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An Algorithm for Approximating Third-Order Ordinary Differential Equations with Applications to Thin Film Flow and Nonlinear Genesio Problems

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ABSTRACT

Third-order problems have been found to model real-life phenomena such as thick film fluid flow, boundary layer problems, and nonlinear Genesio problems, to name a few. This research focuses on the development of an algorithm using a basis function that combines an exponential function with a power series. This algorithm, called the Exponential Function Induced Algorithm (EFIA), was formulated using the collocation and interpolation techniques. The theoretical analysis of the algorithm was also carried out in this research. The outcome of the analysis showed that the algorithm is consistent, convergent, and zero-stable. The EFIA was applied to solving some real-life third-order problems like thin film flow and nonlinear Genesio problems. The numerical and graphical results obtained showed that the EFIA is accurate, has high precision, and is easy to implement.

1. Introduction

The need to formulate algorithms for the solution of third order differential equations with more accurate approximations has been on the rise owing to the significant applications of this class of differential equations. Another reason for this rise is the fact that some of these third-order differential equations do not have a closed-form solution, hence the need to formulate numerical algorithms that could approximate them.

This research focuses on the derivation and implementation of EFIA for the solution of third order problems of the form,

$$y'''(x) = f\left(x, y(x), y'(x), y''(x)\right), \tag{1}$$

subject to the initial conditions

$$y^{\sigma}\left(x_0\right) = y_0^{\sigma},\ \sigma = 0,1,2, \tag{2}$$

where x_0 is the initial point, y_0 is the solution at the initial point x_0, σ is the order of derivatives of the initial conditions, and f is continuous within the integration interval. Equation (1) is assumed to satisfy the theorem of existence and uniqueness stated below.

Theorem 1.1

Suppose the function f in (1) is differentiable to a desired order in the region $\mathbb{R}$ and $f\left(x, y, y', y''\right)$ satisfies the Lipschitz condition in its second, third and fourth terms as

$$\left.\begin{aligned}
&\left|f\left(x, y_1, y', y''\right) - f\left(x, y_2, y', y''\right)\right| \le L\left|y_1 - y_2\right|, \\
&\left|f\left(x, y, y_1', y''\right) - f\left(x, y, y_2', y''\right)\right| \le L\left|y_1' - y_2'\right|, \\
&\left|f\left(x, y, y', y_1''\right) - f\left(x, y, y', y_2''\right)\right| \le L\left|y_1'' - y_2''\right|,
\end{aligned}\right\} \tag{3}$$

for all points $\left(x, y_j, y', y''\right)$, $\left(x, y_j, y_j', y''\right)$ and $\left(x, y_j, y', y_j''\right)$, $j = 1,2$ in the region $\mathbb{R}$. Then the initial value problem (1) has a unique solution in $\mathbb{R}$, Wend (1967, 1969).

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The importance of third-order differential equations cannot be overemphasized. This is because they find applications in various areas of science and engineering. They are applied in systems that have chaotic behaviors (Genesio and Tesi, 1992), thin-film flows (Duffy and Wilson, 1997), electromagnetic waves (Lee *et al.*, 2014), boundary layer equations (Allogmany and Ismail, 2020), and Korteweg-de Vries equations (Jator *et al.,* 2018). Other areas of applications include the motion of rockets, quantum mechanics, and chemical engineering, among others.

Kamoh and Soomiyol (2019) developed a sixth-order five-step method via the Legendre polynomial basis function for the solution of third-order ordinary differential equations. Adeyeye and Omar (2019) also proposed a fifth-order one-step block method with four equidistant generalized hybrid points for solving third-order differential equations. Kuboye *et al.* (2020) developed first and second block methods of order eight for the solution of third-order ordinary differential equations. Allogmany and Ismail (2020) proposed an implicit block algorithm with three points for the direct solution of initial value problems of third order. The algorithm, which was of order nine, was also applied to solving some real-life application problems. Obarhua (2023) formulated an order-eight, three-step, four-point hybrid block method for the direct solution of third ordinary differential equations. Other researchers that developed numerical methods for solving the third problem of (1) include Awoyemi *et al.* (2014), Mohammed and Adeniyi (2014), and Yakusak *et al.* (2016). According to Adesanya *et al.* (2014), the direct method of solving problems of the form (1) is more efficient than the reduction method (where a higher-order problem is translated into its equivalent system of first-order differential equations before solving the resultant system). The algorithm that shall be derived in this research is for the direct solution of third-order differential equations.

The proposed EFIA is a hybrid block method, which is known for improving the order of accuracy of a method while retaining the same step number. For more details on the hybrid block method, see Allogmany (2020), Kwari *et al.* (2021, 2023), Obarhua (2023), Sunday (2022), and Sunday *et al.* (2011, 2015, 2022, 2024).

The remaining part of the paper is organized as follows: The methodology adopted in the research was presented in Section 2. This section has to do with the derivation and implementation of the EFIA. In the third section, the basic properties of the algorithm were analyzed. Section 4 contains results and discussions, while in the fifth section, conclusions are drawn.

2. Methodology: Derivation and Implementation of the EFIA

2.1 Derivation of the EFIA

The proposed EFIA shall be derived using a basis function that is a combination of exponential function and power series. Consider the basis function,

$$y(x)=\sum_{j=0}^{6}a_jx^j+a_7\sum_{j=0}^{7}\frac{x^j}{j!} \tag{4}$$

Differentiating the basis function (4) three times gives,

$$y'''(x)=\sum_{j=0}^{6}j(j-1)(j-2)a_jx^{j-3}+a_7\sum_{j=0}^{4}\frac{x^j}{j!} \tag{5}$$

Substituting the derivative (5) in equation (1) produces,

$$f\left(x,y(x),y'(x),y''(x)\right)=\sum_{j=0}^{6}j(j-1)(j-2)a_jx^{j-3}+a_7\sum_{j=0}^{4}\frac{x^j}{j!} \tag{6}$$

Equation (4) is interpolated at the points $x=x_{n+s}$, $s=\frac{1}{4},\frac{1}{2},\frac{3}{4}$ while equation (6) is collocated at points $x=x_{n+r}$, $r=0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1$. This gives the system of equations,

$$\begin{pmatrix} 1 & x_{n+\frac{1}{4}} & x^2_{n+\frac{1}{4}} & x^3_{n+\frac{1}{4}} & x^4_{n+\frac{1}{4}} & x^5_{n+\frac{1}{4}} & x^6_{n+\frac{1}{4}} & \left(1+x_{n+\frac{1}{4}}+\frac{x^2_{n+\frac{1}{4}}}{2!}+\frac{x^3_{n+\frac{1}{4}}}{3!}+\ldots+\frac{x^7_{n+\frac{1}{4}}}{7!}\right) \\ 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} & \left(1+x_{n+\frac{1}{2}}+\frac{x^2_{n+\frac{1}{2}}}{2!}+\frac{x^3_{n+\frac{1}{2}}}{3!}+\ldots+\frac{x^7_{n+\frac{1}{2}}}{7!}\right) \\ 1 & x_{n+\frac{3}{4}} & x^2_{n+\frac{3}{4}} & x^3_{n+\frac{3}{4}} & x^4_{n+\frac{3}{4}} & x^5_{n+\frac{3}{4}} & x^6_{n+\frac{3}{4}} & \left(1+x_{n+\frac{3}{4}}+\frac{x^2_{n+\frac{3}{4}}}{2!}+\frac{x^3_{n+\frac{3}{4}}}{3!}+\ldots+\frac{x^7_{n+\frac{3}{4}}}{7!}\right) \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & \left(1+x_n+\frac{x_n^2}{2!}+\frac{x_n^3}{3!}+\frac{x_n^4}{4!}\right) \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{4}} & 60x^2_{n+\frac{1}{4}} & 120x^3_{n+\frac{1}{4}} & \left(1+x_{n+\frac{1}{4}}+\frac{x^2_{n+\frac{1}{4}}}{2!}+\frac{x^3_{n+\frac{1}{4}}}{3!}+\frac{x^4_{n+\frac{1}{4}}}{4!}\right) \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{2}} & 60x^2_{n+\frac{1}{2}} & 120x^3_{n+\frac{1}{2}} & \left(1+x_{n+\frac{1}{2}}+\frac{x^2_{n+\frac{1}{2}}}{2!}+\frac{x^3_{n+\frac{1}{2}}}{3!}+\frac{x^4_{n+\frac{1}{2}}}{4!}\right) \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{3}{4}} & 60x^2_{n+\frac{3}{4}} & 120x^3_{n+\frac{3}{4}} & \left(1+x_{n+\frac{3}{4}}+\frac{x^2_{n+\frac{3}{4}}}{2!}+\frac{x^3_{n+\frac{3}{4}}}{3!}+\frac{x^4_{n+\frac{3}{4}}}{4!}\right) \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x^2_{n+1} & 120x^3_{n+1} & \left(1+x_{n+1}+\frac{x^2_{n+1}}{2!}+\frac{x^3_{n+1}}{3!}+\frac{x^4_{n+1}}{4!}\right) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix} \tag{7}$$

Solving the system (7) for the constants $a_j,\ j=0,1,2,\ldots,7$ and substituting into (4) gives the proposed EFIA,

$$y(x)=\sum_{s=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\alpha_s(x)y_{n+s}+h^3\left[\sum_{j=0}^{1}\beta_j(x)f_{n+j}+\sum_{s=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\beta_s(x)f_{n+s}\right], \tag{8}$$

where $\alpha_s(x),\ \beta_j(x)\ and\ \beta_s(x)$ are

$$\left.\begin{aligned}
&\alpha_{\frac{1}{4}}(x)=8t^2-10t+3\\
&\alpha_{\frac{1}{2}}(x)=-16t^2+16t-3\\
&\alpha_{\frac{3}{4}}(x)=8t^2-6t+1\\
&\beta_0(x)=\frac{16}{315}t^7-\frac{2}{9}t^6+\frac{7}{18}t^5-\frac{25}{72}t^4+\frac{1}{6}t^3-\frac{59}{1440}t^2+\frac{677}{161280}t-\frac{1}{15360}\\
&\beta_{\frac{1}{4}}(x)=-\frac{64}{315}t^7+\frac{4}{5}t^6-\frac{52}{45}t^5+\frac{2}{3}t^4-\frac{53}{320}t^2+\frac{131}{2016}t-\frac{29}{3840}\\
&\beta_{\frac{1}{2}}(x)=\frac{32}{105}t^7-\frac{16}{15}t^6+\frac{19}{15}t^5-\frac{1}{2}t^4-\frac{1}{30}t^2+\frac{403}{8960}t-\frac{21}{2560}\\
&\beta_{\frac{3}{4}}(x)=-\frac{64}{315}t^7+\frac{28}{45}t^6-\frac{28}{45}t^5+\frac{2}{9}t^4-\frac{7}{576}t^2+\frac{1}{2520}t+\frac{1}{3840}\\
&\beta_1(x)=\frac{16}{315}t^7-\frac{2}{15}t^6+\frac{11}{90}t^5-\frac{1}{24}t^4+\frac{1}{480}t^2+\frac{1}{32256}t-\frac{1}{15360}
\end{aligned}\right\} \tag{9}$$

The variable t in equation (9) is given by $t=\left(x-x_n\right)/h$ and $dt/dx=1/h$. Differentiating (8) once and twice give2

$$y'(x)=\sum_{s=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\alpha_s^{'}(x)y_{n+s}+h^2\left[\sum_{j=0}^{1}\beta_j^{'}(x)f_{n+j}+\sum_{s=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\beta_s^{'}(x)f_{n+s}\right], \tag{10}$$

and

$$y''(x)=\sum_{s=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\alpha_s^{''}(x)y_{n+s}+h\left[\sum_{j=0}^{1}\beta_j^{''}(x)f_{n+j}+\sum_{s=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\beta_s^{''}(x)f_{n+s}\right], \tag{11}$$

respectively. Evaluating equations (8), (10) and (11) at $t=\frac{1}{4},\frac{1}{2},\frac{3}{4},1$ give the EFIA with twelve-member equations as follows,

$$\left.\begin{aligned}
&y_{n+\frac{1}{4}}=y_n+\frac{1}{4}hy_n^{'}+\frac{1}{32}h^2y_n^{''}+h^3\left(\frac{113}{71680}f_n+\frac{107}{64512}f_{n+\frac{1}{4}}-\frac{103}{107520}f_{n+\frac{1}{2}}+\frac{43}{107520}f_{n+\frac{3}{4}}-\frac{47}{645120}f_{n+1}\right)\\
&y_{n+\frac{1}{2}}=y_n+\frac{1}{2}hy_n^{'}+\frac{1}{8}h^2y_n^{''}+h^3\left(\frac{331}{40320}f_n+\frac{83}{5040}f_{n+\frac{1}{4}}-\frac{1}{168}f_{n+\frac{1}{2}}+\frac{13}{5040}f_{n+\frac{3}{4}}-\frac{19}{40320}f_{n+1}\right)\\
&y_{n+\frac{3}{4}}=y_n+\frac{3}{4}hy_n^{'}+\frac{9}{32}h^2y_n^{''}+h^3\left(\frac{1431}{71680}f_n+\frac{1863}{35840}f_{n+\frac{1}{4}}-\frac{243}{35840}f_{n+\frac{1}{2}}+\frac{45}{7168}f_{n+\frac{3}{4}}-\frac{81}{71680}f_{n+1}\right)\\
&y_{n+1}=y_n+hy_n^{'}+\frac{1}{2}h^2y_n^{''}+h^3\left(\frac{31}{840}f_n+\frac{34}{315}f_{n+\frac{1}{4}}+\frac{1}{210}f_{n+\frac{1}{2}}+\frac{2}{105}f_{n+\frac{3}{4}}-\frac{1}{504}f_{n+1}\right)
\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}
y'_{n+\frac{1}{4}} &= y'_n + \frac{1}{4}hy''_n + h^2\left(\frac{367}{23040}f_n + \frac{3}{128}f_{n+\frac{1}{4}} - \frac{47}{3840}f_{n+\frac{1}{2}} + \frac{29}{5760}f_{n+\frac{3}{4}} - \frac{7}{7680}f_{n+1}\right)\\
y'_{n+\frac{1}{2}} &= y'_n + \frac{1}{2}hy''_n + h^2\left(\frac{53}{1440}f_n + \frac{1}{10}f_{n+\frac{1}{4}} - \frac{1}{48}f_{n+\frac{1}{2}} + \frac{1}{90}f_{n+\frac{3}{4}} - \frac{1}{480}f_{n+1}\right)\\
y'_{n+\frac{3}{4}} &= y'_n + \frac{3}{4}hy''_n + h^2\left(\frac{147}{2560}f_n + \frac{117}{640}f_{n+\frac{1}{4}} + \frac{27}{1280}f_{n+\frac{1}{2}} + \frac{3}{128}f_{n+\frac{3}{4}} - \frac{9}{2560}f_{n+1}\right)\\
y'_{n+1} &= y'_n + hy''_n + h^2\left(\frac{7}{90}f_n + \frac{4}{15}f_{n+\frac{1}{4}} + \frac{1}{15}f_{n+\frac{1}{2}} + \frac{4}{45}f_{n+\frac{3}{4}} + 0f_{n+1}\right)
\end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned}
y''_{n+\frac{1}{4}} &= y''_n + h\left(\frac{251}{2880}f_n + \frac{323}{1440}f_{n+\frac{1}{4}} - \frac{11}{120}f_{n+\frac{1}{2}} + \frac{53}{1440}f_{n+\frac{3}{4}} - \frac{19}{2880}f_{n+1}\right)\\
y''_{n+\frac{1}{2}} &= y''_n + h\left(\frac{29}{360}f_n + \frac{31}{90}f_{n+\frac{1}{4}} + \frac{1}{15}f_{n+\frac{1}{2}} + \frac{1}{90}f_{n+\frac{3}{4}} - \frac{1}{360}f_{n+1}\right)\\
y''_{n+\frac{3}{4}} &= y''_n + h\left(\frac{27}{320}f_n + \frac{51}{160}f_{n+\frac{1}{4}} + \frac{9}{40}f_{n+\frac{1}{2}} + \frac{21}{160}f_{n+\frac{3}{4}} - \frac{3}{320}f_{n+1}\right)\\
y''_{n+1} &= y''_n + h\left(\frac{7}{90}f_n + \frac{16}{45}f_{n+\frac{1}{4}} + \frac{2}{15}f_{n+\frac{1}{2}} + \frac{16}{45}f_{n+\frac{3}{4}} + \frac{7}{90}f_{n+1}\right)
\end{aligned}\right\} \quad (14)$$

The twelve-member equations in (12)-(14) are applied simultaneously as the EFIA for solving third-order problems of the form (1).

2.2 Implementation of the EFIA

The EFIA was derived using the scientific workplace 5.5 software while its implementations were carried out using MATLAB R2020a. The implementation steps are highlighted below:

Algorithm: Solution strategy of the EFIA

Step 1: Initiate the starting point and end points $\left[x_0, x_N\right]$, take the stepsize $h = x_N - x_0/N$, where N is the number of steps

Step 2: Set the one-step interval $\left[x_0, x_1\right]$

Step 3: Set $n = 0$, then solve for the values of

$$\left[y_{n+1/4}\ y_{n+1/2}\ y_{n+3/4}\ y_{n+1}\ y'_{n+1/4}\ y'_{n+1/2}\ y'_{n+3/4}\ y'_{n+1}\ y''_{n+1/4}\ y''_{n+1/2}\ y''_{n+3/4}\ y''_{n+1}\right]^T$$

simultaneously on the subinterval $\left[x_0, x_1\right]$ as y_0, y'_0 and y''_0 are known from equation (1).

Step 4: Repeat step 3 by setting $n = 1$, then solve for the values of

$$\left[y_{n+1/4}\ y_{n+1/2}\ y_{n+3/4}\ y_{n+1}\ y'_{n+1/4}\ y'_{n+1/2}\ y'_{n+3/4}\ y'_{n+1}\ y''_{n+1/4}\ y''_{n+1/2}\ y''_{n+3/4}\ y''_{n+1}\right]^T$$

simultaneously on the subinterval $\left[x_1, x_2\right]$ as y_0, y'_0 and y''_0 are known from the previous step

Step 5: The process is continued for $n = 2, \dots, N-1$ to obtain the numerical approximations of the third order problem (1) on the subintervals $\left[x_0, x_1\right], \left[x_1, x_2\right], \dots, \left[x_{N-1}, x_N\right]$.

Step 6: Compute the absolute error, that is $\left|y(x_j)-y_j\right|$, at each point in the interval of integration

Step 7: Compute the maximum error, that is $\max\limits_{1\le j\le N}\left|y(x_j)-y_j\right|$

Step 8: Execute the result. End.

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3. Theoretical Analysis of EFIA

Some critical properties of the EFIA will be analyzed in this section. These properties include, order, error constant, consistency, convergence and zero-stability.

3.1 Order and Error Constant

Let $\ell\{y(t):h\}$ be the linear operator associated with the EFIA (8), defined by the expression,

$$\ell\{y(x);h\}=y(x)-\sum_{s=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\alpha_s(x)y_{n+s}+h^3\left[\sum_{j=0}^{1}\beta_j(x)f_{n+j}+\sum_{s=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\beta_s(x)f_{n+s}\right], \tag{15}$$

Expanding (15) in Taylor series about the point x_n and collecting like terms in y and h gives,

$$\ell\{y(x),h\}=c_0y(x)+c_1hy'(x)+\ldots+c_ph^py^p(x)+c_{p+1}h^{p+1}y^{p+1}(x)+c_{p+2}h^{p+2}y^{p+2}(x)+\ldots \tag{16}$$

Definition 3.1 (Lambert, 1991)

The linear difference operator ℓ and the associated third order algorithm are said to be of order p if $c_0=c_1=\ldots=c_p=c_{p+1}=c_{p+2}=0,\ c_{p+3}\neq 0.$ The term c_{p+3} is called error constant with its truncation error defined as,

$$T_{n+k}=c_{p+3}h^{p+3}y^{p+3}(x)+O(h^{p+4}) \tag{17}$$

Proposition 3.1

The proposed EFIA is of the fifth order.

Proof

The Taylor series expansion of the first components of the EFIA , that is $y_{n+1/4}$, $y_{n+1/2}$, $y_{n+3/4}$ and y_{n+1} is given by

$$\left.\begin{array}{l}
\displaystyle\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}h\right)^j}{j!}-y_n-\frac{1}{4}hy_n'-\frac{1}{32}h^2y_n''-\frac{113}{71680}h^3y_n'''-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[\frac{107}{64512}\left(\frac{1}{4}\right)^j-\frac{103}{107520}\left(\frac{1}{2}\right)^j+\frac{43}{107520}\left(\frac{3}{4}\right)^j-\frac{47}{645120}(1)^j\right]=0\\
\displaystyle\sum_{j=}^{\infty}\frac{\left(\frac{1}{2}h\right)^j}{j!}-y_n-\frac{1}{2}hy_n'-\frac{1}{8}h^2y_n''-\frac{331}{40320}h^3y_n'''-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[\frac{83}{5040}\left(\frac{1}{4}\right)^j-\frac{1}{168}\left(\frac{1}{2}\right)^j+\frac{13}{5040}\left(\frac{3}{4}\right)^j-\frac{19}{40320}(1)^j\right]=0\\
\displaystyle\sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}h\right)^j}{j!}-y_n-\frac{3}{4}hy_n'-\frac{9}{32}h^2y_n''-\frac{1431}{71680}h^3y_n'''-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[\frac{1863}{35840}\left(\frac{1}{4}\right)^j-\frac{243}{35840}\left(\frac{1}{2}\right)^j+\frac{45}{7168}\left(\frac{3}{4}\right)^j-\frac{81}{71680}(1)^j\right]=0\\
\displaystyle\sum_{j=0}^{\infty}\frac{(h)^j}{j!}-y_n-hy_n'-\frac{1}{2}h^2y_n''-\frac{31}{840}h^3y_n'''-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[\frac{34}{315}\left(\frac{1}{4}\right)^j+\frac{1}{210}\left(\frac{1}{2}\right)^j+\frac{2}{105}\left(\frac{3}{4}\right)^j-\frac{1}{504}(1)^j\right]=0
\end{array}\right\} \tag{18}$$

Comparing the coefficients of h in the linear difference operator (16) with those of (18), we obtain $c_0=c_1=c_2=\ldots=c_7=0$ with error constant

$$c_8=\begin{bmatrix}\frac{139}{2642411520} & \frac{1}{2949120} & \frac{243}{293601280} & \frac{1}{645120}\end{bmatrix}^T \tag{19}$$

This implies that $p=[5\ 5\ 5\ 5]^T$ and therefore the EFIA is of order five. ◻

3.2 Consistency

For an algorithm to be consistent, its order $p \geq 1$, Lambert (1973). This obviously implies that the EFIA is consistent since it is of the fifth order. Suffice to state that the local truncation error at each step of a computation is controlled by consistency.

3.3 Zero-stability

Definition 3.2 (Fatunla, 1988)

An algorithm is said to be zero-stable, if the roots $\varsigma_{m,} m = 1, 2, \ldots, k$ of its first characteristic polynomial $\rho(\varsigma)$ defined by $\rho(\varsigma) = \det\left(\varsigma A^{(0)} - E\right)$ satisfies $\left|\varsigma_m\right| \leq 1$ and every root satisfying $\left|\varsigma_m\right| = 1$ have multiplicity not exceeding the order of the differential equation under consideration.

For the newly derived EFIA,

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, the first characteristic polynomial of the EFIA is given by,

$$\rho(\varsigma) = \left| \varsigma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} \varsigma & 0 & 0 & -1 \\ 0 & \varsigma & 0 & -1 \\ 0 & 0 & \varsigma & -1 \\ 0 & 0 & 0 & \varsigma - 1 \end{vmatrix} = \varsigma^3 (\varsigma - 1) \tag{20}$$

Equating equation (20) to zero and solving for ς gives $\varsigma = 0, 0, 0, 1$. Hence, the EFIA is zero-stable.

3.3 Convergence

Theorem 3.1 (Yan, 2011)

For an algorithm to be convergent, it is necessary and sufficient for it to be consistent and zero-stable.
Since the EFIA satisfies the conditions of consistency and zero-stability, it is convergent.

3.4 Region of Absolute Stability

Definition 3.3 (Yan, 2011)

Region of absolute stability is a region in the complex z plane, where $z = \lambda h$ for which the method is absolutely stable. It is defined as those values of z such that the numerical solutions of $y''' = -\lambda y$ satisfy $y_j \to 0 \;\; as \;\; j \to \infty$ for any initial condition.

The stability polynomial of the EFIA using the boundary locus method is given by,

$$\begin{aligned} \bar{h}(w) = {} & h^{12}\left[\left(1/4404019200\right)w^4 - \left(41/44040192000\right)w^4\right] - h^9\left[\left(233/77414400\right)w^3 - \left(47/1238630400\right)w^4\right] \\ & + h^6\left[\left(23/6881280\right)w^4 - \left(66841/41287680\right)w^3\right] - h^3\left(13/60\right)w^3 + w^4 - \left(5/2\right)w^3 \end{aligned} \tag{21}$$

The region of absolute stability of the EFIA is illustrated in Figure 1.

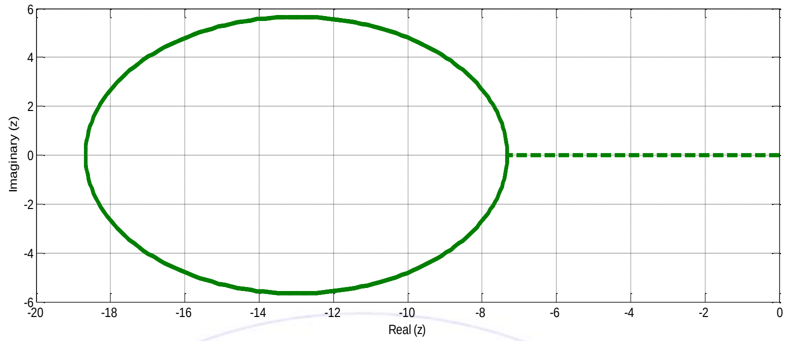

Figure 1 Region of absolute stability of the EFIA

The region of absolute stability of the EFIA in Figure 1 is A-stable. Note that the stable region is the interior of the green-colored contour.

4. Results and Discussions

The newly derived EFIA shall be employed in solving some general third order problems and also applied in solving some real-life third-order application problems of the form (1). The following notations shall be used in Tables 1-5 and Figures 2-8.

x :	point of evaluation
h :	stepsize
NST:	number of steps taken
Error:	absolute error
2SHBM:	two-step hybrid block method of order six developed by Adesanya *et al.* (2013)
ode15s:	Matlab inbuilt solver
I3PA:	implicit three-point algorithm of order nine developed by Allogmany and Ismail (2020)
3S4PM:	three-step four-point hybrid block method of order eight developed by Obarhua (2023)
EFIA:	newly derived exponential function induced algorithm

4.1 Numerical Examples

Problem 1

Consider the third order oscillatory problem,

$$y'''(x) = 3\sin x, \quad y(0) = 1, y'(0) = 0, y''(0) = -2 \tag{22}$$

with the exact solution given by,

$$y(x) = 3\cos x + \frac{x^2}{2} - 2 \tag{23}$$

The proposed EFIA was applied in approximating the oscillatory problem (22). The numerical result obtained in Table 1 clearly showed that the EFIA which is of order five performed better than the 2SHBM of order six developed by Adesanya *et al.* (2013). The solution curves in Figure 2 also showed the convergence of the approximate solution obtained via the EFIA to that of the exact solution.

Table 1 Result for Problem 1 in terms of exact solution, approximate solution and absolute error at $h = 0.01$

x	Exact Solution	EFIA	Error (EFIA)	Error (2SHBM)
0.1	0.99001249583407702204	0.99001249583407702202	2.6171e-19	1.1102e-16
0.2	0.96019973352372511854	0.96019973352372511851	3.0018e-19	0.0000e+00
0.3	0.91100946737681809395	0.91100946737681809392	3.7182e-19	4.4409e-16
0.4	0.84318298200865537950	0.84318298200865537945	4.9171e-19	4.4409e-16
0.5	0.75774768567111827622	0.75774768567111827617	5.2718e-19	7.7716e-16
0.6	0.65600684472903525446	0.65600684472903525440	6.8912e-19	5.5511e-16
0.7	0.53952656185346548057	0.53952656185346548056	1.2888e-20	7.7716e-16
0.8	0.41012012804149655665	0.41012012804149655662	3.1892e-20	1.1657e-15
0.9	0.26982990481199298216	0.26982990481199298212	4.8403e-20	1.4433e-15
1.0	0.12090691760441929503	0.12090691760441929498	5.1733e-20	1.3184e-15

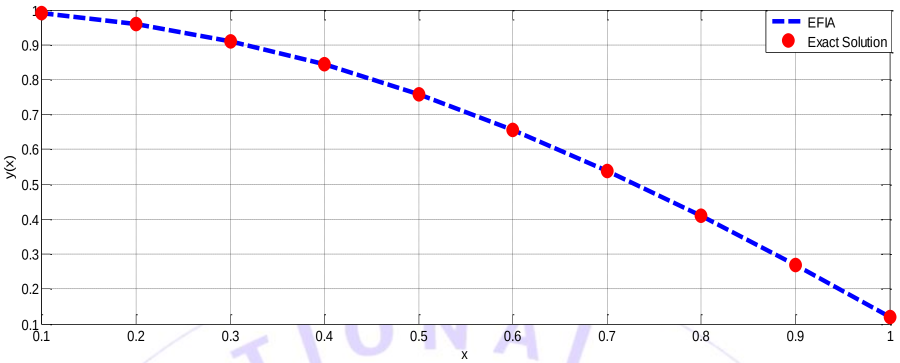

Figure 2 Solution curves of Problem 1

Problem 2

Consider the third order nonlinear system,

$$\left.\begin{aligned} y'''(x) &= \frac{1}{2}e^{4x}y_3(x)y_2'(x), \quad y_1(0)=1, y_1'(0)=-1, y_1''(0)=1, \\ y'''(x) &= \frac{8}{3}e^{2x}y_1(x)y_3'(x), \quad y_2(0)=1, y_2'(0)=-2, y_2''(0)=4, \\ y'''(x) &= 27e^{4x}y_2(x)y_1'(x), \quad y_3(0)=1, y_3'(0)=-3, y_3''(0)=9 \end{aligned}\right\} \tag{24}$$

whose exact solution is given by,

$$\left.\begin{aligned} y_1(x) &= e^{-x}, \\ y_2(x) &= e^{-2x}, \\ y_3(x) &= e^{-3x}. \end{aligned}\right\} \tag{25}$$

Similarly, the EFIA was applied in solving the nonlinear system (24). The approximate solutions obtained for the three components y_1, y_2 and y_3 in Table 2 showed that the EFIA performed better than the order nine I3PA developed by Allogmany and Ismail (2020). Figures 3-5 further illustrate the convergence of the approximate solutions of the three solution components to their respective solutions.

Table 2 Result for Problem 2 in terms of exact solution, approximate solution and absolute error

x	Exact Solution (y_1)	EFIA (y_1)	Error (EFIA)	Error (I3PA)
0.1	0.90483741803595951758	0.90483741803595951758	0.000000e+00	0.000000e+00
0.3	0.74081822068171787610	0.74081822068171787608	2.245162e-19	5.551115e-17
0.5	0.60653065971263342426	0.60653065971263342421	5.001921e-19	1.942890e-16
0.7	0.49658530379140952693	0.49658530379140952678	6.819212e-19	2.498002e-16
0.9	0.40656965974059910973	0.40656965974059910972	1.902122e-20	2.914335e-16

x	Exact Solution (y_2)	EFIA (y_2)	Error (EFIA)	Error (I3PA)
0.1	0.81873075307798182099	0.81873075307798182096	0.000000e+00	0.000000e+00
0.3	0.54881163609402650039	0.54881163609402650036	3.167221e-18	5.551115e-17
0.5	0.36787944117144233402	0.36787944117144233399	3.456312e-18	1.942890e-16
0.7	0.24659696394160648958	0.24659696394160648955	3.781001e-18	2.498002e-16
0.9	0.16529888822158653183	0.16529888822158653180	3.982415e-18	2.914335e-16

x	Exact Solution (y_3)	EFIA (y_3)	Error (EFIA)	Error (I3PA)
0.1	0.74081822068171776507	0.74081822068171776507	0.000000e+00	0.000000e+00
0.3	0.40656965974059916524	0.40656965974059916523	1.672812e-18	5.551115e-17
0.5	0.22313016014842981805	0.22313016014842981802	3.781262e-18	1.942890e-16
0.7	0.12245642825298194700	0.12245642825298194696	4.891222e-18	2.498002e-16
0.9	0.06720551273974975647	0.06720551273974975641	6.172829e-18	2.914335e-16

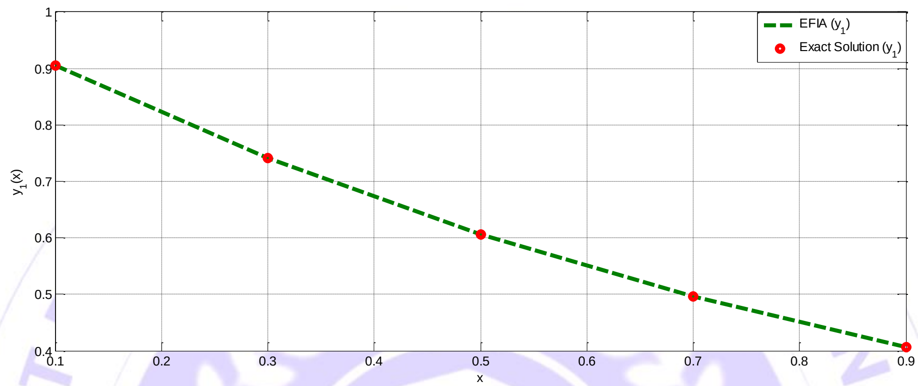

Figure 3 Solution curves of Problem 2 for y_1 component

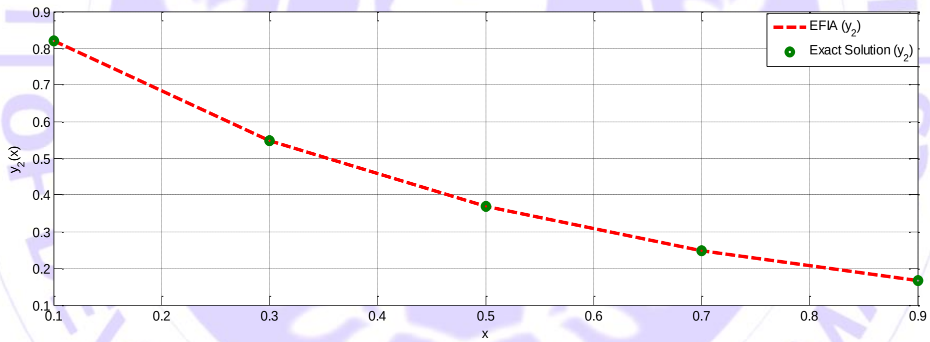

Figure 4 Solution curves of Problem 2 for y_2 component

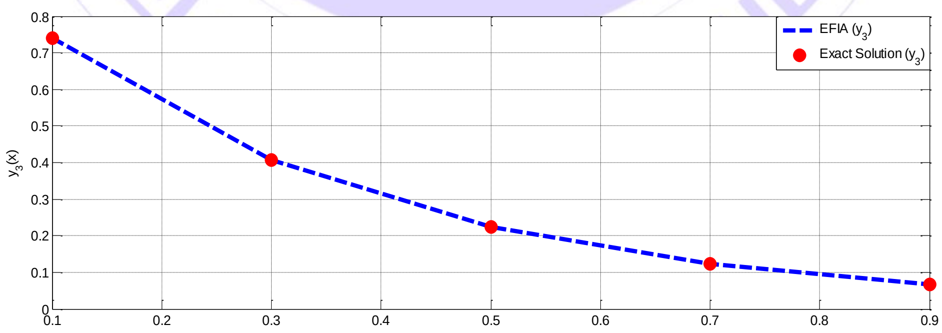

Figure 5 Solution curves of Problem 2 for y_3 component

Problem 3

Consider the third order problem,

$$y'''(x) = -4y'(x) + x, \quad y(0) = y'(0) = 0, y''(0) = 1 \tag{26}$$

with the exact solution given by,

$$y(x)=\left(\frac{3}{16}\right)(1-\cos 2x)+\frac{x^2}{8} \tag{27}$$

The EFIA was also applied to solve the third order problems in (26) at different values of the endpoints $b=5,10,15$ and 20. The same number of steps (NST) adopted in the eight-order 3S4PM developed by Obarhua (2023) were adopted but it was observed that the EFIA (of order five) gave better approximations by virtue of the results presented in Table 3. Figure 6 further buttresses the computational reliability of the EFIA as the approximate solutions converge to those of the exact solutions.

Table 3 Result for Problem 3 in terms of number of steps and absolute error at $h=0.1$

b	Algorithm	NST	Error
5	EFIA	46	2.11e-18
	3S4PM	46	6.44e-16
10	EFIA	61	3.02e-17
	3S4PM	61	1.68e-15
15	EFIA	76	3.15e-16
	3S4PM	76	2.85e-15
20	EFIA	91	5.76e-15
	3S4PM	91	1.10e-14

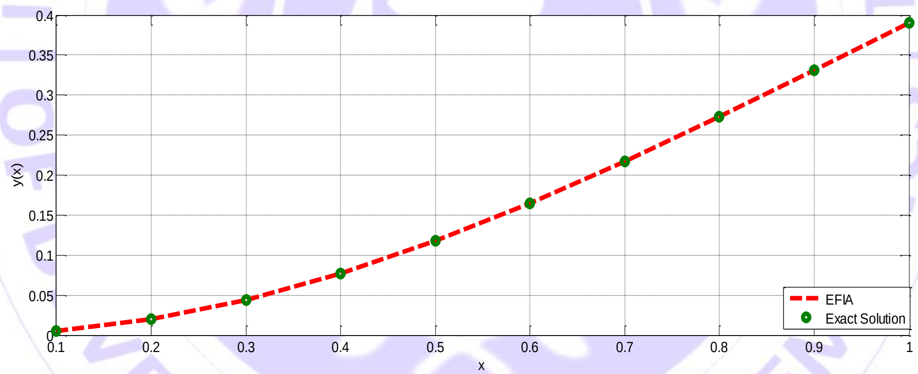

Figure 6 Solution curve for Problem 3

4.2 Applications to Thin Film Flow and Nonlinear Genesio Problems

Problem 4 *(Thin Film Flow Problem)*
Consider the thin film flow problem of a liquid on a surface. This engineering problem is governed by fluid dynamics problem of third-order given by,

$$y'''(x)=f\left(y(x)\right), \tag{28}$$

where $y(x)$ moves with the fluid in a coordinate frame and $f\left(y(x)\right)$ may vary depending on the physical context, such as

$$f\left(y(x)\right)=y^{-2}-1 \tag{29}$$

Equation (29) is a formula for a fluid draining problem on a dry surface, Allogmany and Ismail (2020). Fluid draining problem on a wet surface is given by,

$$f\left(y(x)\right)=\left(1+\psi+\psi^2\right)y^{-2}-\left(\psi+\psi^2\right)y^{-3}-1 \tag{30}$$

where ψ is the thickness of the film and $\psi > 0$. It is important to state that flow of thin film problems with free surface of viscous fluid where surface tension effects play a key role usually results to third order differential equations of the form,

$$y'''(x) = y^{-\lambda}(x), y(x_0) = \kappa_1, y'(x_0) = \kappa_2, y''(x) = \kappa_3, x \geq x_0 \tag{31}$$

where κ_1, κ_2 and κ_3 are real constants. Many authors solved the problem (28) using the initial conditions $y(0) = y'(0) = y''(0) = 1$ and $\lambda = 2$. Duffy and Wilson (1997) presented the parametric exact solution of (28). The newly derived algorithm (EFIA) is applied in solving the thing film problem and the results obtained were compared with those of Allogmany and Ismail (2020), that is I3PA. From the numerical results obtained in Table 4 and solution curves obtained in Figure 7, one can say that the EFIA approximated the problem reasonably.

Table 4 Result for Problem 4 in terms of exact solution, approximate solution and absolute error

x	Exact Solution	EFIA	Error (EFIA)	Error (I3PA)
0.1	1.000000000	1.000000000	0.0000e+00	0.0000e+00
0.2	1.221211030	1.221211033	3.1145e-09	1.0255e-06
0.4	1.488834893	1.488834897	3.7124e-09	1.1310e-07
0.6	1.807361404	1.807361404	1.1004e-11	6.3000e-09
0.8	2.179819234	2.179819234	4.1984e-11	8.0000e-11
1.0	2.608275822	2.608275822	7.0319e-11	9.5440e-07

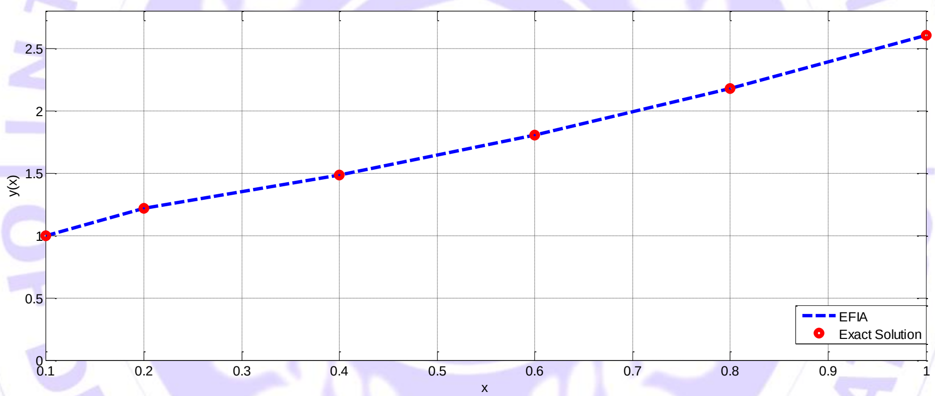

Figure 7 Solution curve for Problem 4

Problem 5 *(The Chaotic Nonlinear Genesio Problem)*
Consider the chaotic Genesio problem given by the third-order problem,

$$y'''(x) + \varepsilon y''(x) + \gamma y'(x) - f\left(y(x)\right) = 0, \tag{32}$$

where

$$f\left(y(x)\right) = \zeta y(x) + y(x)^2, \tag{33}$$

subject to the initial condition,

$$y(x_0) = 0.2, y'(x_0) = -0.3, y''(x_0) = 0.1, 0 \leq x \leq b \tag{34}$$

Note that ε, γ and ζ are positive real constants satisfying $\varepsilon\gamma < \zeta$. The exact solution of this problem is not known, Allogmany and Ismail (2020). The newly derived EFIA shall be used to solve Problem (32) at $\varepsilon = 1.2, \gamma = 2.92, \zeta = 6$ and different values of b.

The numerical solutions obtained in Table 5 clearly showed that the EFIA compares favorably with those of I3PA developed by Allogmany and Ismail (2020) and also the inbuilt Matlab solver (ode15s). The efficiency curves presented in Figure 8 illustrate the fact that the EFIA requires fewer numbers of steps to achieve accuracy higher than those of I3PA and ode15s. It is thus evident that the proposed algorithm is efficient.

Table 5 Result of Problem 5 in terms of number of steps and approximate solution

b	h	Algorithm	NST	Approximate Solution
1.0	0.1	EFIA	4	-0.0540040835391199
		I3PA	4	-0.0540040835391235
		ode15s	12	-0.0540040835065342
	0.01	EFIA	20	-0.0540040835547436
		I3PA	34	-0.0540040835547517
		ode15s	64	-0.0540040835547909
4.0	0.1	EFIA	12	-0.0676305906240667
		I3PA	13	-0.0676305906240893
		ode15s	36	-0.0676305905627812
	0.01	EFIA	118	-0.0676306051591378
		I3PA	133	-0.0676306051591404
		ode15s	256	-0.0676305990645642

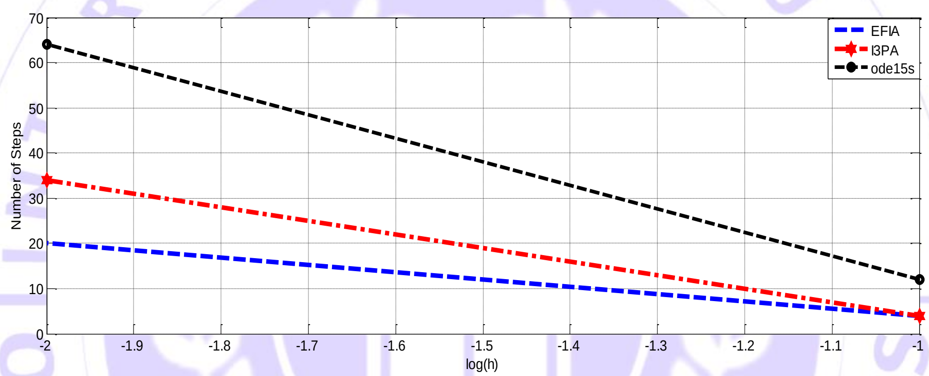

Figure 8 Efficiency curves for Problem 5

5. Conclusions

An algorithm has been derived in this research for the solution of third-order problems of the form (1). The algorithm, EFIA, was developed via the combination of exponential functions with power series. The paper further analyzed some basic properties of the EFIA. The method was applied to solving problems like boundary layers, thin film flow, and some general third-order differential equations. The results obtained clearly illustrated the computational reliability of the EFIA. Further research may focus on the development of optimized algorithms using alternative basis functions.

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