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A Second Derivative Simpson's Method for Solving Initial Value Problems with Stiff Systems

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ABSTRACT

Collocation and interpolation of power series approximation solution is used to develop a continuous hybrid Second Derivative of Simpson's scheme with four off-grid points for the solution of the Stiff System of ordinary differential equations (ODEs). Evaluating the continuous scheme at various grid and off-grid points, the discrete schemes are obtained and written in block form. The block method's fundamental characteristics, including order, zero stability, and stability region, were examined. After testing the block method on a few numerical instances, it was discovered to provide a better approximation than comparable methods reported in the literature.

1. Introduction

In recent decades, there has been a lot of interest in the development of numerical methods for approximating the solution of initial value problems (IVPs) in ordinary differential equations (ODEs). In biology, physics, engineering, and chemistry, a broad study of ODEs is frequently used in mathematical modeling (Alkali *et al.*, 2023). This mathematical modeling turns the problems from an application domain into mathematical formulations whose theoretical and numerical analysis offers insight, solutions, and direction helpful for the original application (Asnor *et al.*, 2019; Adamu, 2023). There are many uses for ordinary differential equations in the sciences and engineering (Kida *et al.*, 2022). Therefore, an adequate numerical method is required to solve the ODEs arising from real-life occurrences, as some of them are extremely non-linear and some of them lack analytical solution (Adamu, 2023; Ajayi *et at.*, 2022).

The branch of mathematics and computer science known as "numerical mathematics" develops analyzes and applies algorithms to numerically solve continuous mathematics issues (Aduroja *et al.*, 2024). These problems, which typically contain variables that change over time, are the basis of real-world applications of algebra, geometry, and calculus (Kida *et al.*, 2022).
This paper develops a numerical method for the solution of first-order ODEs of the form:

$$y' = f(x,y), \quad y(x_0) = y_0, \quad x \in [a,b] \tag{1.1}$$

where $f : \mathrm{R} \times \mathrm{R} \to \mathrm{R}^m$ satisfy a Lipschitz condition (Biala *et al.*, 2015; Adamu *et al.*, 2020). Stiffness issues, which describe coupled physical systems with components that fluctuate with different time scales are common in fields of quantum mechanics, celestial mechanics, biological sciences, and engineering sciences (Biala *et al.*, 2015).

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Numerous numerical techniques for solving ordinary differential equations were developed by academics. The majority of real-world problems that come up in different fields of study are first mathematically modeled before being solved (Mohammed *et al.*, 2012). Differential equations are used to simulate a wide range of problems, including chemical kinetics, orbital dynamics, circuits, and control theory (Alkasassbeh & Zurni, 2017; Biala *et al*., 2015). Biala (2015) develop a Block Hybrid Simpson's Method, which solves stiff systems by using two off-grid points. This is achieved by evaluating the continuous scheme at three different points to generate discrete schemes and combining the three discrete schemes to form the Block Hybrid Simpson's Method (BHSM). The order of the method developed by Biala (2015) can be increase, so also the accuracy of the result need improvement.

Enhancing the numerical solution of ordinary differential equation initial value issues attracted the attention of numerous scholars. One of the results is the development of a class of techniques for solving ordinary differential equations known as block methods, which were first put forth by Milne (1953). who only developed them as a way to get initial values for algorithms that use predictor-correctors. At specific grid points, the Block Method produces an independent solution that doesn't overlap. In comparison to the predictor-corrector approach, it is less costly in terms of the number of functions evaluated; also, it has the Runge-Kutta method's characteristics of self-starting and not requiring initial values.

By building a continuous representation based on the hybrid second derivative method (HSDM) for stiff systems of first order ODEs, Ngwane and Jator (2012) examines a numerical approach for approximation solution. Both initial and boundary problems can be solved directly using the self-starting block approach (Skwame *et al.*, 2018). A second derivative block approach for approximating ordinary differential equations was derived by (Sahi *et al.*, 2012; Ezzeddine & Hojjati, 2012; Adamu *et al.*, 2019).

Later, Adee *et al*. (2005) generated beginning values for the Numerov approach using a hybrid formula of order four. develop a novel hybrid block approach for ODEs, which was discovered to be zero stable, consistent, and convergent. The numerical result demonstrates that the techniques are more accurate than the current methods and are computationally trustworthy.

At specific grid points, the Block Method produces an independent solution that doesn't overlap. In comparison to the predictor-corrector approach, it is less costly in terms of the number of functions evaluated; also, it has the Runge-Kutta method's characteristics of self-starting and not requiring initial values. The block approach for solving functional equations is the subject of (Ngwane & Jato, 2012; Skwame *et al.*, 2018; Alkali *et al.*, 2023; Adamu, 2023).

Hybrid technique and Second derivative terms were introduced to derive integrators which are highly stable and efficient for the solution of differential systems (Adamu *et al.*, 2020). Second derivative terms tend to increase the order of the method and accuracy of the solution.

Therefore, by developing the Second Derivative Hybrid Simpson's Method for the Solution of Stiff Initial Value Problems of Ordinary Differential Equations, this study enhanced the approach of (Biala *et al.*, 2015). The new numerical method will have higher accuracy and wider region of absolute stability and can be adapted to cope with the integration of stiff systems in ODEs.

2. Methodology

Considering the polynomial approximate solution of the form;

$$y(x)=\sum_{i=0}^{7}a_i x^i \quad . \tag{2.1}$$

Interpolate (2.1) at x_n and collocate the first derivative of (2.1) at points $y^{'}(x_{n+j}) = f_{n+j},\ j = \frac{2}{3}, 1, \frac{4}{3}, 2,$ and collocate the second derivative (2.1) at points $y^{''}(x_{n+j}) = g_{n+j},\ j = 1, 2$ to get the system of equation in the form

$$XA = U \tag{2.2}$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7]^T,$$
$$U = \left[y_n, f_n,\ f_{n+\frac{2}{3}}, f_{n+1}, f_{n+\frac{4}{3}}, f_{n+2}\ g_{n+1},\ g_{n+2}\right]^T$$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x_{n+\frac{2}{3}}^2 & 4x_{n+\frac{2}{3}}^3 & 5x_{n+\frac{2}{3}}^4 & 6x_{n+\frac{2}{3}}^5 & 7x_{n+\frac{2}{3}}^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+\frac{4}{3}} & 3x_{n+\frac{4}{3}}^2 & 4x_{n+\frac{4}{3}}^3 & 5x_{n+\frac{4}{3}}^4 & 6x_{n+\frac{4}{3}}^5 & 7x_{n+\frac{4}{3}}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \end{bmatrix}$$

Solving (2.2) for $a_j^{'}s$ with the aid of scientific workplace software and substituting the result back into (2.1) to obtained the continuous hybrid multistep method

$$\begin{aligned} y_{n+t} = {} & \alpha_0(t) y_n + \beta_0(t) f_n + \beta_{\frac{2}{3}}(t) f_{n+\frac{2}{3}} + \beta_1(t) f_{n+1} + \beta_{\frac{4}{3}}(t) f_{n+\frac{4}{3}} + \beta_2(t) f_{n+2} + \\ & \gamma_1(t) g_{n+1} + \gamma_2(t) g_{n+2} \end{aligned} \tag{2.3}$$

where $\alpha_j^{'}s$, $\beta_j^{'}s$ and $\gamma_j^{'}s$ are continuous coefficients expressed as function of t, where $t = \frac{x - x_{n+1}}{h}$, $\frac{dt}{dx} = \frac{1}{h}$. The coefficient of y_{n+j}, f_{n+j}, g_{n+j} are obtained as;

$$\alpha_0 = 1$$

$$\beta_0 = \frac{1}{215\,040} t \begin{pmatrix} 17\,640t + 746\,060t^2 - 1208\,025t^3 + 831\,243t^4 - 269\,955t^5 + \\ 33\,885t^6 - 215\,040 \end{pmatrix}$$

$$\beta_{\frac{2}{3}} = \frac{81}{71\,680} t^2 \left(6300t - 14\,805t^2 + 12\,159t^3 - 4375t^4 + 585t^5 + 840\right)$$

$$\beta_1 = -\frac{1}{105} t^3 \left(-2730t + 2436t^2 - 945t^3 + 135t^4 + 1120\right)$$

$$\beta_{\frac{4}{3}} = -\frac{81}{71\,680} t^2 \left(7140t - 15\,435t^2 + 14\,049t^3 - 5705t^4 + 855t^5 - 840\right)$$

$$\beta_2 = -\frac{1}{215\,040} t^2 \begin{pmatrix} 226\,100t - 549\,255t^2 + 548\,373t^3 - 244\,125t^4 + 39\,555t^5 \\ -17\,640 \end{pmatrix}$$

$$\gamma_1 = -\frac{1}{336} t^2 \left(532t - 735t^2 + 525t^3 - 189t^4 + 27t^5 - 168\right)$$

$$\gamma_2 = \frac{1}{10\,752} t^2 \left(2324t - 5775t^2 + 5901t^3 - 2709t^4 + 459t^5 - 168\right)$$

Evaluating (2.3) at $x = \{ x_{n+\frac{2}{3}}, x_{n+1}, x_{n+\frac{4}{3}},, x'_{n+2}, x_{n+\frac{1}{3}}, x'_{n+\frac{5}{3}} \},$ to obtained the following six discrete schemes:

$$y_{n+\frac{2}{3}} = y_n + \frac{1793}{11340} hf_n + \frac{733}{315} hf_{n+\frac{2}{3}} - \frac{3104}{8505} hf_{n+1} - \frac{1993}{1260} hf_{n+\frac{4}{3}} + \frac{1091}{8505} h\ \ f_{n+2} + \frac{1552}{1701} h^2 g_{n+1} - \frac{41}{1701} h^2 g_{n+2} \tag{2.4}$$

$$y_{n+1} = y_n + \frac{1061}{6720} hf_n + \frac{5427}{2240} hf_{n+\frac{2}{3}} - \frac{16}{105} hf_{n+1} - \frac{3483}{2240} hf_{n+\frac{4}{3}} + \frac{851}{6720} h\ \ f_{n+2} + \frac{37}{42} h^2 g_{n+1} - \frac{1}{42} h^2 g_{n+2} \tag{2.5}$$

$$y_{n+\frac{4}{3}} = y_n + \frac{1342}{8505} hf_n + \frac{766}{315} hf_{n+\frac{2}{3}} + \frac{512}{8505} hf_{n+1} - \frac{454}{315} hf_{n+\frac{4}{3}} + \frac{118}{945} h\ \ f_{n+2} + \frac{512}{567} h^2 g_{n+1} - \frac{40}{1701} h^2 g_{n+2} \tag{2.6}$$

$$y_{n+2} = y_n + \frac{67}{420} hf_n + \frac{81}{35} hf_{n+\frac{2}{3}} - \frac{32}{105} hf_{n+1} - \frac{81}{140} hf_{n+\frac{4}{3}} + \frac{43}{105} hf_{n+2} + \frac{16}{21} h^2 g_{n+1} - \frac{1}{21} h^2 g_{n+2} \tag{2.7}$$

$$y'_{n+\frac{1}{3}} = y_n + \frac{79957}{544320}hf_n + \frac{28531}{20160}hf_{n+\frac{2}{3}} - \frac{1342}{8505}hf_{n+1} - \frac{23539}{20160}hf_{n+\frac{4}{3}} + \frac{5843}{60480}hf_{n+2} + \frac{739}{1134}h^2 g_{n+1} - \frac{31}{1701}h^2 g_{n+2} \tag{2.8}$$

$$y'_{n+\frac{5}{3}} = y_n + \frac{5755}{36288}hf_n + \frac{9575}{4032}hf_{n+\frac{2}{3}} - \frac{250}{1701}hf_{n+1} - \frac{3575}{4032}hf_{n+\frac{4}{3}} + \frac{18175}{108864}h\ f_{n+2} + \frac{2825}{3402}h^2 g_{n+1} - \frac{50}{1701}h^2 g_{n+2} \tag{2.9}$$

Writing (2.4) – (2.9) in block form as

$$A^{(0)}Y_{m+1} = A^{(1)}Y_m + hB^{(1)}F_m + hB^{(0)}F_{m+1} + h^2 C^{(0)} G_{m+1} \tag{2.10}$$

where

$$Y_{m+1} = \left[y_{n+\frac{2}{3}}\ y_{n+1}\ y_{n+\frac{4}{3}}\ y_{n+2}\ y'_{n+\frac{1}{3}}\ y'_{n+\frac{5}{3}} \right]^T, Y_m = \left[y_{n-1}\ y_{n-2}\ y_{n-3}\ y_{n-4}\ y_{n-5}\ y_n \right]^T,$$

$$F_m = \left[f_{n-1,}\ \ f_{n-2}, f_{n-3}, \ f_{n-4}, f_{n-5}, f_n \right]^T, F_{m+1} = \left\lfloor f_{n-1},\ f_{n-2}, f_{n+\frac{2}{3}}, f_{n+1}, f_{n+\frac{4}{3}}, f_{n+2} \right\rfloor^T,$$

$$G_{m+1} = \left[g_{n-1},\ \ g_{n-2}, g_{n-3}, \ g_{n-4}, \ g_{n+1}, g_{n+2} \right]^T,$$

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1793}{11340} \\ 0 & 0 & 0 & 0 & 0 & \frac{1061}{6720} \\ 0 & 0 & 0 & 0 & 0 & \frac{1342}{8505} \\ 0 & 0 & 0 & 0 & 0 & \frac{67}{420} \\ 0 & 0 & 0 & 0 & 0 & \frac{79957}{544320} \\ 0 & 0 & 0 & 0 & 0 & \frac{5755}{36288} \end{bmatrix},$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & \frac{733}{315} & -\frac{3104}{8505} & -\frac{1993}{1260} & \frac{1091}{8505} \\ 0 & 0 & \frac{5427}{2240} & -\frac{16}{105} & -\frac{3483}{2240} & \frac{851}{6720} \\ 0 & 0 & \frac{5427}{2240} & \frac{512}{8505} & -\frac{454}{315} & \frac{118}{945} \\ 0 & 0 & \frac{81}{35} & -\frac{32}{105} & -\frac{81}{140} & \frac{43}{105} \\ 0 & 0 & \frac{28531}{20160} & -\frac{1342}{8505} & -\frac{23539}{20160} & \frac{5843}{60480} \\ 0 & 0 & \frac{9575}{4032} & -\frac{250}{1701} & -\frac{3575}{4032} & \frac{18175}{108864} \end{bmatrix}, C^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1552}{1701} & -\frac{41}{1701} \\ 0 & 0 & 0 & 0 & \frac{37}{42} & -\frac{1}{42} \\ 0 & 0 & 0 & 0 & \frac{512}{567} & -\frac{40}{1701} \\ 0 & 0 & 0 & 0 & \frac{16}{21} & -\frac{1}{21} \\ 0 & 0 & 0 & 0 & \frac{739}{1134} & -\frac{31}{1701} \\ 0 & 0 & 0 & 0 & \frac{2825}{3402} & -\frac{50}{1701} \end{bmatrix}.$$

3. Stability Properties of the Block Method

3.1 Order of the block method

Evaluation (2.4)-(2.9) in Taylor series about x_n give

$$L[y(x);h] = y_{n+t} - \alpha_0 y_n - \beta_0(t) f_n - \beta_{\frac{2}{3}}(t) f_{n+\frac{2}{3}} - \beta_1(t) f_{n+1} - \\ \beta_{\frac{4}{3}}(t) f_{n+\frac{4}{3}} + \beta_2(t) f_{n+2} - \gamma_1(t) g_{n+1} - \gamma_2(t) g_{n+2} = 0 \tag{2.11}$$

$$\begin{bmatrix} 1 \\ \frac{(th)^1}{1!} \\ \frac{(th)^2}{2!} \\ \frac{(th)^3}{3!} \\ \frac{(th)^4}{4!} \\ \frac{(th)^5}{5!} \\ \frac{(th)^6}{6!} \\ \frac{(th)^7}{7!} \\ \frac{(th)^8}{8!} \end{bmatrix} - \alpha_0(t) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \beta_0(t) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \beta_{\frac{2}{3}}(t) \begin{bmatrix} 0 \\ 1 \\ \frac{(\frac{2}{3}h)^1}{1!} \\ \frac{(\frac{2}{3}h)^2}{2!} \\ \frac{(\frac{2}{3}h)^3}{3!} \\ \frac{(\frac{2}{3}h)^4}{4!} \\ \frac{(\frac{2}{3}h)^5}{5!} \\ \frac{(\frac{2}{3}h)^6}{6!} \\ \frac{(\frac{2}{3}h)^7}{7!} \end{bmatrix} - \beta_1(t) \begin{bmatrix} 0 \\ 1 \\ \frac{(1h)^1}{1!} \\ \frac{(1h)^2}{2!} \\ \frac{(1h)^3}{3!} \\ \frac{(1h)^4}{4!} \\ \frac{(1h)^5}{5!} \\ \frac{(1h)^6}{6!} \\ \frac{(1h)^7}{7!} \end{bmatrix} - \beta_{\frac{4}{3}}(t) \begin{bmatrix} 0 \\ 1 \\ \frac{(\frac{4}{3}h)^1}{1!} \\ \frac{(\frac{4}{3}h)^2}{2!} \\ \frac{(\frac{4}{3}h)^3}{3!} \\ \frac{(\frac{4}{3}h)^4}{4!} \\ \frac{(\frac{4}{3}h)^5}{5!} \\ \frac{(\frac{4}{3}h)^6}{6!} \\ \frac{(\frac{4}{3}h)^7}{7!} \end{bmatrix} - \beta_2(t) \begin{bmatrix} 0 \\ 1 \\ \frac{(2h)^1}{1!} \\ \frac{(2h)^2}{2!} \\ \frac{(2h)^3}{3!} \\ \frac{(2h)^4}{4!} \\ \frac{(2h)^5}{5!} \\ \frac{(2h)^6}{6!} \\ \frac{(2h)^7}{7!} \end{bmatrix} - \gamma_1(t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(h)^1}{1!} \\ \frac{(h)^2}{2!} \\ \frac{(h)^3}{3!} \\ \frac{(h)^4}{4!} \\ \frac{(h)^5}{5!} \\ \frac{(h)^6}{6!} \end{bmatrix} - \gamma_2(t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{(2h)^1}{1!} \\ \frac{(2h)^2}{2!} \\ \frac{(2h)^3}{3!} \\ \frac{(2h)^4}{4!} \\ \frac{(2h)^5}{5!} \\ \frac{(2h)^6}{6!} \end{bmatrix} = 0$$

$h^{p+n} \neq 0$, $p+n=9$

where n is the order of the differential equation. Therefore, the order of the BHSDSM is $p=[7,7,7,7,7,7]^T$ with error constant

$$\text{Error Constant} = \begin{bmatrix} \frac{649}{43401015}h^8, \frac{113}{7620480}h^8, \frac{128}{8680203}h^8, \frac{1}{59535}h^8, \frac{64937}{5555329920}h^8, \\ \frac{17365}{1111065984}h^8 \end{bmatrix}^T$$

3.2 Zero Stability of the block method

The block method (2.10) is zero stable since the roots $z_s, s=1,2,3,\ldots,n$ of the first characteristics polynomial $\rho(\lambda)$ is defined by

$$\rho(\lambda) = \det\left[A^{(0)}\lambda - A^{(1)}\right] = 0$$

where

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(\lambda) = \det \left[\lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] = \det \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{bmatrix}$$

$$\lambda^6 - \lambda^5 = \lambda^5(\lambda - 1) = 0$$

Solving for λ we have $\lambda = [0,0,0,0,0,1]$. Hence the block method (2.10) is zero stable.

3.3 Consistency of the block method

Since the methods have order $p = 7 > 1$, therefore, the block method is consistent.

3.4 Convergence of the method

The BHSDSM (2.10) is also consistent since each of its numerical integrator has order $p > 1.$ According to [7], we can assert the convergence of the method.

3.5 Region of absolute stability

The region of absolute stability of the block method is $A - stable,$ because the region consists of the complex plane outside the enclosed figure as shown in Figure 1.

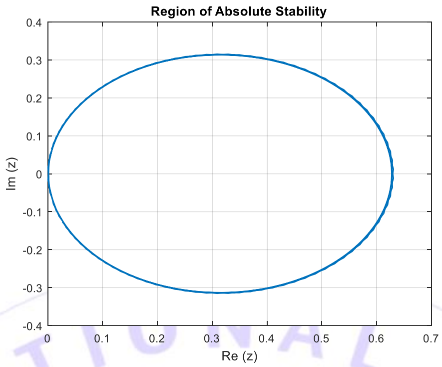

Figure 1: Region of Absolute Stability of the Block Method

4. Numerical Examples

In order to compare the findings with those of other approaches in the literature, a few numerical examples that were established in the literature were solved. All calculations are performed with MATLAB 2015a.

Abbreviation and its Meaning

[10] = Biala *et al.*, 2015

[13] = Ngwane & Jato, 2012

[15] = Sahi *et al.*, 2012

[19] = Jackson *et al.*, 1974

[20] = Cash, 1981

[21] = Wu & Xia, 2001

Example 1 We consider the following IVP which was also solved by (Biala *et al.*, 2015; Jackson *et al.*, 1974; Cash, 1981) on the range $0 \leq x \leq 1$

$y_1' = -y_1 + 95y_{2,} y_1(0) = 1,\ \ y_2' = -y_1 - 97y_2, y_2(0) = 1$, where the analytical solution is given by

$$y_1(x) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x},\ \ y_2(x) = -\frac{1}{47}e^{-2x} + \frac{48}{47}e^{-96x}$$

Table 1: Comparing the absolute Errors for example 1

H	Ey□	[19]	[20] (p=4)	[20] (p=5)
0.0625	Ey_1	3×10^{-7}	3×10^{-7}	1×10^{-8}
	Ey_2	4×10^{-7}	3×10^{-7}	1×10^{-8}
0.03125	Ey_1	1×10^{-8}	1×10^{-8}	-
	Ey_2	1×10^{-8}	1×10^{-8}	-

Table 2: Comparing the absolute Errors for example 1

H	Ey□	[15]	[10]	BHSDSM
0.0625	Ey_1	9×10^{-11}	4×10^{-10}	5.09×10^{-9}
	Ey_2	1×10^{-8}	8×10^{-10}	5.36×10^{-11}
0.03125	Ey_1	4×10^{-12}	7×10^{-12}	2.31×10^{-13}
	Ey_2	4×10^{-12}	7×10^{-14}	2.43×10^{-15}

The tables show that increasing the number of off-step points caused the method's results to converge more quickly. This confirms the method's consistency and zero stability and supports the idea that the procedure becomes more accurate as the step size increases. Table 1 below displays the errors in the solution derived using the block method with fixed step-sizes. Comparable outcomes were seen in (Biala *et al.*, 2015; Sahi et al., 2012; Jackson *et al.*, 1974; Cash, 1981). In Problem 1, it is evident that the new block method outperforms those in (Jackson *et al.*, 1974; Cash, 1981) and is fiercely competitive with (Biala *et al.*, 2015; Sahi *et al.*, 2012).

Example 2 Consider the following nonlinear IVP which was also solved by (Biala *et al.*, 2015; Ngwane & Jato, 2012; Sahi *et al.*, 2012; Wu & Xia, 2001) using different step sizes.

$$y_1' = -1002 y_1 + 1000 y_2^2,\ y_1(0) = 1,\ \ y_2' = y_1 - y_2(1 + y_2),\ \ y_2(0) = 1$$

where the exact solution is given by $y_1(x) = e^{-2x}, y_2(x) = e^{-x}$.

Table 3: Comparison of Exact Errors for example 2

X	H	Ey□	N	[21]	h	N	BHSDSM
1	0.002	Ey₁	500	3.3588×10^{-9}	0.1	5	2.79×10^{-10}
		Ey₂		2.3048×10^{-11}			1.67×10^{-09}
10	0.001	Ey₁	1000	4.7728×10^{-22}	0.01	500	2.61×10^{-23}
		Ey₂		5.6175×10^{-18}			5.11×10^{-19}

Table 4: Comparison of Exact Errors for example 2

X	H	Ey□	N	[13]	N	[10]	BHSDSM
1	0.1	Ey₁	10	5.6763×10^{-13}	5	3.3588×10^{-9}	2.79×10^{-10}
		Ey₂		6.5675×10^{-13}		2.3048×10^{-11}	1.67×10^{-09}
10	0.01	Ey₁	1000	7.0972×10^{-22}	500	4.7728×10^{-22}	2.61×10^{-23}
		Ey₂		7.8198×10^{-18}		5.6175×10^{-18}	5.11×10^{-19}

Comparing example 2 to the technique in (Wu & Xia, 2001), which employed step sizes of {0.002, 0.001}, it is clear from the numerical findings in table 3 that the BHSDSM method works very well for smaller step sizes h = {0.1, 0.01}. It was demonstrated that the approach presented in (Biala *et al*., 2015; Ngwane & Jato, 2012; Wu & Xia, 2001) performed exceptionally well on stiff problems. The computational results in table 4 clearly show that BHSDSM outperforms the method in (Wu & Xia, 2001) in terms of accuracy and step size, and that it is highly competitive in terms of accuracy but outperforms those in (Biala *et al*., 2015; Ngwane & Jato, 2012) in terms of computational efficiency with fewer steps.

5. Conclusion

Interpolation and collocation approach is used to develop Simpson's Method for solving Systems of Initial Value Problems in ordinary differential equations. It is discovered that the new approach is A-stable, convergent, consistent, and zero stable. Two numerical examples of stiff initial value problems of first order ODEs were used to test the proposed approach. The errors arising from Example 1 and Example 2 using the new method were compared to some of the methods that were already in use. The results clearly demonstrate that the newly developed method outperforms the current one. The method's high order of accuracy makes it desirable as well. Additionally, the developed approach is convergent, zero-stable, and consistent.

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