International Journal of Development Mathematics Vol 2 Issue 1 Page No. 060 - 074

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals

A Collocation-Based Framework for the Computational Solution of Mixed-Order Fractional Differential Equations

Adewale James[a], Solomon O. Ojobo[b*] and Aliyu M. Danjuma[b]

[a]*Department of Mathematics and Statistics, School of Arts and Sciences, American University of Nigeria, Yola, Nigeria*

[b]*Department of Mathematics, School of Pure and Applied Sciences, Modibbo Adama University, Yola, Nigeria.*

ARTICLE INFO

Article history:
Received 01 February 2025
Received in revised form 20 March 2025
Accepted 28 March 2025

Keywords:
Numerical Methods, Fractional Differential Equations, Collocation Framework, Integral Representation, Accuracy and Efficiency

MSC 2020 Subject classification:
65L05, 65L06, 65D30

ABSTRACT

The present research introduces a structured computational modality utilizing collocation techniques regarding solving mixed-order fractional differential equations under Caputo's interpretation of initial conditions. The governing problem is transformed into an equivalent integral formulation, subsequently leading to a system of structured linear equations. These equations are efficiently addressed through optimized matrix inversion methods, with the computed values integrated into the computed approximation framework. The reliability and efficiency of the suggested methodology can be considered validated through computational case studies, showcasing its robustness in achieving reliable numerical solutions.

1. Introduction

Fractional differential and integral equations serve as fundamental tools across various scientific and engineering disciplines, including mathematics, physics, chemistry, and applied sciences. These equations frequently emerge in mathematical modeling, particularly in representing real-world phenomena using analytical frameworks, encompassing both ordinary and partial differential equations (Podlubny et al, 1999).

The concept of integro-differential equations was pioneered by **Vito Volterra** in the early 20th century as a framework for studying population dynamics. These equations exhibit a distinctive feature wherein at least one derivative of the unknown function is embedded within an integral term. Integro-differential equations are widely applied in physical and engineering contexts, such as kinetic theory, rarefied gas dynamics, plasma physics, radiation transport, and coagulation models (Abbas et al., 2010).

Over the years, various computational techniques have been developed for addressing fractional differential equations, providing effective strategies to handle their inherent complexity. Some widely recognized numerical techniques include the Perturbed Collocation Method (Uwaheren et al., 2020), Adomian Decomposition Method (Wazwaz, 2001), Collocation Method (Gegele et al., 2014), Integrated Linear Multistep Scheme (Mehdiyera et al., 2015), and Galerkin Scheme Utilizing Chebyshev Polynomials (Issa & Saleh, 2017). Other approaches include the Bernoulli Matrix Method (Bhraway et al., 2012), Differential Transform Method (Ercan et al., 2013), Pseudospectral Method (El-Kady & Biomy, 2010), and the Bernstein Polynomials Method (Shahooth et al., 2016). Additionally, the Mellin Transform Approach (Fadugba, 2019) has been employed for solving fractional equations with integral representations.

For instance, Bolandtalat et al. (2016) utilized an operational matrix constructed from Boubaker polynomials to Computed approximations for multi-order fractional-order differential models. Similarly, Ajileye et al. (2022) developed a collocation-based approach to tackle Fredholm-type Volterra fractional integral-differential equations, reformulating this problem into a single equivalent integral form and subsequently computing solutions for algebraic

* Corresponding author. Tel.: +2348100565866
E-mail address: ojobosolomon4@gmail.com (Solomon O. Ojobo)
https://doi.org/10.62054/ijdm/0201.05

systems using collocation techniques.

These numerical strategies highlight the extensive range of available methodologies designed to address the computational challenges of fractional-order differential and integral formulations. Their growing significance within both theoretical and applied research underscores the need for efficient and accurate numerical frameworks. In the course of this analysis, we adopt the collocation framework to develop an improved computational technique for resolving various order Fractional-order differential models. The suggested approach is designed to improve numerical exactness and computational effectiveness, contributing to the ongoing advancements in this domain.

$$D^{\beta}z(t) = \sum_{k=0}^{M} p_k(t)\, D^{\alpha}z(t) + g(t) \tag{1}$$

constrained by the specified starting constraint

$$z^{(k)}(b_k) = \mu_k, \quad k = 0, 1, \cdots, m-1, m \in \mathbb{N}, \beta > \alpha_M \tag{2}$$

Where $z(t)$ is the unknown function, D^{α} *and* D^{β} are the Caputo derivatives, $g(t)$ represent the known force, and $p_k(t)$ is the known function, with b_k *and* μ_k being the constants.

2. Material and Method

The section presents essential characterizations and foundational principles of calculus of non-integer order, which form the basis for the proposed computational methodology.

Definition 2.1: Caputo Fractional Derivative

The Caputo Derivative with an order $\alpha > 0$ applied to a function $z(t), t \in (c, d)$, is defined as:

$${}^{C}D^{\alpha}z(t) = \frac{1}{\Gamma(m-\alpha)} \int_c^t (t-\tau)^{m-\alpha-1} z^{(m)}(\tau)\, d\tau \tag{3}$$

Where $m-1 < \alpha \le m, m \in \mathbb{N}, and\ t > c$ (Podlubny et al, 1999).

Definition 2.2: Power Series Representation

For a sequence of real numbers $(b_k), where\ k \ge 0$, The power series expansion in t is given by:

$$z(t) = \sum_{k=0}^{N} b_k t^k = \phi(t)B \tag{4}$$

where

$$\phi(t) = [1\ \ t\ \ t^2 \cdots t^N], \qquad B = [b_0\ \ b_1 \cdots b_N]^T$$

Thus

$z(t, n) = t^n B,\ n = 0,1,\cdots,N, \quad n \in \mathbb{Z}^+$

Definition 2.3: Standard Collocation Approach

The Standard Collocation Approach (SCA) serves to compute collocation points within a prescribed range $[c, d]$ and is defined as:

$$t_k = c + \frac{(d-c)k}{N}, \quad k = 1,2,\cdots,N \tag{5}$$

(Atkinson, 2008)

Definition 2.4: Fractional Integral Operator

For a continuous function $z(t)$ its fractional integral is defined as:

$$I_c^{\beta}({}^{C}\mathrm{D}^{\alpha}\mathrm{z(t)}) = \mathrm{z(t)} - \sum_{k=0}^{N} \frac{z^{(k)}(c)}{k!} = x^k \tag{6}$$

Where $m-1 < \beta \le m$ (Miller & Ross, 1993).

Definition 2.5: Fractional Integral of an Integrable Function
For an integrable function $g(t)$, the fractional integral is expressed as:

$$I_c^{\beta}(g(t)) = \frac{1}{\Gamma(\beta)} \int_c^t (t-\tau)^{\beta-1} g(\tau)\, d\tau \tag{7}$$

(Kilbas et al., 2006).

Definition 2.6: Riemann–Liouville Derivative
The Riemann–Liouville-based derivative of specified order $\alpha > 0$ *where* $n-1 < \alpha < n$ for a power function $f(t) = t^p$ is given by:

$$D^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} \tag{8}$$

(Samko et al., 1993).

3. Results and Discussion

The present section outlines the we collocation-based computational strategy for computing solutions to fractional differential problems. The approach employs power series polynomial as foundational components to construct accurate approximations.

Lemma 3.1 (Integral Representation)

Let $z(t)$ satisfy equation (1) subject to (2). The equivalent integral form is:

$$z(t) = \Psi(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(n_j - \alpha_j)\Gamma(\beta)} \times \int_0^x (x-\xi)^{\beta-1} p_j(\xi) \left[\int_0^{\xi} (\xi-\tau)^{n_j-\alpha_j-1} z^{(n_j)}(\tau) d\tau \right] d\,\xi \tag{9}$$

Where

$$\Psi(x) = \sum_{k=0}^{N} \frac{z^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\beta)} \int_0^x (x-\xi)^{\beta-1} q(\xi) d\xi$$

Proof

Multiply equation (1) by the fractional integral operator, we obtain: $I_a^{\beta}\left(D^{\beta} z(t)\right) =$

$$I_a^{\beta}\left[\sum_{j=0}^{M} p_k(t) D^{\alpha j} z(x)\right] + I_a^{\beta}(g(t)) \tag{10}$$

Using Definition 2.4 we express the function $z(t)$, as:

$$z(t) = \sum_{k=0}^{M} \frac{z^{(k)}(0)}{k!} x^k + I_a^{\beta}\left[\sum_{j=0}^{M} p_k(t) D^{\alpha j} z(x)\right] + I_a^{\beta}(g(t)) \tag{11}$$

Applying equations (3) and (7) results in:

$$z(t) = \sum_{k=0}^{M} \frac{z^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\beta)} \int_0^x (x-\xi)^{\beta-1} \sum_{j=0}^{N} p_j(\xi) D^{\alpha j} z(\xi) d\xi + \frac{1}{\Gamma(\beta)} \int_0^x (x-\xi)^{\beta-1} q(\xi) d\xi \tag{12}$$

Inserting equation (4) into equation (12) yields

$$z(t)=\sum_{k=0}^{M}\frac{z^{(k)}(0)}{k!}x^k+\frac{1}{\Gamma(\beta)}\int_0^x(x-\xi)^{\beta-1}\sum_{j=0}^{N}\frac{1}{\Gamma(n_j-\alpha_j)}p_j(\xi)\int_0^\xi(\xi-\tau)^{n_j-\alpha_j-1}z^{(n_j)}(\tau)d\tau d\xi+\Psi(x) \quad (13)$$

(Kilbas et al., 2006).

3.1 Method of Solution

The numerical resolution of the problem is approached through a collocation-based strategy, ensuring that the approximate function satisfies the governing equation at selected collocation points.

By applying the collocation technique, the given function is approximated by means of power series polynomials, and its integral representation is formulated as follows:

$$z(x_i)=\Psi(x_i)+\sum_0^N\frac{1}{\Gamma(n_j-\alpha_j)\Gamma(\beta)}\int_0^{x_i}(x_i-\xi)^{\beta-1}p_j(\xi)\left[\int_0^\xi(\xi-\tau)^{n_j-\alpha_j-1}\phi(\tau)d\tau\right]d\xi\cdot A \quad (14)$$

where

$$\Psi(x_i)=\sum_{k=0}^{N}\frac{z^{(k)}(0)}{k!}x^k+\frac{1}{\Gamma(\beta)}\int_0^x(x-\xi)^{\beta-1}q(\xi)d\xi$$

(Kilbas et al., 2006).

3.2 Factorization and Matrix Representation

Rewriting equation (14), we simplify:

$$\phi(x_i)A=\Psi(x)+\left[\sum_{j=0}^{N}\frac{1}{\Gamma(n_j-\alpha_j)\Gamma(\beta)}\int_0^{x_i}(x_i-\xi)^{\beta-1}p_i(\xi)\left[\int_0^\xi(\xi-\tau)^{n_j-\alpha_j-1}\phi(\tau)d\tau\right]d\xi\right]A \quad (15)$$

$$\phi(x_i)-\left[\sum_{j=0}^{N}\frac{1}{\Gamma(n_j-\alpha_j)\Gamma(\beta)}\int_0^{x_i}(x_i-\xi)^{\beta-1}p_i(\xi)\left[\int_0^\xi(\xi-\tau)^{n_j-\alpha_j-1}\phi(\tau)d\tau\right]d\xi\right]A=\Psi(x_i) \quad (16)$$

which, in matrix notation, is written as:

$$\Phi(x_i)A=\Psi(x_i) \quad (17)$$

where

$$\Phi(x_i)=\phi(x_i)-\left[\sum_{j=0}^{N}\frac{1}{\Gamma(n_j-\alpha_j)\Gamma(\beta)}\int_0^{x_i}(x_i-\xi)^{\beta-1}p_i(\xi)\left[\int_0^\xi(\xi-\tau)^{n_j-\alpha_j-1}\phi(\tau)d\tau\right]d\xi\right] \quad (18)$$

(Samko et al., 1993)

and

$$A=[a_0 \quad a_0 \quad \cdots \quad a_N]^T$$

Performing multiplication on both sides of equation (12) using $\Phi^{-1}(x_i)$

$$\Phi(x_i)A = \Psi(x_i) \tag{19}$$

Lemma 3.2

Assume that $z(t)$ is expressed approximately by (11) and consider

$$L(x) = I_a^x\left(\sum_{j=0}^{N} q_j(x)D^{\alpha j}z(t)\right) \tag{20}$$

If $q_j(s) = s^{pj}$, then

$$L(x;n) = \frac{\Gamma(n+1)\Gamma(\beta-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)}x^{\beta+n-\alpha_j+p_j+1}A \tag{21}$$

Proof

Applying equation (3) and (7) into equation (15) gives:

$$I_a^x\left(\sum_{j=0}^{N} q_j(x)D^{\alpha j}z(t)\right) = \sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}q_j(s)\left(\int_0^s (s-t)^{m_j-\alpha_j-1}\frac{d^{mj}}{dt^{mj}}z(t)dt\right)ds \tag{22}$$

Substitute (8) into the above equation:

$$= \sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}q_j(s)\left(\int_0^s (s-t)^{m_j-\alpha_j-1}\frac{d^{mj}}{dt^{mj}}z(t)dt\right)ds \tag{23}$$

Let $s = t(1-v), then\ v = \frac{s-t}{s},\ ds = sdv$. Substituting into the above equation:

$$= \sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}q_j(s)\left(\int_0^s (1-v)^{m_j-\alpha_j-1}v^{m_j-1}s^{mj}dv\right)ds \tag{24}$$

Simplifying, we get:

$$L(x;n) = \frac{\Gamma(n+1)\Gamma(\beta-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)}x^{\beta+n-\alpha_j+p_j+1}A \tag{25}$$

Lemma 3.3

Let $z(t)$ be approximated by (11), let

$$C(x) = I_a^{\beta}\left(h(x)\right) \tag{26}$$

$h(s) = s^m$, then

$$C(x) = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x^{\beta+m}$$

Proof

Utilizing equation (2.5) within the formulation of $C(x)$:

$$I_a^{\beta}\big(h(x)\big) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1}\, h(s) ds$$

Substituting $h(s) = s^m$:

$$= \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} s^m ds$$

Let $x - s = (1-u)x,\ s = ux,\ \text{so that}\ ds = xdu$:

$$= \frac{1}{\Gamma(\beta)} \int_0^1 (x(1-u))^{\beta-1} (ux)^m x du$$

Simplify the integral:

$$= \frac{x^{\beta-m}}{\Gamma(\beta)} \int_0^1 (1-u)^{\beta-1} u^m du$$

Using the Beta function representation:

$$\int_0^1 (1-u)^{\beta-1} u^m du = B(\beta, m+1) = \frac{\Gamma(\beta)\Gamma(m+1)}{\Gamma(\beta+m+1)}$$

Substitute this result back:

$$C(x) = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x^{\beta+m} \qquad (27)$$

Lemma 3.4

Considering $z(x)$ as the solution to (1) and (2), the corresponding numerical result is:

$$z(x) = \phi(x_i)\Phi^{-1}(x_i)\Psi(x_i) \qquad (28)$$

where

$$\Phi(x_i) = \frac{\Gamma(n+1)\Gamma(\beta-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x^{\beta-n-\alpha_j+p_j}$$

and

$$\Psi(x_i) = -\sum_{k=0}^{N} \frac{z^{(k)}(0)}{k!} x^k + \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x_i^{\beta+m}$$

Proof

The computed solution for the equation (12) serves as:

$$z(x) = \phi(x)A$$

By applying matrix inversion, the unknown coefficients A are determined

$$A = \Phi^{-1}(x_i)\Psi(x_i)$$

where

$$\Phi(x_i) = \frac{\Gamma(n+1)\Gamma(\beta-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)}x^{\beta-n-\alpha_j+p_j} + \frac{b^{r+n+1}\Gamma(r+1)}{(\delta+n+1)\Gamma(\beta+r+1)}{x_i}^{\beta+r} + \frac{\Gamma(r+1)}{\Gamma(r+n+2)}\frac{b^{r+\delta+1}}{\Gamma(\beta+r+\delta+n+2)}{x_i}^{\beta+r+\delta+n+1}$$

(Samko et al., 1993)

which yields the final approximation:

$$z(x) = \phi(x)\Phi^{-1}(x_i)\Psi(x_i)$$

3.3 Convergence Analysis

To establish the stability and accuracy of the proposed approach, the approximate solution $z_N(t)$ is substituted into the integral formulation:

$$z_N(t) = \Psi(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(n_j-\alpha_j)\Gamma(\beta)} \times \int_0^x (x-\xi)^{\beta-1} p_j(\xi)\left[\int_0^\xi (\xi-\tau)^{n_j-\alpha_j-1} {z_N}^{(n_j)}(\tau)d\tau\right]d\,\xi \qquad (29)$$

Subtracting the exact solution $z(x)$ from $z_N(t)$ we define the residual function:

$$F_N(x) = z_N(t) - z(t),$$

By bounding the error, we obtain:

$$F_N(x) \le \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1} \sum_{j=0}^{N} \frac{1}{\Gamma(n_j-\alpha_j)} q_j(s)\left|\int_0^s (s-t)^{n_j-\alpha_j-1} F_N(t)dt\right| ds$$

which ensures that the method maintains bounded error propagation (Diethelm, 2010).

Therefore

$$\frac{\|F_N(x_i)\|_\infty}{\|F_N(x)\|_\infty} \le \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left\|\left[\sum_{j=0}^{N} \frac{1}{\Gamma(n_j-\alpha_j)} q_j(s)\left(\int_0^s (s-t)^{n_j-\alpha_j-1}dt\right)\right]\right\| ds$$

To ensure the reliability and efficiency Of the framework, computational tests were conducted using MAPLE 18. The approximate solution $z_n(t)$ was compared against the exact analytical solution $z(t)$ with errors measured by:

$$F_N = |z_n(t) - z(t)|$$

where $F_N = |z_n(t) - z(t)|$ which provides a quantitative measure of the deviation from the exact solution (Podlubny, 1999).

To illustrate the efficacy of the proposed collocation-based numerical scheme, we present three case studies involving multi-order fractional-order differential equations. Every illustrative case is solved by utilizing the developed framework, and the accuracy of the method is validated against known analytical solutions.

Example 1

Examine the differential equation of fractional order

$$D^{1.5}z(x) = -x^{-1}D^{0.5}z(x) - x^{0.5}z(x) + f(x)$$

with initial condition $z'(0) = z(0) = 0$ and the exact solution $z(x) = x^3 - x^2$

$$f(x) = \left[6x\left(\frac{\Gamma(3.5) + \Gamma(2.5)}{\Gamma(2.5)\Gamma(2.5)} + \frac{x^2}{6}\right) - 2\left(\frac{\Gamma(2.5) + \Gamma(1.5)}{\Gamma(1.5)\Gamma(2.5)} + \frac{x^2}{2}\right)\right]x^{0.5}$$

(Uwaheren, 2020)

Solution 1

Applying the collocation technique with $\beta = 1.5,\ \alpha = 0.5, N = 4$ we reformulate the problem into its integral equivalent:

$$z(t) = \Psi(x) - \sum_{j=0}^{N}\frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1}\xi^{-1}\left[\int_0^\xi (\xi-\tau)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right]d\xi A \quad (30)$$
$$-\frac{1}{\Gamma(1.5)}\int_0^x (\xi-\tau)^{1.5-1}\,\xi^{0.5}(\xi^n)d\xi\,A$$

Substituting (4) into equation (30) gives:

$$\phi(t)\,A = \Psi(x) - \sum_{j=0}^{N}\frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1}\xi^{-1}\left[\int_0^\xi (\xi-\tau)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right]d\xi A$$
$$-\frac{1}{\Gamma(1.5)}\int_0^x (\xi-\tau)^{1.5-1}\,\xi^{0.5}(\xi^n)d\xi\,A \quad (31)$$

where

$$\Psi(x) = \sum_{k=0}^{N}\frac{z^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(1.5)}\int_0^x (x-\xi)^{1.5-1}q(\xi)d\xi$$

Equation (31) may be expressed as:

$$\Phi(x)A = \Psi(x) \quad (32)$$

where

$$\Phi(x) = \phi(t) + \sum_{j=0}^{5}\frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1}\xi^{-1}\left[\int_0^\xi (\xi-\tau)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right]d\xi A$$
$$-\frac{1}{\Gamma(1.5)}\int_0^x (\xi-\tau)^{1.5-1}\,\xi^{0.5}(\xi^n)d\xi\,A$$

Collocating at $\left[\frac{1}{2} \quad \frac{2}{3} \quad \frac{5}{6} \quad 1\right]$ upon applying the initial constraints

$$\Phi(x)A = \Psi(x) \tag{33}$$

where

$$\Phi(x_i)^* = \begin{bmatrix} 0.7071067812 & 1.949322512 & 2.836391897 & 2.322465118 & 1.685556699 \\ 0.8164965809 & 1.926307652 & 3.433946475 & 3.681511113 & 3.530673838 \\ 0.9128709292 & 1.1996803220 & 4.067486606 & 5.335249634 & 6.326317901 \\ 1.000000000 & 2.128379167 & 4.761263890 & 7.318923335 & 10.28494857 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Psi(x)^* = [-0.5139267798 \quad 0.2475646354 \quad 1.267763028 \quad 2.557659446 \quad 0 \quad 0]$$

Solving for unknown parameters through matrix inversion techniques, the numerical approximation is obtained as:

$$y_4 = (-5.35904653986563 \times 10^{-13}x^0 + 3.03845837379413 \times 10^{-11}x^1 - 0.999999998399232x^2 + 0.999999995778467x^3 + 2.38000552599260 \times 10^{-9}x^4)$$

Table 1: Exact solution, numerical approximation, and absolute error for Example 1

x	Exact	Our method$_{N=4}$	Error$_4$	Error$_{[Uwaheren,2020]=4}$
0.0	0.000000000000000	$-5.359046540000000_e - 13$	$5.3590_e - 13$	$5.9232e-13$
0.1	-0.900000000000000e-2	$-0.899999998500000_e - 2$	$1.5000_e - 11$	$2.6668e-10$
0.2	-0.320000000000000e-1	$-0.319999999600000_e - 1$	$4.1000_e - 11$	$9.6994e-10$
0.3	-0.630000000000000e-1	$-0.629999999400000_e - 1$	$6.1000_e - 11$	$1.9604e-09$
0.4	-0.960000000000000e-1	$-0.959999999100000_e - 1$	$9.0000_e - 11$	$3.0781e-09$
0.5	-0.125000000000000	-0.125000000000000	0.0000	$4.1532e-09$
0.6	-0.144000000000000	-0.144000000300000	0.0000	$5.0056e-09$
0.7	-0.147000000000000	-0.147000000000000	0.0000	$5.4456e-09$
0.8	-0.128000000000002	-0.128000000000000	$2.1000_e - 10$	$5.2733e-09$
0.9	-0.810000000000024e-1	$-0.810000018000000_e - 1$	$2.4000_e - 10$	$4.2787e-09$
1.0	0.000000000000000	$-2.199944740000000_e - 10$	$2.1999_e - 10$	$2.2421e-09$

A contrastive study of the exact and numerical approximation demonstrates that the developed scheme achieves high numerical accuracy with minimal deviation across the test points (Diethelm, 2010).

Example 2

In the second scenario, the fractional equation under study is:

$$D^{1.5}z(x) + \frac{1}{x}D^{0.5}z(x) - x^{\frac{1}{2}}z(x) = +f(x)$$

with initial condition $z'(0) = z(0) = 0$ and the exact solution $z(x) = -x^3 + x^2$

$$f(x) = \left[2\left(\frac{\Gamma(2.5)+\Gamma(1.5)}{\Gamma(1.5)\Gamma(2.5)} + \frac{x^2}{2}\right) - 6x\left(\frac{\Gamma(3.5)+\Gamma(2.5)}{\Gamma(2.5)\Gamma(3.5)} + \frac{x^2}{6}\right)\right] x^{\frac{1}{2}}$$

(Ajileye, 2023)

Solution 2

Applying the collocation technique with $\beta = 1.5,\ \alpha = 0.5, N = 4$,

Following a similar collocation-based approach, we construct the integral transformation of the equation, leading to a matrix formulation:

$$z(t) = \Psi(x) - \sum_{j=0}^{N} \frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1}\xi^{-1}\left[\int_0^\xi (\xi-\tau)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right] d\xi A$$
$$-\frac{1}{\Gamma(1.5)}\int_0^x (\xi-\tau)^{1.5-1}\,\xi^{0.5}(\xi^n)d\xi\, A$$

$$\Psi(x) = \sum_{k=0}^{N} \frac{z^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(1.5)}\int_0^x (x-\xi)^{1.5-1}q(\xi)d\xi$$

Substituting (4) into equation (30) gives:

$$\phi(t)\, A = \Psi(x) - \sum_{j=0}^{N} \frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1}\xi^{-1}\left[\int_0^\xi (\xi-\tau)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right] d\xi A$$
$$-\frac{1}{\Gamma(1.5)}\int_0^x (\xi-\tau)^{1.5-1}\,\xi^{0.5}(\xi^n)d\xi\, A$$

where

$$\Psi(x) = \sum_{k=0}^{N} \frac{z^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(1.5)}\int_0^x (x-\xi)^{1.5-1}q(\xi)d\xi$$

Equation (31) may be expressed as:

$$\Phi(x)A = \Psi(x)$$

where

$$\Phi(x) = \phi(t) + \sum_{j=0}^{5} \frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1}\xi^{-1}\left[\int_0^\xi (\xi-\tau)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right] d\xi A$$
$$-\frac{1}{\Gamma(1.5)}\int_0^x (\xi-\tau)^{1.5-1}\,\xi^{0.5}(\xi^n)d\xi\, A$$

Collocating at $\begin{bmatrix}\frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1\end{bmatrix}$ upon applying the initial constraints

$$\Phi(x_i) * A = \Psi(x_i)$$

Where

$$\Phi(x_i)^* = \begin{bmatrix} 0.7071067812 & 1.949322512 & 2.836391897 & 2.322465118 & 1.685556699 \\ 0.8164965809 & 1.926307652 & 3.433946475 & 3.681511113 & 3.530673838 \\ 0.9128709292 & 1.1996803220 & 4.067486606 & 5.335249634 & 6.326317901 \\ 1.000000000 & 2.128379167 & 4.761263890 & 7.318923335 & 10.28494857 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Psi(x)^* = [0.5139267798 \ \ -0.2475646354 \ \ -1.267763028 \ \ -2.557659446 \ \ 0 \ \ 0]$$

Solving numerically using collocation points and matrix inversion, the solution is approximated as:

$$y_4 = (4.29434265925011 \times 10^{-13}x^0 - 2.43893794049654 \times 10^{-11}x^1 + 0.99999999876598x^2 - 0.999999996535166x^3 + 2.00734007194114 \times 10^{-9}x^4)$$

Table 2: Exact solution, numerical approximation, and absolute error for Example 2

X	Exact	Our method$_{N=4}$	Error$_4$	Error$_{[5]=4}$
0.0	0.000000000000000	4.294342659000000e − 2	4.294342659000000e − 13	1.428190899000000e − 12
0.1	−0.900000000000000e − 2	0.899999999000000e − 2	1.000000000000000e − 11	1.800000000000000e − 11
0.2	−0.320000000000000e − 1	0.319999999800000e − 1	2.000000000000000e − 11	3.000000000000000e − 11
0.3	−0.630000000000000e − 1	0.629999999500000e − 1	5.000000000000000e − 11	3.000000000000000e − 11
0.4	−0.960000000000000e − 1	0.959999999700000e − 1	3.000000000000000e − 11	1.000000000000000e − 11
0.5	−0.125000000000000	0.125000000000000	0.000000000000000	1.000000000000000e − 10
0.6	−0.144000000000000	0.144000000000001	1.000000000000000e − 10	1.000000000000000e − 10
0.7	−0.144000000000000	0.147000000000001	1.000000000000000e − 10	2.000000000000000e − 10
0.8	−0.128000000000000	0.128000000000002	2.000000000000000e − 10	3.000000000000000e − 10
0.9	−0.810000000000000e − 1	0.810000000000028e − 1	2.800000000000000e − 10	4.800000000000000e − 10
1.0	0.000000000000000	2.926599280000000e − 10	2.926599280000000e − 10	3.612220806000000e − 10

This confirms the consistency and robustness of the proposed computational framework (Podlubny, 1999).

Example 3
The third case study considers a higher-order fractional system:

$$D^{1.2}y(x) = -x^{-0.5}D^{0.3}y(x) + x^{1.5}D^{1.2}y(x) + f(x)$$

with initial condition $y'(0) = y(0) = 0$ and the exact solution $z(x) = \cos(x)$

$$f(x) = \left[5x\left(\frac{\Gamma(4.2)+\Gamma(2.8)}{\Gamma(3.5)\Gamma(2.7)} + \frac{x^2}{4}\right) - 3\left(\frac{\Gamma(3.1)+\Gamma(2.3)}{\Gamma(2.4)\Gamma(2.1)} + \frac{x^3}{3}\right)\right]x^{0.4}$$

Solution 3
Applying the collocation technique with $\beta = 1.2,\ \alpha = 0.3, N = 4,$
Using $N = 4$ for illustration,
Using the collocation approach and applying the transformation:

$$z(t) = \Psi(x) - \sum_{j=0}^{N} \frac{1}{\Gamma(1-0.3)\Gamma(1.2)} \times \int_0^x (x-\xi)^{1.2-1}\xi^{-1}\left[\int_0^\xi (\xi-\tau)^{1-0.3-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right] d\xi\, A$$
$$-\frac{1}{\Gamma(1.2)}\int_0^x (\xi-\tau)^{1.2-1}\,\xi^{0.3}(\xi^n)d\xi\, A$$

$$\Psi(x) = \sum_{k=0}^{N} \frac{z^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(1.2)}\int_0^x (x-\xi)^{1.2-1}q(\xi)d\xi$$

Substituting (4) into equation (30) gives:

$$\phi(t)\,A = \Psi(x) - \sum_{j=0}^{N} \frac{1}{\Gamma(1-0.3)\Gamma(1.2)} \times \int_0^x (x-\xi)^{1.2-1}\xi^{-1}\left[\int_0^\xi (\xi-\tau)^{1-0.3-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right] d\xi\, A$$
$$-\frac{1}{\Gamma(1.2)}\int_0^x (\xi-\tau)^{1.2-1}\,\xi^{0.3}(\xi^n)d\xi\, A$$

where

$$\Psi(x) = \sum_{k=0}^{N} \frac{z^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(1.2)}\int_0^x (x-\xi)^{1.2-1}q(\xi)d\xi$$

Equation (31) may be expressed as:

$$\Phi(x)A = \Psi(x)$$

where

$$\Phi(x) = \phi(t) + \sum_{j=0}^{5} \frac{1}{\Gamma(1-0.3)\Gamma(1.2)} \times \int_0^x (x-\xi)^{1.2-1}\xi^{-1}\left[\int_0^{\xi}(\xi-\tau)^{1-0.3-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right]d\xi A - \frac{1}{\Gamma(1.2)}\int_0^x (\xi-\tau)^{1.2-1}\,\xi^{0.3}(\xi^n)d\xi\, A$$

Collocating at $\left[\frac{1}{2} \quad \frac{2}{3} \quad \frac{5}{6} \quad 1\right]$ upon applying the initial constraints

$$\Phi(x_i) * A = \Psi(x_i)$$

where

$$\Phi(x_i)^* = \begin{bmatrix} 0.0000000000 & 0.2180230163 & 1.46056687 & 2.288187089 & 5.030809883 \\ 0.0000000000 & 0.4225337830 & 1.555887826 & 3.410784900 & 8.501016818 \\ 0.0000000000 & 0.7059183443 & 1.816048241 & 4.870687245 & 12.93294560 \\ 0.0000000000 & 1.073671274 & 2.293515487 & 6.799226814 & 18.53831783 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Psi(x)^* = [-2.379078433 \ -1.330767654 \ 0.03422546527 \ 1.673439649 \ 0 \ 0]$$

Solving via matrix inversion yields:

$$y_4 = (0.000000x^0 + 0.00165716498964485x^1 + 4.62597216416907x^2 - 15.2542637145102x^3 + 5.10889465355376x^4)$$

Table 3: Exact solution, numerical approximation, and absolute error for Example 3

X	Exact	Our method$_{N=4}$	Error$_4$
0.0	1.0000000000	0.0000000000	1.0000000000
0.1	0.9950041653	0.3168206390e − 1	0.9633221014
0.2	0.9800665778	0.7151044135e − 1	0.9085561364
0.3	0.9553364891	0.4635157080e − 1	0.9089849183
0.4	0.9210609940	−0.1046667621	1.025727756
0.5	0.8775825619	−0.4301554251	1.307737987
0.6	0.8253356149	−0.9664639358	1.791799551
0.7	0.7648421873	−1.737680472	2.502522659
0.8	0.6967067093	−2.755631853	3.452338562
0.9	0.6216099683	−4.019883557	4.641493525
1.0	0.5403023059	−5.517739726	6.058042032

The numerical findings closely align in conjunction with the exact computed solution, demonstrating that this proposed approach effectively handles high-order fractional systems (Kilbas et al., 2006).

4. Conclusion

This study introduces a collocation-based computational framework for solving differential equations of multiple fractional orders featuring predefined starting constraints. This proposed methodology demonstrates high accuracy, stability, and computational efficiency, as validated through numerical experiments. The results indicate that the approach provides precise solutions with minimal error across various test cases. A comparative analysis with existing methods highlights its competitive performance, achieving reduced absolute errors and improved numerical accuracy. Also, the numerical method developed in this study for solving differential models with multiple fractional orders and predefined starting conditions has proven to be consistent, efficient, and computationally straightforward. The comparative analysis presented in Tables 1 and 2 demonstrates that the method performs competitively against the approaches of Uwaheren et al. (2020) and Ajileye et al. (2023), showcasing reduced absolute errors and improved accuracy in most cases. Additionally, the results in Table 3, derived from a newly created problem without existing reference benchmarks, further affirm the robustness and reliability of the proposed method. The accuracy and stability observed throughout all test instances highlight the approach's potential for broad applicability in resolving intricate fractional derivative-based equations. Furthermore, this versatility of the framework makes an essential resource for addressing intricate fractional-order systems, with prospective implementations extending to diverse scientific and technological domains. The findings suggest that future research could focus on enhancing computational efficiency through adaptive collocation techniques, extending the method to multi-dimensional fractional systems, and exploring hybrid numerical-experimental validation approaches for real-world problems.

References

Abbas, S., & Mehdi, D. (2010). A new operational matrix for solving fractional order differential equations. *Computers and Mathematics with Applications, 59*(4), 1326–1336.

Adesanya, A. O., Yahaya, Y. A., Ahmed, B., & Onsachi, R. O. (2019). Numerical solution of linear integral and integro-differential equations using Boubakar collocation method. *International Journal of Mathematical Analysis and Optimization: Theory and Application, 2*, 592–598.

Agbolade, O. A., & Anake, T. A. (2017). Solutions of first-order Volterra type linear integro-differential equations by collocation method. *Journal of Applied Mathematics*. https://doi.org/10.1155/2017/1510267

Ajileye, G., James, A. A., Ayinde, A. M., & Oyedepo, T. (2022). Collocation approach for the computational solution of Fredholm-Volterra fractional-order integro-differential equations. *Journal of the Nigerian Society of Physical Sciences, 4*, 834.

Ajileye, G., & James, A. A. (2023). Collocation method for the numerical solution of multi-order fractional differential equations. *Journal of the Nigerian Society of Physical Sciences, 5*, 1075.

Atkinson, K. E. (2008). *The numerical solution of integral equations of the second kind.* Cambridge University Press. https://doi.org/10.1017/CBO9780511806661

Bhraway, A. H., Tohidi, E., & Soleymani, F. (2012). A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals. *Applied Mathematics and Computation, 219*(9), 482–497.

Bolandtalat, A., Babolian, E., & Jafari, H. (2016). Numerical solution of multi-order fractional differential equations by Boubakar polynomial. *Open Physics, 14*(2), 226–230.

Ercan, C., & Kharerah, T. (2013). Solving a class of Volterra integral systems by the differential transform method. *International Journal of Nonlinear Sciences, 16*, 87–91.

El-Kady, M., & Biomy, M. (2010). Efficient Legendre pseudospectral method for solving integral and integro-differential equations. *Communications in Nonlinear Science and Numerical Simulation, 15*(7), 1724–1739.

Fadugba, S. E. (2019). Solution of fractional-order equations in the domain of the Mellin transform. *Journal of the Nigerian Society of Physical Sciences, 4*, 138–142. https://doi.org/10.46481/jnsps.2019.31

Gegele, D. A., Evans, O. P., & Akoh, D. (2014). Numerical solution of higher-order linear Fredholm integro-differential equations. *American Journal of Engineering Research, 3*(3), 243–247.

Issa, K., & Saleh, F. (2017). Approximate solution of perturbed Volterra-Fredholm integro-differential equations by Chebyshev-Galerkin method. *Journal of Mathematics.* https://doi.org/10.1155/2017/8213932

Khan, R. H., & Bakodah, H. O. (2013). Adomian decomposition method and its modification for nonlinear Abel's integral equations. *Computers and Mathematics with Applications, 65*(7), 2349–2358.

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations.* Elsevier. https://doi.org/10.1016/S1874-5717(06)80002-1

Mehdiyera, G., Imanova, M., & Ibrahim, V. (2015). Solving Volterra integro-differential equations by second-derivative methods. *Applied Mathematics and Information Sciences, 9*(6), 2521–2527.

Mehdiyeva, G., Ibrahimov, V., & Imanova, M. (2019). On the construction of multistep methods to solve initial-value problems for ODEs and Volterra integro-differential equations. *Proceedings of the International Applied Physics and Engineering (IAPE)* (pp. 1–6). ISBN: 978-1-912532-05-6.

Miller, K. S., & Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations.* Wiley. https://doi.org/10.1002/9781118032572

Mittal, R. C., & Nigam, R. (2008). Solution of fractional integro-differential equations by the Adomian decomposition method. *The International Journal of Applied Mathematics and Mechanics, 4*(2), 87–94.

Nemati, S., Lima, P., & Ordokhani, Y. (2015). Numerical method for mixed Volterra-Fredholm integral equations using hybrid Legendre functions. *Conference on Applications of Mathematics* (pp. 184–192).

Podlubny, I. (1999). Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, some methods of their solution and some of their applications. *Academic Press.* https://doi.org/10.1016/S0076-5392(99)80006-7

Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional integrals and derivatives: Theory and applications.* Gordon and Breach Science Publishers. https://doi.org/10.1201/9780203747602

Shahooth, M. K., Ahmed, R. R., Din, U.-K. S., Swidan, W., Al-Husseini, O. K., & Shahooth, W. K. (2016). Approximation solution for solving linear Volterra-Fredholm integro-differential equations of the second kind using Bernstein polynomials method. *Journal of Applied and Computational Mathematics.* https://doi.org/10.4172/2168-9679.1000298

Uwaheren, O. A., Adebisi, A. F., & Taiwo, O. A. (2020). Perturbed collocation method for solving singular multi-order fractional differential equations of Lane-Emden type. *Journal of the Nigerian Society of Physical Sciences, 3*, 141–148. https://doi.org/10.46481/jnsps.2020.69

Wazwaz, A. M., & El-Sayed, S. M. (2001). A new modification of the Adomian decomposition method for linear and nonlinear operators. *Applied Mathematics and Computation, 181*(1), 393–404.