1.0×10^{-15}
0.2	7.105427×10^{-15}	0	0
0.3	8.881784×10^{-15}	1.0×10^{-15}	1.0×10^{-15}
0.4	2.120526×10^{-14}	0	2.0×10^{-15}
0.5	1.367795×10^{-13}	1.0×10^{-15}	3.0×10^{-15}
0.6	7.982504×10^{-13}	1.0×10^{-15}	$3.0 \times 10^{-}15$
0.7	3.98819×10^{-12}	1.0×10^{-15}	3.0×10^{-15}

Problem 4

A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, 3000 strands. Find the number of strands of the bacteria present in the culture at time $t \;:\; 0 \leq t \leq 1$. Let Nt , denote the number of bacteria strands in the culture at time t, the initial value problem modeling this problem is given by,

$$\frac{dN}{dt} = 0.366N, N(0) = 694$$

The exact solution is given by

$$N(t) = 694e^{0.366t}$$

Source: Abolarin *et al.*, (2020). The results are shown in Table 6.

Table 6: Absolute errors $|y(x_n) - y_n|$ for problem 4

x	7SBM $(k=7, p=8)$	EBHM5 $(k=1, p=6)$	EBHM6 $(k=1, p=7)$
$\mathbf{0.1}$	6.821210×10^{-13}	1.023×10^{-12}	1.14×10^{-13}
$\mathbf{0.2}$	6.821210×10^{-13}	1.023×10^{-12}	5.68×10^{-13}
$\mathbf{0.3}$	6.821210×10^{-13}	1.819×10^{-12}	1.02×10^{-12}
$\mathbf{0.4}$	3.410605×10^{-13}	2.728×10^{-12}	1.59×10^{-12}
$\mathbf{0.5}$	7.503331×10^{-12}	2.728×10^{-12}	2.27×10^{-12}
$\mathbf{0.6}$	5.024958×10^{-11}	2.956×10^{-12}	3.07×10^{-12}
$\mathbf{0.7}$	2.373781×10^{-10}	3.752×10^{-12}	2.96×10^{-12}

5. Discussion and Conclusion

Two higher-order single-step embedded block hybrid methods (EBHM5 & EBHM6) are developed for solving first-order stiff ordinary differential equations using a multi-step collocation technique. Both methods are consistent (with orders 6 and 7) and zero-stable, hence convergent. The stability analysis revealed that both methods are A-stable, making them viable methods for solving stiff ODEs. Applying the methods on these four problems, although both the EBHM5 (7) and EBHM6 (10) have lower step numbers $k = 1$, they yielded the least absolute errors or maximum absolute errors compared with EHBM (same step number $k = 1$) or even a higher step number $k = 7$ block method on the same problem. It is observed that the new EBHM5 and EBHM6 yielded the least absolute error in problem 1, as it mostly converged to the exact solution as compared with the existing EHBM. Hence, the results from the four examples indicate that the new methods compete favorably with other existing methods

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