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Poisson Sauleh Distribution and its Properties

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ABSTRACT

This study introduces the Poisson-Sauleh distribution (PSuD), a novel statistical model designed to effectively handle overdispersed and heavy-tailed count data. Traditional models, such as the Poisson and Negative Binomial distributions, often fall short in accurately modeling real-world data characterized by greater variability than the mean and heavy-tailed count data. The PSuD is a mixture of the Poisson and Sauleh distributions, while Sauleh distribution is also a mixture of Exponential and Gamma distributions, to enhance flexibility and fit for complex datasets. We derive the PSuD and explore its statistical properties such as moments about the mean, variance, standard deviation, coefficient of variance, skewness, kurtosis and index of dispersion. The distribution was applied to two real-life datasets: the number of epileptic seizures and the quantity of red mites on apple leaves. Goodness-of-fit test demonstrates that the PSuD outperforms Poisson, Negative Binomial, Generalized Poisson, Poisson Janardan and Poisson Lindley Distributions, providing a more accurate representation of the underlying data characteristics. The findings indicate that the Poisson-Sauleh Distribution (PSuD) serves as a flexible and effective tool for modeling count data, particularly in cases of overdispersion and other forms of data variability across diverse fields.

1. Introduction

The Poisson distribution is a widely used statistical model for modelling the number of events occurring within a fixed interval of time or space, given that these events happen independently and with a known constant mean (Alomair & Ahsan-ul Haq, 2023). However, in many real-world scenarios, the assumption of a constant rate may not hold true. For instance, the rate of occurrence of events may vary due to underlying heterogeneity in the population or external factors influencing the event rate. This variability can lead to overdispersion, where the observed variance exceeds the mean, a situation inadequately addressed by the standard Poisson model (Hilbe, 2014).

To address variability in event rates, the mixed Poisson distribution, also known as the compound Poisson distribution, has been developed. This model incorporates a random effect into the Poisson framework, allowing the rate parameter to vary according to a specified distribution, such as gamma or log-normal (Moller, *et al.,* 1998). This flexibility enables the mixed Poisson distribution to capture inherent variability in event rates across different contexts. Its applications span various fields, including epidemiology, insurance, and telecommunications, where event rates are influenced by unobserved factors (Woolson & Clarke, 2011).

Various researchers have derived over-dispersed and under-dispersed discrete distributions by compounding the Poisson distribution with continuous distributions to create mixed Poisson distributions. Examples include the discrete Lindley distribution (Shafiq, *et al.,* 2022), the Poisson-Chris-Jerry distribution (Ahmad & Wani, 2023), the discrete Poisson Bilal distribution (Ahsan, *et al.,* 2023), the Poisson entropy-based weighted exponential distribution

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(Alomair & Ahsan-ul Haq, 2023), the semi-Poisson distribution (Almuhayfith *et al.,* 2023), and the Poisson-Ram Awadh distribution (Shukla, Shanker, & Tiwari, 2022). Recently, the two-parameter Poisson Rani distribution (Bamigbala *et al*., 2025), Poisson XRani Distribution (Borbye, Nasiru & Ajongba, 2024), Poisson-Pranav Distribution (Hassan, Wani, & Shafi, 2020), and Size-biased Poisson-Pranav Distribution (Shukla *et a*l., 2023) have also been introduced.

Ahmed, Afaq, and Rajnee (2020) proposed the Sauleh distribution, formulated mixture model combining an Exponential distribution (with a scale parameter) and a Gamma distribution (with shape parameter and a scale parameter), weighted by a mixing proportion. In this study, we propose a novel statistical model, the Poisson-Sauleh distribution, designed to handle over-dispersed and heavy-tailed count data. This model combines the Poisson distribution with the Sauleh distribution. The aim is to provide a better alternative to traditional models like Poisson distribution, Negative Binomial Distribution (NBD) and other similar models. The study outlines the model's derivation, statistical properties, parameter estimation, and application to real datasets.

2. Methods

Derivation of Poisson-Sauleh Distributions

The probability density function (pdf) of Sauleh distribution with parameter η is:

$$f\left(y;\eta\right)=\frac{\eta^{4}}{\eta^{4}+2\eta+6}\left(\eta+y^{2}+y^{3}\right)e^{-\eta y} \qquad for \qquad y>0, \quad \eta>0 \tag{1}$$

If $y \sim P\left(\lambda\right)$ is Poisson distribution given by

$$f\left(y/\lambda\right)=\frac{\lambda^{y}e^{-\lambda}}{y!} \qquad y=0,1,2,3,\ldots \qquad \lambda>0$$

where λ is the parameter of the distribution and follows Sauleh distribution in equation (1), then,

$$h\left(\lambda;\eta\right)=\frac{\eta^{4}}{\eta^{4}+2\eta+6}\left(\eta+\lambda^{2}+\lambda^{3}\right)e^{-\eta\lambda} \qquad \lambda>0, \quad \eta>0 \tag{2}$$

Therefore, Poisson-Sauleh Distribution (PSuD) is derived as follows:

$$P_{Y}\left(y\right)=\int_{0}^{\infty} f\left(y/\lambda\right)h\left(\eta,\lambda\right)d\lambda=\int_{0}^{\infty}\frac{\lambda^{y}e^{-\lambda}}{y!}.\frac{\eta^{4}}{\eta^{4}+2\eta+6}\left(\eta+\lambda^{2}+\lambda^{3}\right)e^{-\eta\lambda}d\lambda \tag{3}$$

$$=\frac{\eta^{4}}{y!\left(\eta^{4}+2\eta+6\right)}\int_{0}^{\infty} e^{-\lambda(1+\eta)}\left(\eta\lambda^{y}+\lambda^{y+2}+\lambda^{y+3}\right)d\lambda \tag{4}$$

$$=\frac{\eta^{4}}{y!\left(\eta^{4}+2\eta+6\right)}\left[\frac{\eta\Gamma\left(y+1\right)}{\left(1+\eta\right)^{y+1}}+\frac{\Gamma\left(y+3\right)}{\left(1+\eta\right)^{y+3}}+\frac{\Gamma\left(y+4\right)}{\left(1+\theta\right)^{y+4}}\right] \tag{5}$$

$$=\frac{\eta^4}{\left(\eta^4+2\eta+6\right)}\left[\frac{\eta}{(1+\eta)^{y+1}}+\frac{y^2+3y+2}{(1+\eta)^{y+3}}+\frac{y^3+6y^2+11y+6}{(1+\eta)^{y+4}}\right] \tag{6}$$

$$P_Y(y)=\frac{\eta^4}{\left(\eta^4+2\eta+6\right)}\left[\frac{\eta(1+\eta)^3+\left(y^2+3y+2\right)(1+\eta)+y^3+6y^2+11y+6}{(1+\eta)^{y+4}}\right],\ y=0,1,2,\ldots;\ \eta>0 \tag{7}$$

Equation (7) is pmf of Poisson-Sauleh Distribution $\left(PSuD(y;\eta)\right)$ and its Cumulative distribution frequency (CDF) is obtained as:

$$F(y)=\sum_{t=0}^{y}P_t(t)$$

$$=\sum_{t=0}^{y}\frac{\eta^4}{\left(\eta^4+2\eta+6\right)}\left[\frac{\eta(1+\eta)^3+\left(t^2+3t+2\right)(1+\eta)+t^3+6t^2+11t+6}{(1+\eta)^{t+4}}\right]$$

$$F(y)=1-\frac{\begin{pmatrix}\eta^7+3\eta^6+3\eta^5+\left(y^2+5y+7\right)\eta^4+(y+4)(y+3)^2\eta^3\\+\left(3y^2+23y+44\right)\eta^2+(6y+26)\eta+6\end{pmatrix}}{\left(\eta^4+2\eta+6\right)(1+\eta)^{y+4}} \tag{8}$$

Properties of Probability Mass Function

(i) $0\le P_Y(y)\le 1$ (ii) $\sum_{y=0}^{\infty}P_Y(y)=1$

It is obvious that $P_Y(y)$ is non negative. In the second property show that

$$P_Y(y)=A\sum_{y=0}^{\infty}(1+\eta)^{-x}\left[\eta(1+\eta)^3+\left(y^2+3y+2\right)(1+\eta)+y^3+6y^2+11y+6\right]$$

Where

$$A=\frac{\eta^4}{\left(\eta^4+2\eta+6\right)(1+\eta)^4}$$

$$P_Y(y) = A\left[\begin{array}{l}\eta(1+\eta)^3\sum_{y=0}^{\infty}(1+\eta)^{-y} + (1+\eta)\sum_{y=0}^{\infty}y^2(1+\eta)^{-y} + 3(1+\eta)\sum_{y=0}^{\infty}y(1+\eta)^{-y} + 2(1+\eta)\sum_{y=0}^{\infty}(1+\eta)^{-y} \\ +\sum_{y=0}^{\infty}y^3(1+\eta)^{-y} + 6\sum_{y=0}^{\infty}y^2(1+\eta)^{-y} + 11\sum_{y=0}^{\infty}y(1+\eta)^{-y} + 6\sum_{y=0}^{\infty}(1+\eta)^{-y}\end{array}\right] \tag{9}$$

From equation (9), solve the summation separately

$$\sum_{y=0}^{\infty}6(1+\eta)^{-y} = 6\sum_{y=0}^{\infty}\frac{1}{(1+\eta)^y} = 6\sum_{y=0}^{\infty}\left(\frac{1}{1+\eta}\right)^y = 6\left(1+\left(\frac{1}{1+\eta}\right)+\left(\frac{1}{1+\eta}\right)^2+\left(\frac{1}{1+\eta}\right)^3+\ldots\right)$$

Using sum of Geometric Progression to infinity (Stewart, 2015)

$$S_\infty = \frac{a}{1-r};\ a=1,\ r=\frac{1}{1+\eta}$$

$$S_\infty = \frac{a}{1-\left(\frac{1}{1+\eta}\right)} = \frac{1}{\frac{\eta+1-1}{1+\eta}} = \frac{1+\eta}{\eta}$$

$$\sum_{y=0}^{\infty}\left(\frac{1}{1+\eta}\right)^y = \left(1+\left(\frac{1}{1+\eta}\right)+\left(\frac{1}{1+\eta}\right)^2+\left(\frac{1}{1+\eta}\right)^3+\ldots\right) = \left(\frac{1+\eta}{\eta}\right)$$

Therefore,

$$6\sum_{y=0}^{\infty}\left(\frac{1}{1+\eta}\right)^y = 6\left(\frac{\eta+1}{\eta}\right) \tag{10}$$

$$3(1+\eta)\sum_{y=0}^{\infty}y(1+\eta)^{-y} = \frac{3(1+\eta)^2}{\eta^2} \tag{11}$$

$$(1+\eta)\sum_{y=0}^{\infty}y^2(1+\eta)^{-y} = \frac{(1+\eta)^2(\eta+2)}{\eta^3} \tag{12}$$

$$2(1+\eta)\sum_{y=0}^{\infty}(1+\eta)^{-y} = \frac{2(1+\eta)^2}{\eta} \tag{13}$$

$$\sum_{y=0}^{\infty} y^3 (1+\eta)^{-y} = \frac{(\eta^2+6\eta+6)(1+\eta)}{\eta^4} \tag{14}$$

$$6\sum_{y=0}^{\infty} y^2 (1+\eta)^{-y} = \frac{6(\eta+2)(1+\eta)}{\eta^3} \tag{15}$$

$$11\sum_{y=0}^{\infty} y (1+\eta)^{-y} = \frac{11(1+\eta)}{\eta^2} \tag{16}$$

Substitute equation (10) to (16) into equation (9) to get

$$P_Y(y) = \frac{\eta^4}{(1+\eta)^4(\eta^4+2\eta+6)} \left(\begin{array}{l} (1+\eta)^4 + \frac{3(1+\eta)^2}{\eta^2} + \frac{(\eta+1)^2(\eta+2)}{\eta^3} + \frac{2(\eta+1)^2}{\eta} + \\ \frac{(\eta^2+6\eta+6)(\eta+1)}{\eta^4} + \frac{11(\eta+1)}{\eta^2} + \frac{6(\eta+2)(\eta+1)}{\eta^3} + \frac{6(\eta+1)}{\eta} \end{array} \right)$$

Multiply both side by $\frac{\eta^4}{(1+\eta)}$ to get

$$= \frac{1}{(\eta^7+3\eta^6+3\eta^5+3\eta^4+12\eta^3+24\eta^2+20\eta+6)} (\eta^7+3\eta^6+3\eta^5+3\eta^4+12\eta^3+24\eta^2+20\eta+6)$$

$$P_Y(y) = 1$$

Therefore, Poisson-Sauleh Distribution $(PSuD(y;\eta))$ is a valid pmf

Statistical Properties of $PSuD(y;\eta)$

(i) Moments of $PSuD(y;\eta)$

The r^{th} moment of the $PSuD(y;\eta)$ denoted by $\mu_{(r)}'$ is given by:

$$\mu_{(r)}' = E(y^r) = \sum_{y=0}^{\infty} y^r P_Y(y) \tag{17}$$

$$\mu_{(r)}' = \frac{\eta^4}{(\eta^4+2\eta+6)(1+\eta)^4} \sum_{y=0}^{\infty} y^r \left[\frac{\eta(1+\eta)^3 + (y^2+3y+2)(1+\eta) + y^3 + 6y^2 + 11y + 6}{(1+\eta)^y} \right] \tag{18}$$

when $r = 1, 2, 3, 4$ in eqn (18), the first 4 moments about mean of $PSuD(y;\eta)$ are given below

$r = 1$; $$\mu_1 = \frac{\eta^4 + 6\eta + 24}{\eta(\eta^4 + 2\eta + 6)}$$

$r = 2$; $$\mu_2 = \frac{\eta^5 + 2\eta^4 + 6\eta^2 + 48\eta + 120}{\eta^2(\eta^4 + 2\eta + 6)}$$

$r = 3$; $$\mu_3 = \frac{\eta^6 + 6\eta^5 + 6\eta^4 + 6\eta^3 + 96\eta^2 + 480\eta + 720}{\eta^3(\eta^4 + 2\eta + 6)}$$

$r = 4$; $$\mu_4 = \frac{\eta^7 + 14\eta^6 + 36\eta^5 + 30\eta^4 + 192\eta^3 + 1560\eta^2 + 5040\eta + 5040}{\eta^4(\eta^4 + 2\eta + 6)}$$

(ii) Variance (σ^2)

$$\sigma^2 = \mu'_{(2)} - \left(\mu'_{(1)}\right)^2$$

$$\sigma^2 = \frac{\eta^5 + 2\eta^4 + 6\eta^2 + 48\eta + 120}{\eta^2(\eta^4 + 2\eta + 6)} - \left(\frac{\eta^4 + 6\eta + 24}{\eta(\eta^4 + 2\eta + 6)}\right)^2$$

$$\sigma^2 = \frac{\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144}{\eta^2(\eta^4 + 2\eta + 6)^2}$$

(iii) Standard Deviation (σ)

$$\sigma = \sqrt{\frac{\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144}{\eta^2(\eta^4 + 2\eta + 6)^2}}$$

$$\sigma = \frac{\sqrt{\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144}}{\eta(\eta^4 + 2\eta + 6)}$$

Coefficient of variation (CV), Skewness, Kurtosis and Index of Dispersion of $PSuD(y;\eta)$ are obtained below

(iv) Coefficient of Variation (CV)

$$CV = \frac{\sqrt{\sigma^2}}{\mu_1} = \frac{\sigma}{\mu_1}$$

$$CV = \frac{\sqrt{\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144}}{\eta\left(\eta^4 + 2\eta + 6\right)} \div \frac{\eta^4 + 6\eta + 24}{\eta\left(\eta^4 + 2\eta + 6\right)}$$

$$CV = \frac{\sqrt{\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144}}{\eta\left(\eta^4 + 2\eta + 6\right)} \times \frac{\eta\left(\eta^4 + 2\eta + 6\right)}{\eta^4 + 6\eta + 24}$$

$$CV = \frac{\sqrt{\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144}}{\eta^4 + 6\eta + 24}$$

(v) Index of Dispersion (ID)

$$ID = \frac{Variance}{Mean} = \frac{\sigma^2}{\mu_1} = \frac{\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144}{\eta^2\left(\eta^4 + 2\eta + 6\right)^2} \div \frac{\eta^4 + 6\eta + 24}{\eta\left(\eta^4 + 2\eta + 6\right)}$$

$$ID = \frac{\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144}{\eta\left(\eta^4 + 6\eta + 24\right)\left(\eta^4 + 2\eta + 6\right)}$$

(vi) Coefficient of Kurtosis (K_s)

$$K_s = \frac{\mu_4}{\left(\mu_2\right)^2}$$

$$K_s = \frac{\left(\eta^7 + 14\eta^6 + 36\eta^5 + 30\eta^4 + 192\eta^3 + 1560\eta^2 + 5040\eta + 5040\right)\left(\eta^4 + 2\eta + 6\right)^3}{\left(\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144\right)^2}$$

(vii) **Coefficient of Skewness (S_k)**

$$S_k = \frac{\mu_3}{\left(\mu_2\right)^{\frac{3}{2}}}$$

$$S_k = \frac{\left(\eta^6 + 6\eta^5 + 6\eta^4 + 6\eta^3 + 96\eta^2 + 480\eta + 720\right)\left(\eta^4 + 2\eta + 6\right)^2}{\left(\eta^9 + \eta^8 + 8\eta^6 + 46\eta^5 + 84\eta^4 + 12\eta^3 + 96\eta^2 + 240\eta + 144\right)^{3/2}}$$

(viii) Generating Functions (Pgf, Mgf, Cgf) of $PSuD(y;\eta)$

Probability generating function $P_Y(t)$ is defined as:

$$P_Y(t) = E(t) = \sum_{y=0}^{\infty} t^y P_Y(y) \tag{19}$$

$$P_Y(t) = \frac{\eta^5}{(\eta^4 + 2\eta + 6)(1+\eta)^4} \sum_{y=0}^{\infty} t^y \frac{\left(\eta(1+\eta)^3 + (y^2 + 3y + 2)(1+\eta) + y^3 + 6y^2 + 11y + 6\right)}{(1+\eta)^y} \tag{20}$$

$$\eta(1+\eta)^3 \sum_{y=0}^{\infty} \left(\frac{t}{1+\eta}\right)^y = \frac{\eta(1+\eta)^4}{(\eta - t + 1)} \tag{21}$$

$$(1+\eta)\sum_{y=0}^{\infty} y^2 \left(\frac{t}{1+\eta}\right)^y = \frac{(1+\eta)^2 t(1+\eta+t)}{(\eta - t + 1)^3} \tag{22}$$

$$3(1+\eta)\sum_{y=0}^{\infty} y \left(\frac{t}{1+\eta}\right)^y = \frac{3(1+\eta)^2 t}{(\eta - t + 1)^2} \tag{23}$$

$$2(1+\eta)\sum_{y=0}^{\infty} \left(\frac{t}{1+\eta}\right)^y = \frac{2(1+\eta)^2}{(\eta - t + 1)} \tag{24}$$

$$6\sum_{y=0}^{\infty} y^2 \left(\frac{t}{1+\eta}\right)^y = \frac{6(1+\eta)^2 t(1+\eta+t)}{(\eta - t + 1)^3} \tag{25}$$

$$11\sum_{y=0}^{\infty} y \left(\frac{t}{1+\eta}\right)^y = \frac{11(1+\eta)t}{(\eta - t + 1)^2} \tag{26}$$

$$6\sum_{y=0}^{\infty} \left(\frac{t}{1+\eta}\right)^y = \frac{6(1+\eta)t}{(\eta - t + 1)} \tag{27}$$

$$\sum_{y=0}^{\infty} y^3 \left(\frac{t}{1+\eta}\right)^y = \frac{6(1+\eta)t\left(t^2 + 4t\theta + \theta^2 + 4t + 2\theta + 1\right)}{(\eta - t + 1)^4} \tag{28}$$

Substitute eqn (21) to (28) into eqn (20) to get

$$P_Y(t)=\frac{\eta^4(1+\eta)}{(\eta^4+2\eta+6)(1+\eta)^4}\left[\begin{array}{l}\frac{\eta(1+\eta)^3}{(\eta-t+1)}+\frac{(1+\eta)^2t(1+\eta+t)}{(\eta-t+1)^3}+\frac{3(1+\eta)^2t}{(\eta+1-t)^2}+\frac{2(1+\eta)^2t}{(\eta+1-t)}\\ \frac{6(1+\eta)t(t^2+4t\eta+\eta^2+4t+2\eta+1)}{(\eta+1-t)^4}+\frac{6(1+\eta)^2t(1+\eta+t)}{(\eta+1-t)^3}\\ \frac{11(1+\eta)t}{(\eta+1-t)^2}+\frac{6(1+\eta)t}{(\eta+1-t)}\end{array}\right]$$

$$P_Y(t)=\frac{\eta^4}{(\eta^4+2\eta+6)(1+\eta)^3}\left[\begin{array}{l}\frac{\eta(1+\eta)^3}{(\eta-t+1)}+\frac{(1+\eta)^2t(1+\eta+t)}{(\eta-t+1)^3}+\frac{3(1+\eta)^2t}{(\eta+1-t)^2}+\frac{2(1+\eta)^2t}{(\eta+1-t)}\\ \frac{6(1+\eta)t(t^2+4t\eta+\eta^2+4t+2\eta+1)}{(\eta+1-t)^4}+\frac{6(1+\eta)^2t(1+\eta+t)}{(\eta+1-t)^3}\\ \frac{11(1+\eta)t}{(\eta+1-t)^2}+\frac{6(1+\eta)t}{(\eta+1-t)}\end{array}\right] \tag{29}$$

(ix) Moment Generating Function (Mgf)

If Y has Poisson Sauleh Distribution $(Y;\eta)$ then the moment generating function $M_Y(t)$ has the following form

$$M_Y(t)=\sum_{y=0}^{\infty}e^{ty}P_Y(y) \tag{30}$$

$$=\frac{\eta^5}{(\eta^4+2\eta+6)(1+\eta)^4}\left[\sum_{y=0}^{\infty}e^{ty}\left(\frac{\left(\eta(1+\eta)^3+(y^2+3y+2)(1+\eta)+y^3+6y^2+11y+6\right)}{(1+\eta)^y}\right)\right]$$

$$M_Y(t)=\frac{\eta^4}{(\eta^4+2\eta+6)(1+\eta)^3}\left[\begin{array}{l}\frac{\eta(1+\eta)^3}{(\eta-e^t+1)}+\frac{(1+\eta)^2e^t(1+\eta+e^t)}{(\eta-e^t+1)^3}+\frac{3(1+\eta)^2e^t}{(\eta-e^t+1)^2}+\frac{2(1+\eta)^2e^t}{(\eta-e^t+1)}\\ \frac{6(1+\eta)e^t(e^{2t}+4e^t\eta+\eta^2+4e^t+2\eta+1)}{(\eta-e^t+1)^4}+\frac{6(1+\eta)^2e^t(1+\eta+e^t)}{(\eta-e^t+1)^3}\\ \frac{11(1+\eta)e^t}{(\eta-e^t+1)^2}+\frac{6(1+\eta)e^t}{(\eta-e^t+1)}\end{array}\right] \tag{31}$$

(x) Characteristics generating function (Cgf) given by

$$C_Y(t)=\phi_Y(it)=\sum_{y=0}^{\infty}e^{ity}P_Y(y) \tag{32}$$

$$C_Y(t)=\frac{\eta^4}{(\eta^4+2\eta+6)(1+\eta)^3}\left[\begin{array}{l}\frac{\eta(1+\eta)^3}{(\eta-e^{it}+1)}+\frac{(1+\eta)^2e^{it}(1+\eta+e^{it})}{(\eta-e^{it}+1)^3}+\frac{3(1+\eta)^2e^{it}}{(\eta-e^{it}+1)^2}+\frac{2(1+\eta)^2e^{it}}{(\eta-e^{it}+1)}\\ \frac{6(1+\eta)e^{it}(e^{2it}+4e^{it}\eta+\eta^2+4e^{it}+2\eta+1)}{(\eta-e^{it}+1)^4}+\frac{6(1+\eta)^2e^{it}(1+\eta+e^{it})}{(\eta-e^{it}+1)^3}\\ \frac{11(1+\eta)e^{it}}{(\eta-e^{it}+1)^2}+\frac{6(1+\eta)e^{it}}{(\eta-e^{it}+1)}\end{array}\right] \tag{33}$$

3. Maximum Likelihood Estimation (MLE) of Poisson-Sauleh Distribution

Suppose $y_1, y_2, ..., y_n$ is a random sample of size n for $PSuD(y;\eta)$. The likelihood function L of Poisson-Sauleh is as:

$$L(\eta / y_i)=\left(\frac{\eta^4}{\eta^4+2\eta+6}\right)^n\prod_{i=1}^{n}\left[\frac{\eta(1+\eta)^3+(y_i^3+3y_i+2)(1+\theta)+y_i^3+6y_i^2+11y_i+6}{(1+\theta)^{y_i+4}}\right] \tag{34}$$

log likelihood function is given by

$$\log_e L(\eta / y_i)=4n\log_e\eta-n\log_e(\eta^4+2\eta+6)+\sum_{i=1}^{n}\begin{pmatrix}\eta(1+\eta)^3+(y_i^3+3y_i+2)(1+\eta)\\ +y_i^3+6y_i^2+11y_i+6\end{pmatrix}-\sum_{i=1}^{n}(1+\eta)^{y_i+4}$$

Differentiate w.r.t. to η , we get

$$\frac{\partial \log_e L\left(\eta / y_i\right)}{\partial \eta}=\frac{4n}{\eta}-\frac{4n\eta^3+2n}{\left(\eta^4+2\eta+6\right)}+\sum_{i=1}^{n}\left(\frac{\left(1+\eta\right)^2\left(1+4\eta\right)+\left(y_i^3+3y_i+2\right)}{\begin{array}{l}\eta\left(1+\eta\right)^3+\left(y_i^3+3y_i+2\right)\left(1+\eta\right)+y_i^3\\+6y_i^2+11y_i+6\end{array}}\right)-\sum_{i=1}^{n}\left(\frac{y_i+4n}{\left(1+\eta\right)}\right)$$

$$\frac{\partial \log_e L\left(\eta / y_i\right)}{\partial \eta}=\frac{4n}{\eta}-\frac{4n\eta^3+2n}{\left(\eta^4+2\eta+6\right)}+\sum_{i=1}^{n}\left(\frac{3\eta\left(\eta^3+4\eta^2+\eta+6\right)+\left(y_i^3+3y_i+2\right)}{\begin{array}{l}\theta\left(1+\theta\right)^3+\left(y_i^3+3y_i+2\right)\left(1+\eta\right)+y_i^3\\+6y_i^2+11y_i+6\end{array}}\right)-\sum_{i=1}^{n}\left(\frac{y_i+4n}{\left(1+\eta\right)}\right) \quad (35)$$

Substituting $\frac{\partial \log_e L\left(\eta / y_i\right)}{\partial \eta}=0$ in equation (35) which does not have a closed-form solution and it must be solved using iterative techniques such as Newton- Raphson, Bisection method, Regula-Falsi method etc. to get the required $\hat{\eta}_{mle}$.

Model Selection

The Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) were used for model comparison. The formula for AIC and BIC are given by:

$$AIC=2p-2\log L$$

$$BIC=-2\log L+p\log\left(n\right)$$

Where p is the number of parameters in the statistical model, n is the sample size and $-2\log L$ is the maximized value of the log-likelihood function under the considered model.

List of Competing distributions used in this work

The list below is the competing distributions, each defined by their probability mass functions in equations (36)–(40), which will be compared against the proposed distribution.

Poisson (PD)

$$f\left(x\right)=\frac{\lambda^x e^{-\lambda}}{x!} \qquad \lambda>0, \quad x=0,1,2,... \quad (36)$$

Negative Binomial (NBD)

$$f(x)=\binom{x+r-1}{x}p^r q^x \qquad r>0,\ 0<p<1,\quad x=0,1,2,... \tag{37}$$

Generalized Poisson (GPD)

$$f(x)=\frac{\lambda_1(\lambda_1+x\lambda_2)^{x-1}e^{(-\lambda_1-x\lambda_2)}}{x!} \qquad \lambda_1>0\ \ and\,|\lambda_2|<1,\qquad x=0,1,2,... \tag{38}$$

Poisson Janardan (PJD)

$$f(x)=\left(\frac{\theta}{\theta+\alpha}\right)^2\left(\frac{\alpha}{\theta+\alpha}\right)^x\left(1+\frac{\alpha(1+\alpha x)}{\theta+\alpha^2}\right),\quad x=0,1,2,...,\ \ \theta>0,\ \alpha>0 \tag{39}$$

Poisson Lindley (PLD)

$$f(x)=\frac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}},\quad x=0,1,2,...,\ \ \theta>0 \tag{40}$$

4. Result and Discussion

Figure 1 displays the probability mass function (pmf) plots of $PSuD(y;\eta)$ for selected parameter values. It is seen from Figure 1 that the $PSuD(y;\eta)$ can be right-skewed, unimodal and approximately symmetric.

Figure 2 shows the CDF plots for the Poisson-Sauleh distribution with parameters 0.5 to 3 show a smooth transition from gradual to rapid accumulation of probability. At lower parameters, like 0.5, the CDF rises slowly, requiring larger y values to approach 1. As the parameter increases, the CDF becomes steeper and shifts leftward, indicating that most probability mass concentrates at smaller y values. By parameter 3, the CDF quickly reaches 1. The graph shows the validity of the Poisson-Sauleh distribution. Each CDF curve begins close to 0, increases steadily without decreasing, and approaches 1 as y becomes large, which are essential properties of any valid cumulative distribution function. Across different parameter values (0.5 to 3), the curves consistently demonstrate this behavior, indicating that the distribution properly accumulates probability mass. The smooth, non-decreasing nature of the CDFs and their convergence to 1 confirm that the Poisson-Sauleh distribution satisfies the fundamental requirements of a valid probability model.

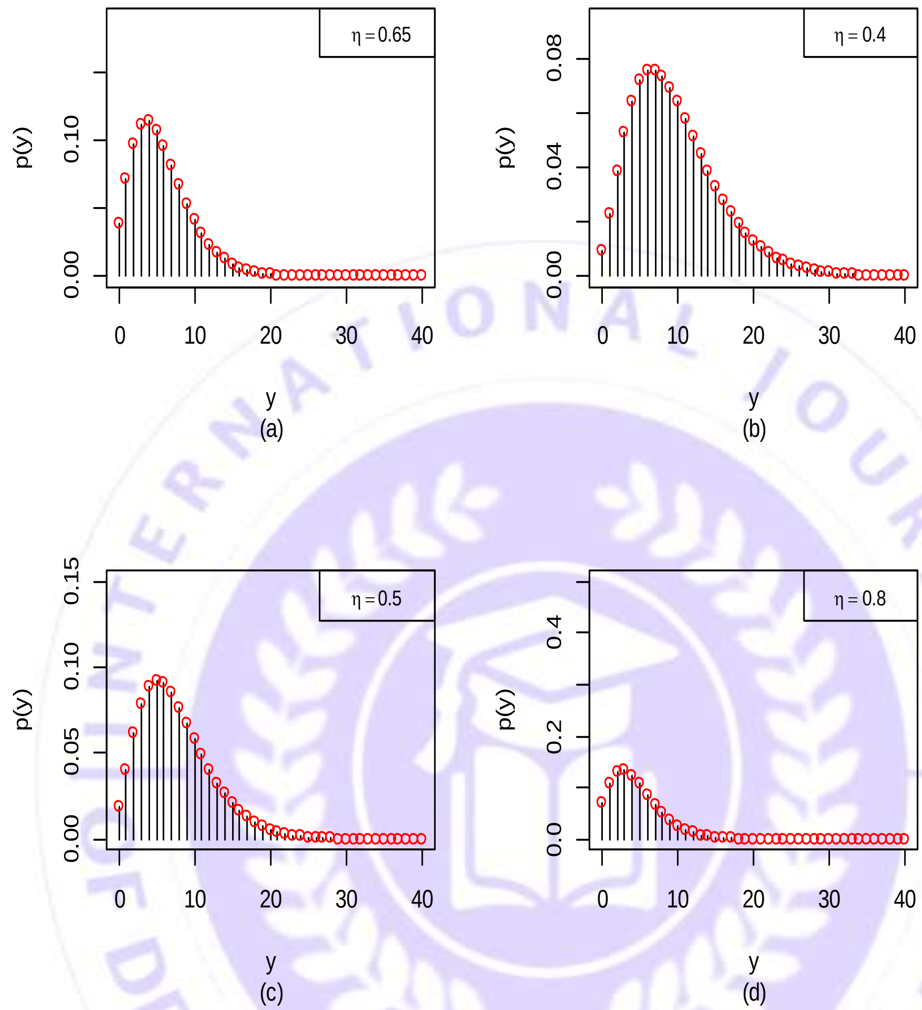

Figure I: Probability Mass Function plot for Poisson Sauleh Distribution with parameters values 0.65, 0.4, 0.5 and 0.8

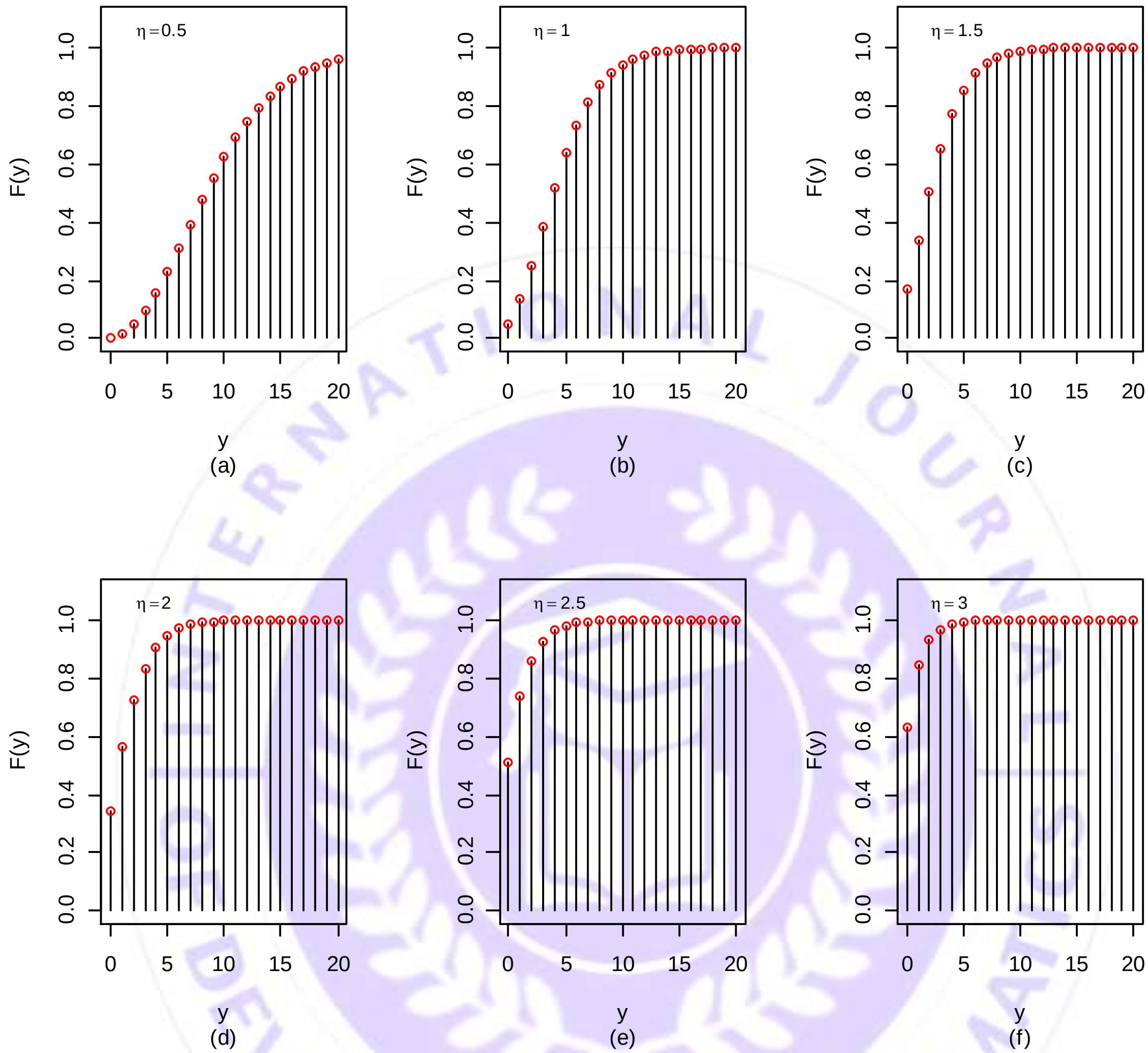

Figure II: Cumulative Density Function plot for Poisson-Sauleh Distribution with parameters values 0.5, 1, 1.5, 2, 2.5 and 3

Table 1: Studying Characteristics of $PSuD\left(y;\eta\right)$ by varying η = 0.05 to 5 with an interval of 0.15

Theta (η)	Mean	Variance	Skewness	Kurtosis	Index of Dispersion	CV
0.05	79.67207	47371.43	0.094667	0.061901	594.5801	2.73182
0.2	19.68383	2850.239	0.105211	0.073453	144.801	2.712258
0.35	11.11158	896.6457	0.116447	0.086454	80.69474	2.694851
0.5	7.663717	422.3186	0.128603	0.101329	55.10624	2.681518
0.65	5.776247	238.6984	0.14207	0.11883	41.32414	2.674724
0.8	4.55853	148.9203	0.15742	0.140146	32.66849	2.677023
0.95	3.68587	98.33863	0.175407	0.16701	26.6799	2.690433
1.1	3.016233	67.09664	0.196953	0.201821	22.24518	2.715723
1.25	2.481685	46.63734	0.223105	0.247739	18.79261	2.751821
1.4	2.047752	32.77249	0.254988	0.30877	16.00413	2.795615
1.55	1.69498	23.21035	0.293754	0.389793	13.69359	2.842342
1.7	1.41001	16.56498	0.340533	0.496514	11.74813	2.886511
1.85	1.181712	11.93149	0.396385	0.635314	10.09679	2.923047
2	1.00000	8.692308	0.462248	0.812945	8.692308	2.948272
2.15	0.855804	6.418845	0.538866	1.036085	7.500372	2.960427
2.3	0.741284	4.813445	0.626701	1.310722	6.493386	2.95967
2.45	0.649948	3.670423	0.725832	1.641459	5.64726	2.947676
2.6	0.576593	2.848388	0.835873	2.030805	4.94003	2.927049
2.75	0.517156	2.250398	0.955918	2.478604	4.351492	2.900738
2.9	0.468505	1.809966	1.084544	2.98172	3.863277	2.871578
3.05	0.428254	1.481337	1.219884	3.534093	3.459014	2.842009
3.2	0.394584	1.232845	1.359758	4.12716	3.124416	2.813938
3.35	0.366113	1.042418	1.501837	4.750601	2.847258	2.788726
3.5	0.341784	0.894533	1.643829	5.393251	2.617247	2.767238
3.65	0.320786	0.778172	1.783629	6.044021	2.425829	2.749934
3.8	0.302493	0.685438	1.919447	6.692692	2.265963	2.736963
3.95	0.286416	0.61061	2.049867	7.330477	2.131897	2.72825
4.1	0.272174	0.549505	2.17387	7.950333	2.01895	2.723577
4.25	0.259462	0.49903	2.290816	8.547041	1.923325	2.722635
4.4	0.248039	0.456876	2.400392	9.117107	1.841948	2.725074
4.55	0.237712	0.421303	2.502557	9.658547	1.772327	2.730528
4.7	0.228321	0.390986	2.597476	10.17062	1.71244	2.73864
4.85	0.219738	0.364907	2.685463	10.65355	1.660647	2.749073
5	0.211856	0.342278	2.76693	11.10828	1.615611	2.761516

The Poisson-Sauleh distribution, a proposed lifetime distribution, exhibits notable variations in its statistical moments as the parameter η increases from 0.05 to 5. Initially, both the mean and variance are high, with values of

79.67207 and 47371.43 respectively at $\eta = 0.05$, indicating a broad spread of data. As η increases, these values decrease significantly, reaching 0.211856 (mean) and 0.342278 (variance) at $\eta = 5$, suggesting a concentration of data around the mean. Skewness, which measures the asymmetry of the distribution, starts at a low value of 0.094667, indicating near symmetry, and increases to 2.76693, reflecting a pronounced right skew at higher η values. Kurtosis, indicating the "tailedness" of the distribution, also rises from 0.061901 to 11.10828, suggesting a transition from a flatter to a more peaked distribution with heavier tails. The index of dispersion, representing the ratio of variance to mean, decreases from 594.5801 to 1.615611, aligning with the characteristics of a Poisson distribution where this index approaches 1. The coefficient of variation (CV), expressing the extent of variability relative to the mean, remains relatively stable around 2.7 across the η range, indicating a consistent level of relative dispersion. These trends highlight the flexibility of the Poisson-Sauleh distribution in modeling data with varying degrees of dispersion, skewness, and kurtosis, making it potentially valuable for diverse applications in reliability analysis and lifetime data modeling.

Simulation Study

This section compares various estimates of η using simulation. The average absolute biases (Bias), variance (VAR), mean square error (MSEs) and root mean square error (RMSEs) were calculated for η =0.3, 0.6, 1.0, 1.5 for n = 10, 30, 50, 100, 200, 500 and 1000.

$$Bias(\hat{\eta}) = E(\hat{\eta} - \eta) \qquad MSE(\hat{\eta}) = E(\hat{\eta} - \eta)^2 \qquad RMSE(\hat{\eta}) = \sqrt{MSE(\hat{\eta})}$$

The simulation results for the Poisson-Sauleh distribution in Table 2 show how different initial estimates (η = 0.3, 0.6, 1.0, and 1.5) and varying sample sizes (n = 10 to 1000) affect the bias, variance, mean squared error (MSE), and root mean squared error (RMSE) of parameter estimates. As sample sizes increase, bias and variance decrease, leading to lower MSE and RMSE values. This indicates improved accuracy and precision of the estimators. For example, with η = 0.3, bias decreases from 0.4001 to 0.3001, and variance from 0.01541 to 0.00031 as n increases from 10 to 1000. Similarly, for η = 1.5, bias drops from 2.1680 to 1.5003, and variance from 0.03342 to 0.00043. These trends suggest that the estimators are efficient (low variance) and consistent (bias near zero), especially for large sample sizes, aligning with the principles of large numbers in probability theory. Overall, the performance of the estimators improves significantly with larger sample sizes, making them reliable for modeling complex data patterns.

Table 2: Simulation results for the $PSuD(y;\eta)$

Initial estimate η	**Sample size (n)**	$\hat{\eta}$	Bias	Var	MSE	RMSE
0.3	10	0.4001	0.01541	0.00329	0.00353	0.059414
	30	0.3080	0.01508	0.00112	0.00141	0.03755
	50	0.3181	0.00536	0.00055	0.00058	0.024083
	100	0.3088	0.00480	0.00034	0.00039	0.019748
	200	0.2846	0.00244	0.00021	0.00027	0.016432
	500	0.3012	0.00043	0.00008	0.00010	0.01
	1000	0.3001	0.00031	0.00002	0.00005	0.007071
0.6	10	0.5097	0.03603	0.01476	0.01606	0.126728
	30	0.6735	0.02316	0.00488	0.00542	0.073621
	50	0.6366	0.02263	0.00289	0.00340	0.05831
	100	0.5724	0.01888	0.00159	0.00194	0.044045
	200	0.6310	0.01817	0.00072	0.00105	0.032404
	500	0.6215	0.01486	0.00031	0.00053	0.023022
	1000	0.6025	0.00157	0.00014	0.00038	0.019494
1.0	10	1.2041	0.08471	0.05094	0.05812	0.241081
	30	1.2011	0.05159	0.01438	0.01704	0.130537
	50	1.1670	0.05512	0.00739	0.01043	0.102127
	100	1.1083	0.04886	0.00455	0.00693	0.083247
	200	1.0493	0.04526	0.00213	0.00418	0.064653
	500	1.0245	0.04168	0.00094	0.00314	0.056036
	1000	1.0008	0.00452	0.00041	0.00032	0.017889
1.5	10	2.1680	0.03342	0.18109	0.26723	0.516943
	30	2.7337	0.03189	0.04349	0.10056	0.317112
	50	1.5592	0.02216	0.02816	0.08681	0.294635
	100	1.5496	0.00616	0.01284	0.06862	0.261954
	200	1.5289	0.00027	0.00707	0.06009	0.245133
	500	1.5109	0.00068	0.00265	0.05494	0.234393
	1000	1.5003	0.00043	0.00135	0.05354	0.231387

Application of Poisson-Sauleh Distribution

We fitted the proposed model on two datasets and showed that it gives better results than some other related models.

Dataset I: Number of Epileptic Seizures (Chakraborty, 2010; Aderoju, 2020)

Number of epileptic seizures	0	1	2	3	4	5	6	7	8
Count	126	80	59	42	24	8	5	4	3

Dataset II: Quantity of European Red Mites on Apples (Rama & Kemlesh, 2018; Bliss, 1953)

No. of Red Mites on Apple Leaves	0	1	2	3	4	5	6	7
Count	70	38	17	10	9	3	2	1

Table 3: Descriptive Statistics of Datasets I and II

Datasets	Mean	Standard Deviation	Variance	Skewness	Kurtosis	Dispersion index
I	1.54416	1.697953	2.883044	1.317	1.749	1.867
II	1.1476	1.50786	2.274	1.545	2.061	1.982

The descriptive statistics in Table 3 show that Dataset I has a mean of 1.54416, a standard deviation of 1.697953, variance of 2.883044, skewness of 1.317, and kurtosis of 1.749. Dataset II have a mean of 1.1476, standard deviation of 1.50786, variance of 2.274, skewness of 1.545, and kurtosis of 2.061. Both datasets exhibit positive skewness, indicating a longer right tail and suggesting that higher values are more spread out. Their kurtosis values, being less than 3, classify them as platykurtic distributions, which are flatter with lighter tails compared to a normal distribution. In these datasets, the variances are indeed greater than the means, indicating the presence of overdispersion. The index of dispersion is greater than unity for all datasets, indicating that all datasets are over-dispersed.

Table 4: Comparison of proposed model and competing models using Chi-Square, P-value, AIC and BIC for dataset I

No. of Epileptic seizures	Count	PD	NBD	GPD	PJD	PLD	PSuD
0	126	74.9354	120.2186	118.1120	121.8803	128.6806	127.9701
1	80	115.7122	92.9959	95.8102	90.9409	87.1358	83.5433
2	59	89.3390	59.1734	59.8863	58.7411	55.2666	55.9689
3	42	45.9846	34.9449	34.4855	35.2131	33.6361	35.7728
4	24	17.7518	19.8373	19.2356	30.1647	19.8977	21.5612
5	8	5.4823	10.9889	10.5921	11.1955	11.5286	12.3257
6	5	1.4109	5.9867	5.8060	6.0785	6.5745	6.7404
7	4	0.3112	3.2224	3.1805	3.2449	3.7028	3.5536
8	3	0.0600	1.7187	1.7446	1.7095	2.0645	1.8175
Estimate (Standard Error)		λ=1.5441 (0.0663)	$\hat{r}$=1.5501 (0.2775)	λ_1=1.5441 (0.0940)	$\hat{\theta}$=3.239 (3.467)	$\hat{\theta}$=0.973 (0.053)	$\hat{\eta}$=1.616 (0.049)
			p=0.4990 (0.0472)	λ_2=2.0101 (0.2082)	$\hat{\alpha}$=2.904 (2.806)		
Chi-Square (χ^2)		256.5398	6.510867	7.826975	5.7473	5.7228	4.4975
P-Value		0.0000	0.2596	0.166032	0.3316	0.4549	0.6097
AIC		1274.091	1193.8838	1195.6555	1192.9636	1192.3621	1189.7506
BIC		1277.952	1201.6054	1203.3771	1200.6852	1196.2228	1193.6114

Table 4 show goodness-of-fit analysis for Dataset I evaluates the performance of six statistical distributions: Poisson (P), Negative Binomial (NBD), Generalized Poisson (GPD), Poisson Janardan (PJD), Poisson Lindley (PLD), and Poisson-Sauleh (PSuD). The observed counts of epileptic events are compared against the expected counts predicted by each distribution across different event frequencies.

In terms of the Chi-Square statistic (χ^2), lower values indicate a better fit. The PSuD exhibits the lowest χ^2 value at 4.4975, suggesting it aligns most closely with the observed data. This is followed by PLD (χ^2 = 5.7228), PJD (χ^2 = 5.7473), GPD (χ^2 = 7.826975), and NBD (χ^2 = 6.510867). The standard Poisson distribution shows a significantly higher χ^2 value of 256.5398, indicating a poor fit. Regarding p-values, which assess the statistical significance of the fit, PSuD achieves the highest p-value at 0.6097, suggesting a strong fit. PLD and PJD also show acceptable fits with p-values of 0.4549 and 0.3316, respectively. GPD and NBD have p-values of 0.166032 and 0.2596, respectively. The Poisson distribution has a p-value of 0.0000, reinforcing its inadequacy for this dataset. The Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are measures used to compare models, with lower values indicating a better balance between fit and complexity. PSuD records the lowest AIC (1189.7506) and BIC (1193.6114) values, followed closely by PLD and PJD. The Poisson distribution again performs poorly with the highest AIC (1274.091) and BIC (1277.952) values. These results suggest that the Poisson-Sauleh distribution provides the best fit for Dataset I, effectively capturing the data's characteristics.

Table 5: Comparison of proposed model and competing models using Chi-Square, P-value, AIC and BIC for dataset II

No. of Red Mites on Apple Leaves	Count	P	NBD	GPD	PJD	PLD	PSuD
0	70	47.6540	69.4870	68.8529	68.8949	67.2611	69.3604
1	38	54.6433	37.6001	38.8948	37.8215	38.8876	36.6306
2	17	31.3288	20.1013	20.0415	20.4262	21.2444	20.5823
3	10	11.9745	10.7028	10.3468	10.8907	11.1864	11.4021
4	9	3.4327	5.6870	5.4282	5.7468	5.7400	6.0730
5	3	0.7872	3.0181	2.8968	3.0067	2.8894	3.1007
6	2	0.1504	1.6004	1.5700	1.5619	1.4332	1.5235
7	1	0.0246	0.8481	0.8623	0.8064	0.7026	0.7243
Estimate		$\lambda =$ 1.1466 (0.0874)	$\hat{r} = 1.024$ (0.275) $p = 0.528$ (0.072)	$\lambda_1 = 1.1466$ (0.1288) $\lambda_2 = 2.1685$ (0.3800)	$\hat{\theta} = 0.387$ (1.178) $\hat{\alpha} = 0.356$ (0.958)	$\hat{\theta} = 1.260$ (0.1139)	$\hat{\eta} = 1.879$ (0.093)
Chi-Square (χ^2)		99.0137	2.5897	3.0065	2.6769	3.3114	2.5209
P-Value		0.0000	0.6286	0.5567	0.6132	0.6521	0.7733
AIC		487.619	448.8743	449.4905	448.7653	447.0217	446.2140
BIC		490.630	454.8955	455.5117	454.7866	450.0324	449.2246

Table 5 shows the goodness-of-fit test results for the number of red mites on apple leaves, comparing several distributions: Poisson (P), Negative Binomial Distribution (NBD), Generalized Poisson Distribution (GPD), Poisson Janardan Distribution (PJD), Poisson Lindley Distribution (PLD), and Poisson-Sauleh Distribution (PSuD).

The Poisson distribution shows a poor fit, particularly for counts of 0 mites (expected count of 47.6540 vs. observed 70) and higher counts, as indicated by a high Chi-Square statistic (99.0137) and a p-value of 0.0000. The Poisson Janardan, Poisson Lindley, and Poisson-Sauleh distributions demonstrate better fits, with lower Chi-Square values (2.5209 to 3.3114) and higher p-values (0.6132 to 0.7733). The Poisson-Sauleh distribution has the lowest AIC (446.2140) and BIC (449.2246), indicating it is the most suitable model for this dataset. Overall, these results suggest that alternative distributions, particularly the Poisson Sauleh, offer a more accurate representation of the data than the Poisson distribution.

5. Conclusion

The Poisson-Sauleh distribution (PSuD) presents a significant advancement in modeling over-dispersed and heavy-tailed count data. Through rigorous derivation and analysis, we have shown that the PSuD effectively captures the complexities of real-world datasets, outperforming traditional models such as the Poisson and Negative Binomial distributions. The application of the PSuD to datasets involving epileptic seizures and red mites on apple leaves highlights its practical utility and robustness. The goodness-of-fit tests indicate that the PSuD provides a superior fit, as evidenced by lower Chi-Square values and better AIC and BIC scores. Future work may explore further applications of the PSuD and its potential extensions to other types of data, reinforcing its role as a versatile tool in statistical modeling.

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