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Perturbed Collocation Hybrid Methods with Fixed Stepsize for Nonlinear Systems of Initial Value Problems

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ABSTRACT

The perturbed collocation methods are effective numerical methods for solving stiff system of ordinary differential equations. In this study, we develop new numerical methods based on perturbation of the collocation process and discuss some applications and illustrative examples for system of ordinary differential equations. This class incorporates some off-grid points into the numerical schemes, extending the implicit linear multistep methods developed hybrid type. Absolute and residual error analysis are used to qualitatively and objectively examine the stability, convergence, and accuracy of the suggested procedures. These methods provide high-accuracy approximations of system of equations across the integration interval, for example, when compared to the conventional methods. Effective utilization of the new integrators' applications yields physical interpretations of what the complex systems of ordinary differential equations represent in natural occurrences, that is, the genuine representation of the systems in real life. These are first demonstrated as phase plots with strange and novel characteristics. The surface phase plots that are obtained represent segments or subsets of a system's phase space and frequently illustrate facts observed in the actual world. The discrepancies between the exact solutions and the numerical solutions which are presented in tabular and graphical forms can be used to evaluate how accurate the numerical solutions are. The results in Tables and Figures obtained support our conclusion, highlighting the revolutionary potential of the perturbed collocation methods in furthering the numerical approximation of system of ordinary differential equations in the physical world.

1. Introduction

An equation involving a function of one variable and its derivatives with regard to that variable is known as an ordinary differential equation (ODE). An ordinary differential equation is one type of differential equation (DE) that has been used in the physical world since before the invention of digital computers. In various fields of science, engineering, and economics, they are commonly found in mathematical models. Ordinary differential equations have recently emerged in a number of fields of study, including the Kepler problems, chemical kinetics problems (see, Butcher (2003)), medicine, pharmacokinetics models (see, Yakubu et al. (2023)) aircraft stability, nuclear reactions, and many more. The general form of a system of ordinary differential equation is given as,

$$y'(x) = f(x, y(x)), \tag{1}$$

where x is a real variable, $y : R \to R^n$ is a vector-valued function of x, $f : R^{n+1} \to R^n$, and $y'(x) = dy(x)/dx$ denotes the derivative with respect to x, that is,

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$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix} = \begin{bmatrix} dy_1(x)/dx \\ dy_2(x)/dx \\ \vdots \\ dy_n(x)/dx \end{bmatrix}. \tag{2a}$$

Thus, we have a system of coupled differential equations in which the function f is known and we wish to determine the unknown function y . An important special case, which we will often consider for simplicity, is $n = 1$, that is, a single scalar ODE. If the first derivative is the highest order derivative of the solution function appearing in the equation, an ODE is said to be of *first order*. Equations with higher-order derivatives occur frequently in practice but can be transformed into an equivalent first-order system as follows. For example, given an n^{th} order scalar equation

$$u^{(n)} = f(x, u, u', \ldots, u^{(n-1)}).$$

Define the n new unknowns $y_1(x) = u, y_2(x) = u', \ldots, y_n(x) = u^{(n-1)}$, so that the original equation becomes the first-order system of n equations as follows,

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_{n-1}'(x) \\ y_n'(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ f(x, y_1, y_2, \ldots, y_{n+1}) \end{bmatrix}. \tag{2b}$$

Thus, in general, a scalar ODE of order n is equivalent to a system of n first-order ODEs. If f is nonlinear in y , then the ODE is called nonlinear, otherwise the ODE is a linear, which can be expressed in the following form:

$$y'(x) = A(x)y(x) + q(x), \qquad a < x < b, \tag{3a}$$

where $A(x) \in R^{n \times n}$ and $q(x) \in R^n$. In both linear and nonlinear cases, the interval ends a and b can be finite or infinite. Without loss of generality, we consider first order linear and nonlinear ODEs here. When $q(x) = 0$, the ODE is called homogeneous, otherwise it is non-homogeneous.

In general, an nth order system will have n linearly independent solutions and in order to impose uniqueness, further conditions must be specified on the solution. Conditions fall into two main categories- Initial Conditions and Boundary Conditions and give rise to Initial Value Problems (IVPs) and Boundary Value Problems (BVPs) respectively. An IVP which is the main concern of the study has the following general form:

$$\begin{cases} y'(x) = f(x, y(x)), & a < x < b, \\ y(a) = y_0. \end{cases} \tag{3b}$$

The theory and numerical techniques dealing with IVPs are compared with those of BVPs. The unique solution is guaranteed to exist under very mild assumptions. It follows from the first assumption,

Assumption 1: *In the ODE (1), the function* f *belongs to* C^1 *-class, and therefore, satisfies the Lipschitz condition with the constant L. That is, if the estimation*

$$\|f(x,y)-f(x,\tilde{y})\| \le L\|y-\tilde{y}\|,$$

holds, L is called the Lipschitz constant.

Definition 1.1: Collocation methods: *Collocation methods can be described as methods, which involve the determination of approximate solution in a suitable set of functions, sometimes called basis functions. The approximate solution is required to satisfy the differential equation of the type in (1) and its supplementary conditions at certain points in the range of interest, called the collocation points.*

Definition 1.2: Lambert (1973). Hybrid methods: *Two extreme approaches for developing numerical solutions for ordinary differential equations are extrapolation and substitution. Here, "substitution methods" refers to Runge-Kutta-type methods and "extrapolation methods" refers to linear multistep methods. Thus, the introduction of hybrid formula as an important step, may be regarded as a step into the no man's land. Using the notation* $f_{n+v} = f(x_{n+v}, y_{n+v})$ *in order to implement the formula of the* k *-step hybrid method, even when it is explicit (that is* $\beta_k = 0$*) a special predictor to estimate* y_{n+v} *is necessary.*

2.0 Construction of the solution methods by perturbation of the collocation points

The development of numerical methods for approximating solutions of system of initial value problems (IVPs) in ordinary differential equations (ODEs) has attracted considerable attention in the recent decades and many individuals have shown interest in constructing efficient methods with good stability properties for the numerical integration of ordinary differential equations. Although, a very wide variety of numerical methods have been proposed, the number of methods with high order and good stability properties remains relatively small. Among the methods for the numerical solutions of ODEs, the discrete variable methods (DVMs), which systematically generate the approximate solutions $\{y_n\}$ along the step-points $\{x_n\}$, $x_n = a + nh$, are the most popular(see, Mitsui and Yakubu (2011)). Here h denotes the positive stepsize and y_n is assumed to approximate $y(x_n)$. Many of the existing methods such as the Euler method, Runge-Kutta methods and linear multistep methods fall into this category of discrete variable methods. They are respectively written as,

Euler method:

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad (4a)$$

Runge-Kutta methods:

$$\begin{cases} Y_i = y_n + h\sum_{j=1}^{i-1}\alpha_{ij} f(x_n + c_j h, Y_j), \\ y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i), \end{cases} \qquad (4b)$$

Linear multistep methods:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}) \cdot \tag{4c}$$

Note that in the above formulations the parameters $\alpha_{ij}, b_i, c_i, \alpha_j$ and β_j characterize a particular method together with the number of stages, for instance s in the Runge-Kutta case, or with the number of steps k in the case of linear multistep method. The above referenced methods are very popular because of their flexible capabilities for the solution of ODEs. A conventional discrete variable method has the following policy. Let k be the number of steps incorporated in a method (a k-step method), and we assume that the back values $y_n, y_{n+u}, \ldots, y_{n+k-1}$ are available. At the present step-point x_{n+k-1}, we compute y_{n+k} in a constructive way using the information y_{n+j} and $f(x_{n+j}, y_{n+j}), \quad j = 0,1,2,\ldots,k-1$. The discrete variable methods have been widely applied because of their flexible capabilities for the solutions of general IVPs in ODEs. Also we should remark some features of the methods. First, a DVM should usually be 'linear' with respect to the functional values of y and f to cope effectively with the systems of ODEs. Secondly, the stepsize h is assumed to be a constant, in particular for theoretical analysis, whereas an adaptive *stepsize control* is significant for practice.

In what follows, a new class of perturbed collocation hybrid methods for the numerical solution of systems of initial-value problems of ordinary differential equations is proposed. This is an extension of linear multistep hybrid methods by the inclusion of the off-grid points in the numerical schemes. Basically, it consists of members in a block and their short description are given as follows. We assume a k-step method with the back values $y_n, y_{n+u}, \ldots, y_{n+k-1}$. Further, we suppose a certain approximation of the value $y_{n+k}^{[0]}$ is available. Then we compute the value y_{n+k+1} by a linear multistep hybrid scheme employing $y_n, y_{n+u}, \ldots, y_{n+k}^{[0]}$ and their derivatives. The process of iteration will continue till the solution becomes sufficiently small in magnitude, that is, till the estimate,

$$\left\| y_{n+k}^{[m+1]} - y_{n+k}^{[m]} \right\| \le \delta_{TOL} , \tag{5}$$

is attained for a pre-assigned δ_{TOL}. The process of iteration is restarted on the right-shifted step-points $x_{n+u}, y_{n+u}, y_{n+w}, \ldots, x_{n+k+1}$. The previously calculated $y_{n+k+1}^{[m]}$ plays the role of the initial guess for y_{n+k+1} and so on.

2.1 Perturbed collocation hybrid method of order 3 with two off-step points

This section specify certain perturbed collocation of the hybrid methods, that is, through careful selection of collocation points, which will make the methods zero-stable and have smaller error constants than their equivalents in the literature. This is due to the fact that when integrating differential equations, hybrid approaches with smaller error constants, *A-stable* and *L-stable*, are preferred. In order to increase the methods' accuracy, collocation points known as intra-step points are introduced. They are represented by u, v and have been chosen in particular to guarantee that the methods can retain zero-stability and appropriately integrate linear and nonlinear system of initial value problems (see, Fatunla, 1991). The first perturbed collocation hybrid method is obtained in the sense of Yakubu et al. (2017), and the coefficients are compactly displayed as follows,

$$\begin{bmatrix} y_{n+u} \\ y_{n+v} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} y_n \\ y_n \\ y_n \end{bmatrix} + h \begin{bmatrix} \frac{15-45\sqrt{5}y'_n}{120} \\ \frac{15+45\sqrt{5}y'_n}{120} \\ \frac{105y'_n}{60} \end{bmatrix} + h \begin{bmatrix} \frac{-35+17\sqrt{5}}{120} & \frac{55+23\sqrt{5}}{120} & \frac{-95-55\sqrt{5}}{120} \\ \frac{55-23\sqrt{5}}{120} & \frac{-35-17\sqrt{5}}{120} & \frac{-95+55\sqrt{5}}{120} \\ \frac{-40+21\sqrt{5}}{60} & \frac{-40-21\sqrt{5}}{60} & \frac{35}{60} \end{bmatrix} \begin{bmatrix} y'_{n+u} \\ y'_{n+v} \\ y'_{n+1} \end{bmatrix}. \qquad (6)$$

2.2 Perturbed collocation hybrid method of order 4 with three off-step points

In order to have a better and more accurate method of higher order with some of the requirements mentioned above in the first subsection, we carefully select the collocation points in the second method. Consequently, we choose three precise interior collocation points rather than the two interior collocation points above and denote them by u, v and w respectively, as indicated below. The coefficients of the method obtained are shown below:

$$\begin{bmatrix} y_{n+u} \\ y_{n+w} \\ y_{n+v} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} y_n \\ y_n \\ y_n \\ y_n \end{bmatrix} + h \begin{bmatrix} \frac{695-125\sqrt{5}y'_n}{600} \\ \frac{5465y'_n}{4800} \\ \frac{695+125\sqrt{5}y'_n}{600} \\ \frac{1105y'_n}{300} \end{bmatrix} + h \begin{bmatrix} \frac{1405-631\sqrt{5}}{600} & \frac{-4960-800\sqrt{5}}{600} & \frac{3655+1631\sqrt{5}}{600} & \frac{-1095-375\sqrt{5}}{600} \\ \frac{6835-3042\sqrt{5}}{4800} & \frac{-15520}{4800} & \frac{6835+3042\sqrt{5}}{4800} & -\frac{1215}{4800} \\ \frac{3655-1631\sqrt{5}}{600} & \frac{-4960+800\sqrt{5}}{600} & \frac{1405+631\sqrt{5}}{600} & \frac{-1095+375\sqrt{5}}{600} \\ \frac{2120-939\sqrt{5}}{300} & -\frac{4640}{300} & \frac{2120+939\sqrt{5}}{300} & -\frac{405}{300} \end{bmatrix} \begin{bmatrix} y'_{n+u} \\ y'_{n+w} \\ y'_{n+v} \\ y'_{n+1} \end{bmatrix}. \qquad (7)$$

Compared to their equivalents in the literature, the methods so derived, which comprise a few members in a block, generally provide more accurate approximations to the exact solution of systems of high order differential equations. They have smaller error constants compared to the conventional techniques of the same order in use (the linear multistep methods and the Runge–Kutta methods). The members $[y_{n+u}, y_{n+w}, y_{n+v}, y_{n+1}]$ are only defined implicitly by the methods in the block. Therefore in their applications to solve system of ordinary differential equations, a nonlinear equation has to be solved at each time step.

3.0 Analysis of the numerical properties of the perturbed collocation hybrid methods

3.1 Order of the perturbed collocation hybrid methods

From zero-stability and consistency, order of convergence and stability characteristics, order of a method is one of the essential aspects of a newly constructed method. Thus, we determine the order of the derived methods here. Following Lambert (1973), we show how to calculate order using method (7) in subsection 2.2 above.

$$\vartheta_0 = -\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vartheta_u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vartheta_w = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vartheta_v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vartheta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\upsilon_0 = \begin{bmatrix} \frac{695-125\sqrt{5}}{600} \\ \frac{5465}{4800} \\ \frac{695+125\sqrt{5}}{600} \\ \frac{1105}{300} \end{bmatrix}, \quad \upsilon_u = \begin{bmatrix} \frac{1405-631\sqrt{5}}{600} \\ \frac{6835-3042\sqrt{5}}{4800} \\ \frac{3655-1631\sqrt{5}}{600} \\ \frac{2120-939\sqrt{5}}{300} \end{bmatrix}, \quad \upsilon_w = \begin{bmatrix} \frac{4960-800\sqrt{5}}{600} \\ \frac{-15520}{4800} \\ \frac{-4960+800\sqrt{5}}{600} \\ \frac{-4640}{300} \end{bmatrix},$$

$$\upsilon_v = \begin{bmatrix} \frac{3655+1631\sqrt{5}}{600} \\ \frac{6835+3042\sqrt{5}}{4800} \\ \frac{1405+631\sqrt{5}}{600} \\ \frac{2120+939\sqrt{5}}{300} \end{bmatrix}, \qquad \upsilon_1 = \begin{bmatrix} \frac{-1095-375\sqrt{5}}{600} \\ -\frac{1215}{4800} \\ \frac{-1095+375\sqrt{5}}{600} \\ \frac{-405}{300} \end{bmatrix}.$$

To obtain the order, we substitute the above matrices into the below equations and evaluate recursively to have,

$$C_0 = \vartheta_0 + \vartheta_u + \vartheta_w + \vartheta_v + \vartheta_1 = 0$$

$$C_1 = \left[\vartheta_1 + (u)\vartheta_u + (w)\vartheta_w + (v)\vartheta_v\right] - \left(\upsilon_0 + \upsilon_u + \upsilon_w + \upsilon_v + \upsilon_1\right) = 0$$

$$C_2 = \tfrac{1}{2!}\left[\vartheta_1 + (u)^2\vartheta_u + (w)^2\vartheta_w + (v)^2\vartheta_v\right] - \left(\upsilon_1 + (u)\upsilon_u + (w)\upsilon_w + (v)\upsilon_v\right) = 0$$

$$C_3 = \tfrac{1}{3!}\left[\vartheta_1 + (u)^3\vartheta_u + (w)^3\vartheta_w + (v)^3\vartheta_v\right] - \left(\tfrac{1}{2}\right)\left[\upsilon_1 + (u)^2\upsilon_u + (w)^2\upsilon_w + (v)^2\upsilon_v\right] = 0$$

$$C_4 = \tfrac{1}{4!}\left[\vartheta_1 + (u)^4\vartheta_u + (w)^4\vartheta_w + (v)^4\vartheta_v\right] - \left(\tfrac{1}{3!}\right)\left[\upsilon_1 + (u)^3\upsilon_u + (w)^3\upsilon_w + (v)^3\upsilon_v\right] = 0$$

$$C_5 = \tfrac{1}{5!}\left[\vartheta_1 + (u)^5\vartheta_u + (w)^5\vartheta_w + (v)^5\vartheta_v\right] - \left(\tfrac{1}{4!}\right)\left[\upsilon_1 + (u)^4\upsilon_u + (w)^4\upsilon_w + (v)^4\upsilon_v\right] \neq 0$$

and after routine algebraic simplifications we have,

$$C_0 = C_1 = C_2 = C_3 = C_4 = 0, \text{ but } C_5 \neq 0.$$

This shows that the highest order of the newly derived perturbed collocation hybrid methods is four (4) with error constants displayed in Table 1 below.

3.2 Zero-stability analysis of the perturbed collocation hybrid methods

A numerical method is said to be zero-stable if it advances the solution in time for system of higher order equations, or dynamical systems of ordinary differential equations while preserving the stability of the trivial solution. This suggests that some minor adjustments to the starting conditions won't cause the solution to expand or become infinite, but will instead remain small while the solution is being calculated. Zero stability is an important characteristic of a numerical method since it ensures that the method will produce meaningful results even when the initial conditions are simply approximate. A non-zero-stable numerical method may produce solutions that are unbounded or diverge from the true solution during calculation. In these cases, the method may be considered unreliable and may need to be improved in order to produce more accurate results.

Writing the perturbed collocation hybrid method in (7) in a matrix form, the following equations are produced to ascertain the hybrid methods' zero-stability:

$$U^{(1)}Y_{\mu} = U^{(0)}Y_{\mu-1} + h\left(V^{(0)}F_{\mu-1} + V^{(1)}F_{\mu}\right), \tag{8}$$

where

$$Y_{\mu} = \left(y_{n+u}, y_{n+w}, y_{n+v}, y_{n+1}\right)^{T},\ Y_{\mu-1} = \left(y_{n-u}, y_{n-w}, y_{n-v}, y_{n}\right)^{T},$$

$$F_{\mu} = \left(f_{n+u}, f_{n+w}, f_{n+v}, f_{n+1}\right)^{T},\ F_{\mu-1} = \left(f_{n-u}, f_{n-w}, f_{n-v}, f_{n}\right)^{T},$$

and $U^{(1)}, U^{(0)}, V^{(1)}$ and $V^{(0)}$ in this case are maximum of 4 by 4 matrices with their corresponding entries given by the coefficients of the method in subsection (2.2). The matrices for the method are given as below where $U^{(1)}$ is the matrix identity and of dimension 4,

$$U^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, U^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{9}$$

$$V^{(0)} = \begin{bmatrix} \frac{695-125\sqrt{5}}{600} \\ \frac{5465}{4800} \\ \frac{695+125\sqrt{5}}{600} \\ \frac{1105}{300} \end{bmatrix}, \quad V^{(1)} = \begin{bmatrix} \frac{1405-631\sqrt{5}}{600} & \frac{-4960-800\sqrt{5}}{600} & \frac{3655+1631\sqrt{5}}{600} & \frac{-1095-375\sqrt{5}}{600} \\ \frac{6835-3042\sqrt{5}}{4800} & \frac{-15520}{4800} & \frac{6835+3042\sqrt{5}}{4800} & -\frac{1215}{4800} \\ \frac{3655-1631\sqrt{5}}{600} & \frac{-4960+800\sqrt{5}}{600} & \frac{1405+631\sqrt{5}}{600} & \frac{-1095+375\sqrt{5}}{600} \\ \frac{2120-939\sqrt{5}}{300} & -\frac{4640}{300} & \frac{2120+939\sqrt{5}}{300} & -\frac{405}{300} \end{bmatrix}$$

The zero-stability is always the stability of the difference system by taking the limit when h tends to zero. Hence, as $h \to 0$, the method in (7) becomes,

$$U^{(1)}Y_{\mu} - U^{(0)}Y_{\mu-1} = 0, \tag{10}$$

with $\rho(\eta)$ as the first characteristic polynomial of the system written as

$$\rho(\eta) = \det\left(\eta U^{(1)} - U^{(0)}\right). \tag{11}$$

Substituting the matrices $U^{(1)}$ and $U^{(0)}$ from equation (9), into (11) we have

$$\rho(\eta) = \left(\eta \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right). \tag{12}$$

Simplifying (12) to obtain,

$$\rho(\eta) = \eta^3(\eta - 1) = 0, \quad \Rightarrow \eta = (0,0,0,1). \qquad (13)$$

Since $\eta = (0,0,0,1)$ the method in (7) is zero stable.

Similarly, it is possible to show that the method in equation (6) is zero-stable following the same calculation.

Definition 3.1: Lambert (1973) (Zero-stable). *The family of methods in section (2) are said to be zero-stable if the roots condition*

$$\rho(\lambda) = \det\left[\sum_{i=0}^{k} A^i \lambda^{k-1}\right] = 0, \qquad (14)$$

satisfies the inequality $\left|\lambda_j\right| \le 1,\ j = 1,\ 2,\ 3,\ldots,k$. *But when the inequality reduces to* $\left|\lambda_j\right| = 1$, *then the multiplicity should not be greater than one.*

According to definition 3.1, the developed perturbed collocation hybrid methods are zero-stable, making them appealing for solving problems involving dynamical systems and system of high order equations. Based on our calculations, the orders and error constants of the derived perturbed collocation methods are shown in Table 1. It is clear from the Table that the perturbed collocation hybrid techniques are higher-order and, as a result, more accurate than some of the collocation methods currently used in the literature.

Table 1: Order and Error Constants of the Perturbed Collocation Hybrid Methods

Method	Order	Error constants
Method (6)	i) y_{n+u}, P = 3 ii) y_{n+v} ,P = 3 iii y_{n+1}, P = 3	$C_4 = 4.8611\times10^{-2}$ $C_4 = 9.3274\times10^{-2}$ $C_4 = 8.7894\times10^{-2}$
Method (7)	i) y_{n+u}, P = 4 ii) y_{n+w}, P = 4 iii) y_{n+v} ,P = 4 iv) y_{n+1}, P = 4	$C_5 = 1.1218\times10^{-2}$ $C_5 = 2.1762\times10^{-3}$ $C_5 = 2.1762\times10^{-2}$ $C_5 = 1.3661\times10^{-2}$

Definition 3.2: Lambert (1991) (Consistency). *The family of methods in section (2) are consistent if the order* p *of the method satisfy the inequality* $p \ge 1$,

(i) $\rho(1) = 0$ *and*

(ii) $\rho'(1) = \sigma(1)$, *where* $\rho(z)$ *and* $\sigma(z)$ *are 1st and 2nd characteristic polynomials which are assumed to have no common factor.*

From Table 1 and definition 3.1, we can attest that the derived perturbed collocation hybrid methods are consistent.

Definition 3.3: Butcher (1987) (*A(* α *)-stable). The family of methods in section (2) are A(* α *)-stable if the inequality* $0 < \alpha \le \pi / 2$ *and the angular sector*

$$S_\alpha = \{z \in C : |\arg(-z)| < \alpha, z \ne 0\},$$

is contained in the stability domain $\Re$ *. When the interval of absolute stability consists of the negative real axis, the method is said to be* $A_0 - stable$.

Definition 3.4: Ramos and Patricio (2014). *The family of methods given in section (2) are said to be irreducible if the continuous coefficients* $\vartheta_j(x)$ *and* $\upsilon_j(x)$ *have no common factor.*

3.3 Order of convergence of the perturbed collocation hybrid methods

In this section we investigate the order of convergence of the newly derived perturbed collocation block hybrid numerical integrators. To help us examine the methods' convergence, we provide few definitions.

Definition 3.5: Lambert (1973). *A function* $f(x, y)$ *satisfies the Lipschitz condition on the domain* $D \subseteq \Re^2$ *if there exists a constant* $L > 0$ *such that,*

$$|f(x, y) - f(x, y^*)| \le L|y - y^*|,$$

for all $(x, y), (x, y^*) \in D$.

Definition 3.6: *A linear multistep method (LMM) is said to be convergent, if for all initial value problem subject to the above Lipschitz condition:*

$$\lim_{h \to 0} y_n = y^*(x_n),$$

for all solutions $\{y_n\}$ *of the method (see, Lambert, 1973,page 22).*

For illustration we consider the block method given in section (2.2), that is, method (7). Assuming that the exact solution of y_{n+1} is y^*_{n+1} which is written precisely as

$$y^*_{n+1} = y^*_n + \frac{1105\, hf_n}{300} + \frac{\left(2120 - 939\sqrt{5}\right) hf_{n+u}}{300} - \frac{4640\, hf_{n+w}}{300} + \frac{\left(2120 + 939\sqrt{5}\right) hf_{n+v}}{300} - \frac{405\, hf_{n+1}}{300}$$
$$+ \frac{5}{366} h^5 y^{*(5)}(\xi_n) + R_6,$$

where R_6 is the remainder term. Subtract y_{n+1} from y^*_{n+1} to obtain

$$
\begin{aligned}
y^*_{n+1} - y_{n+1} = y^*_n - y_n &+ \frac{1105\,h}{300}\left[f\left(x_n, y^*_n\right) - f\left(x_n, y_n\right)\right] + \frac{\left(2120 - 939\sqrt{5}\right)h}{300}\left[f\left(x_{n+u}, y^*_{n+u}\right) - f\left(x_{n+u}, y_{n+u}\right)\right] \\
&- \frac{4640\,h}{300}\left[f\left(x_{n+w}, y^*_{n+w}\right) - f\left(x_{n+w}, y_{n+w}\right)\right] + \frac{\left(2120 + 939\sqrt{5}\right)h}{300}\left[f\left(x_{n+v}, y^*_{n+v}\right) - f\left(x_{n+v}, y_{n+v}\right)\right] \\
&- \frac{405\,h}{300}\left[f\left(x_{n+1}, y^*_{n+1}\right) - f\left(x_{n+1}, y_{n+1}\right)\right] + \frac{5h^5}{366} y^{*(5)}\left(\xi_n\right) + R_6 \ .
\end{aligned}
$$

Letting $d_n = y^*_n - y_n, \ldots, d_{n+v} = y^*_{n+v} - y_{n+v}, d_{n+1} = y^*_{n+1} - y_{n+1}$, then we have

$$
\begin{aligned}
\left|d_{n+1}\right| \le \left(1 + \frac{1105\,h}{300} L\right)\left|d_n\right| &+ \frac{\left(2120 - 939\sqrt{5}\right)h}{300} L\left|d_{n+u}\right| - \frac{4640\,h}{300} L\left|d_{n+w}\right| + \frac{\left(2120 + 939\sqrt{5}\right)h}{300} L\left|d_{n+v}\right| \\
&- \frac{405\,h}{300} L\left|d_{n+1}\right| + \frac{5h^5}{366} y^{*(5)}\left(\xi_n\right) + R_6 \ .
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
\left(1 + \frac{405\,h}{300} L\right)\left|d_{n+1}\right| \le \left(1 + \frac{1105\,h}{300} L\right)\left|d_n\right| &+ \frac{\left(2120 - 939\sqrt{5}\right)h}{300} L\left|d_{n+u}\right| - \frac{4640\,h}{300} L\left|d_{n+w}\right| \\
&+ \frac{\left(2120 + 939\sqrt{5}\right)h}{300} L\left|d_{n+v}\right| + \frac{5h^5}{366} y^{*(5)}\left(\xi_n\right) + \mathrm{O}\left(h^6\right)
\end{aligned}
$$

As $h \to 0$ with the exception of $\left|d_n\right|$ every other term on the right hand side tends to zero while on the left hand side we are left with $\left|d_{n+1}\right|$ and by definition of convergence, we have

$$
\lim_{h \to 0} y_{n+1} = y^*_{n+1} \ .
$$

The convergence of y_{n+1} is established in agreement with some authors (see, [Ismail et al. (2020) and Akinola and Akoh (2023)]). Therefore it can be shown that the other members of the block, y_{n+u}, y_{n+w} and y_{n+v} converge. Hence, we have shown that the new perturbed collocation block hybrid numerical integrators are convergent, the remainder term R_6 which is defined as

$$
R_{p+2} = C_{p+2} h^{p+2} y^{*(p+2)}\left(\xi\right) = \mathrm{O}\left(h^6\right).
$$

Therefore, the highest order of convergence of the new perturbed collocation block hybrid methods is 6.

3.4 Regions of Absolute Stability of the Perturbed Collocation Hybrid Methods

To study the linear stabiity properties of the perturbed collocation hybrid methods we reformulate them as general linear methods (see, Burrage and Butcher, 1979,1980). Hence, we use the notations introduced by Butcher (2008), where a general linear method is represented by a partition $(s + r) \times (s + r)$ matrices (containing A,U,B,V),

$$
\begin{bmatrix} Y^{[n]} \\ y^{[n-1]} \end{bmatrix} = \left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] \begin{bmatrix} hf\left(Y^{[n]}\right) \\ y^{[n]} \end{bmatrix}, \quad n = 1, 2, \ldots, N \tag{15}
$$

where

$$Y^{[n]}=\begin{bmatrix} Y_1^{[n]} \\ \vdots \\ Y_2^{[n]} \\ Y_s^{[n]} \end{bmatrix},\quad y^{[n-1]}=\begin{bmatrix} y_1^{[n-1]} \\ \vdots \\ y_2^{[n-1]} \\ y_r^{[n-1]} \end{bmatrix},\quad f\left(Y^{[n]}\right)=\begin{bmatrix} f\left(Y_1^{[n]}\right) \\ \vdots \\ f\left(Y_2^{[n]}\right) \\ f\left(Y_s^{[n]}\right) \end{bmatrix},\quad y^{[n]}=\begin{bmatrix} y_1^{[n]} \\ \vdots \\ y_2^{[n]} \\ y_r^{[n]} \end{bmatrix},$$

and r denotes quantities as output from each step and input to the next step and s denotes stage values used in the computation of the steps $Y_1, Y_2, \cdots, Y_s$. The coefficients of these matrices indicate the relationship between the various numerical quantities that arise in the computation of the stability regions. To simplify the calculation of the region of absolute stability we swap out the matrices V and U for C and D respectively. The elements of the matrices A, C, B and D are substituted into the stability matrix. In the sense of Dahlquist (1963), we apply (15) to the linear test equation $y' = \lambda y,\ x \geq 0$ and $\lambda \in \mathbb{C}$ ($\mathbb{C}$ is a complex number) which leads to the recurrent relation

$$y^{[n+1]} = M(z)y^{[n]}, n = 1,2,\ldots, N-1,\ z = \lambda h\,,$$

where the stability matrix $M(z)$ is defined by

$$M(z) = C + zB(I - zA)^{-1}D\,.$$

For the stability regions, we adopt the approach of Akinola and Akoh (2023), where the perturbed collocation hybrid methods are expressed as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} y_{n+u} \\ \vdots \\ y_{n+1} \end{pmatrix}=\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}\begin{pmatrix} y_{n-u} \\ \vdots \\ y_{n} \end{pmatrix}+\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}\begin{pmatrix} f_{n+u} \\ \vdots \\ f_{n+1} \end{pmatrix}+$$
$$\begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{pmatrix}\begin{pmatrix} f_{n-u} \\ \vdots \\ f_{n} \end{pmatrix}$$

with

$$AY_m = BY_{m-1} + hCF_m + hDF_{m-1}\,. \tag{16}$$

Using method (7) to show the calculation of the region of absolute stability, following Akinola and Akoh (2023) we have the method expressed as

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+u} \\ y_{n+w} \\ y_{n+v} \\ y_{n+1} \end{bmatrix},\qquad B=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-u} \\ y_{n-w} \\ y_{n-v} \\ y_{n} \end{bmatrix},$$

$$C=\begin{bmatrix} \frac{1405-631\sqrt{5}}{600} & \frac{-4960-800\sqrt{5}}{600} & \frac{3655+1631\sqrt{5}}{600} & \frac{-1095-375\sqrt{5}}{600} \\ \frac{6835-3042\sqrt{5}}{4800} & \frac{-15520}{4800} & \frac{6835+3042\sqrt{5}}{4800} & -\frac{1215}{4800} \\ \frac{3655-1631\sqrt{5}}{600} & \frac{-4960+800\sqrt{5}}{600} & \frac{1405+631\sqrt{5}}{600} & \frac{-1095+375\sqrt{5}}{600} \\ \frac{2120-939\sqrt{5}}{300} & -\frac{4640}{300} & \frac{2120+939\sqrt{5}}{300} & \frac{-405}{300} \end{bmatrix}\begin{bmatrix} f_{n+u} \\ f_{n+w} \\ f_{n+v} \\ f_{n+1} \end{bmatrix},\ D=\begin{bmatrix} 0 & 0 & 0 & \frac{695-125\sqrt{5}}{600} \\ 0 & 0 & 0 & \frac{5465}{4800} \\ 0 & 0 & 0 & \frac{695+125\sqrt{5}}{600} \\ 0 & 0 & 0 & \frac{1105}{300} \end{bmatrix}\begin{bmatrix} f_{n-u} \\ f_{n-w} \\ f_{n-v} \\ f_{n} \end{bmatrix}.$$

Substituting the matrices (ABCD) into the characteristics equation we have

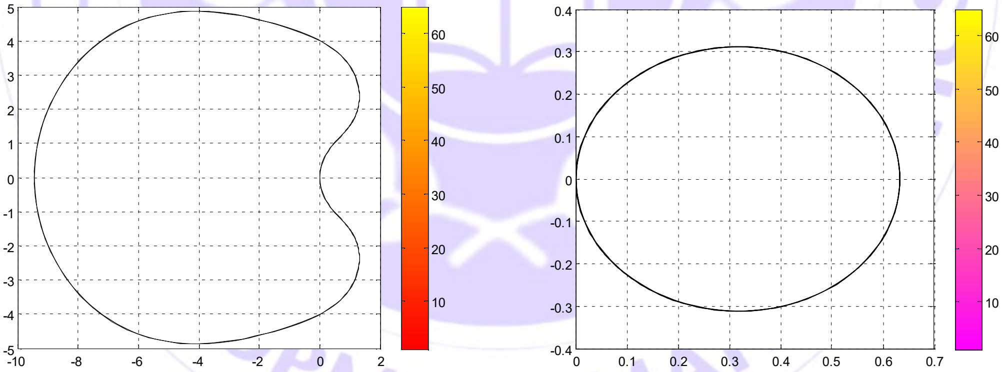
$$\rho(\eta, z) = \det\left(\eta I - M(z)\right) = \det(\ r(A - Cz - DIz) - B). \tag{17}$$

The determinant produces the stability polynomial of the method. Going by the stability polynomial of the method, the region of absolute stability of the method $\Re$ is given by

$$\Re = x \in C : \rho(\eta, z) = 1 \Rightarrow |\eta| \leq 1.$$

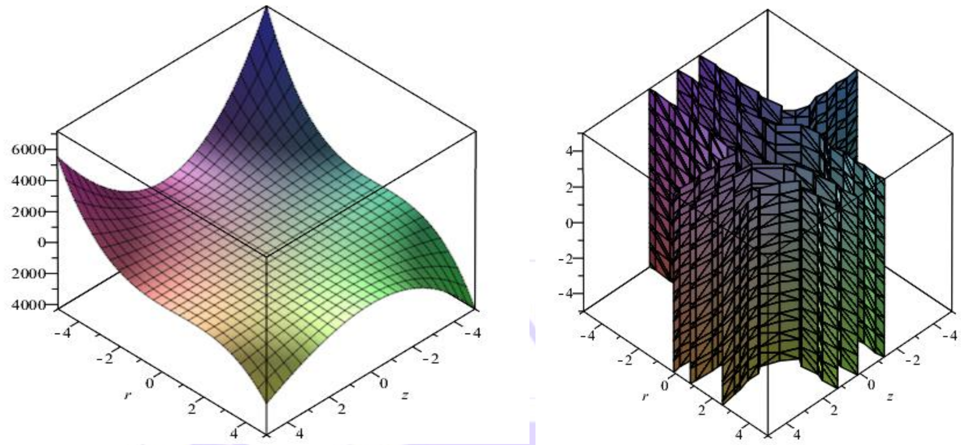
These are plotted to produce the required graphs of the regions of absolute stability of the perturbed collocation hybrid methods as shown in Figure 1.

Remark 1: *The idea taking into account to obtain regions of absolute stability is not new for a method with a nonlinear structure. These days, the new idea of the graphic surface of a nonlinear function has replaced the original idea. Theoretical aspects of the graphic surface were introduced in some of the well-known studies by Qureshi et al. (2021) and Ramos et al. (2021) and are currently regarded as the most effective method for the design and analysis of numerical schemes.*



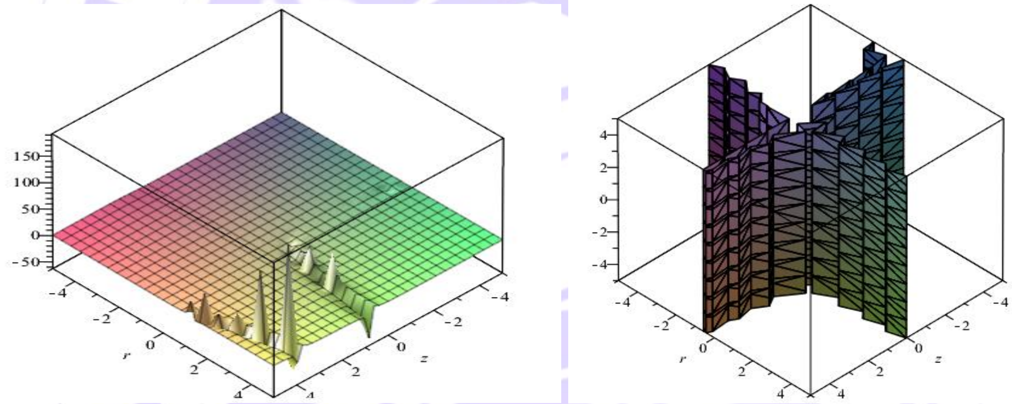
Method (6) Method (7)

(i) Region of Absolute Stability of the Perturbed Collocation Hybrid Methods



(a) (b)

(ii) Graphic Surface Curves of the Perturbed Collocation Hybrid Method (6) using 3D plot (a) and log mode implicit plot (b) (D-Dimension)



(a) (b)

(iii) Graphic Surface Curves of the Perturbed Collocation Hybrid Method(7) obtained as 3D plot (a) and log mode implicit plot (b) (D-Dimension)

Figure 1: *Regions of absolute stability of the perturbed hybrid methods with graphic surfaces* ***3D***

Remark 2: *The stability of the derived methods are shown in Figure 1(i), which include the complex plane as the plots for their regions of absolute stability. Figure 1 (ii) and (iii) also display the relevant 3D graphics surfaces (3D stability surfaces) for the proposed third- and fourth-order perturbed collocation hybrid methods, respectively.*

4. Implementation of the perturbed collocation hybrid methods

We provide a brief implementation strategy for the new derived methods in this section, in the sense of Yakubu et al. (2025). Since we have determined all of the new methods' algebraic properties, we provide a brief overview of how to use the newly derived methods by determining the Jacobian of any given system of differential equation. Based on the discussion of Dahlquist's test equation above, we know from the elementary ordinary differential equation that $y'(x) = f(x, y(x))$, which is called the general form of first order ordinary differential equation. Therefore, second order ordinary differential equation is obviously written as,

$$y''(x) = f'(x, y) = g(x, y).$$

Now, we see that the Jacobian of $g(x, y)$ is given as

$$\begin{aligned} g_y(x,y) &= f_x(x,y) + f_y(x,y) f(x,y) \\ &= f_x(x,y) + f(x,y) f_y(x,y) \\ &= f_x + ff_y \end{aligned}$$

where $f_y(x, y)$ is the Jacobian of $f(x, y)$.

Algorithm 1:Pseudo-code for the implementation of the new perturbed collocation hybrid methods with a fixed step size

---

Data: Define initial guess: $f(x)$, $df(x)$, N, h, $[a,b]$, where $f(x)$ is the problem to be solved and $df(x)$ is the derivative function, e is the tolerance, N total number of function evaluations (iterations, here $N = 500$) and h is the step-size defined as $h = \frac{(b-a)}{(N-1)}$, $[a,b]$ is the interval of consideration and y_0 vector of initial conditions for the problem. Calling certain built-in functions (syntax) for comparison is very simple in the program.

Output: $y_{new} = y_{n+1}^{(i+1)}$.

The default is N = b.

tspan=[0,b];vector specifying the interval of integration.

```
**Set** $tol, N, n = 0, a \le x \le b$.
**Define** $x_n = a, y_n = y(a),\ y_{new} = y_{n+1}^{(i+1)}, y_{old} = y_{n+i}^{(i)}, [a,b],$
**for** $n = 1 : N-1$, **do** $a \le x \le b$
$x(n) = x_0 + nh$
while $|(y_{old} - y_{new})| > tol$, **do or** while(abs(ynew-yold)>tol), **do**,
$y_{new} = y_{old}$ - $A^{1}B$, where A is a $(2 \times 2)$ Jacobian matrix and B is system of equations.
```

```
7: Let $y_{new} = y_{old}(1:\dim), i = 0(1)\mathrm{N}-1$, where dim is the dimension of the problem to be solved
8: Goto the end
9: for $n = n+1$, do
10: Goto 4
11: end else
12: if $n \geq N$, then
13: Goto 5
14: end if
15: end while
16: Goto 7
17: end
18: plot(x,y,'-')
19: hold on
20: [t,y]=ode23('nnnt',tspan,y0);
```

5.0 Results and Discussion

In this study, we apply the developed methods to solve system of initial value problems (IVPs), each of which has its own difficulty. Researchers in various scientific, engineering, and technological domains have always found each equation to be a difficult challenge. They are more suited to solving stiff systems in ordinary differential equations because of this characteristic of good stability properties. These techniques have proven to be effective in handling a variety of problems with varying degrees of numerical difficulty. Here, we demonstrate the methods' capability to handle the aforementioned problems, some of which include singularities and which cannot be resolved using the concept of limited regions of absolute stability by applying them with a set step size. We employ Matlab software to construct programs for the solution of systems of initial value problems. In the solution as well as the discussion of results we use nfe to denote the number of function evaluations and Ext to stand for the exact solutions.

Example 1: The first experiment considers a linear system of initial value problem, with exact solution for easy comparison purposes.

$$\begin{cases} u' = u + v, & u(0) = 2, \\ v' = -2u - v, & v(0) = 1, \end{cases}$$

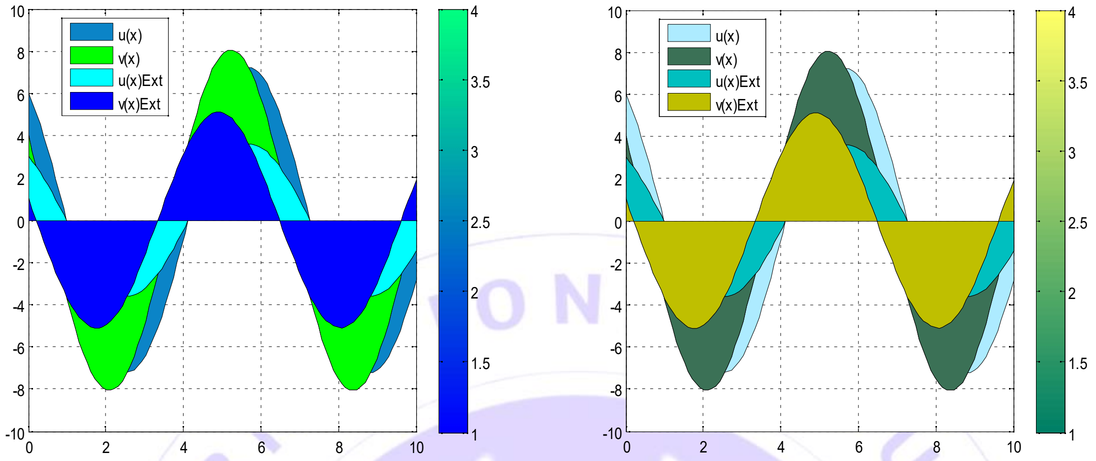
whose exact solution is given by

$$\begin{cases} u(x) = 3\sin(x) + 2\cos(x), \\ v(x) = \cos(x) - 5\sin(x). \end{cases}$$

We apply the perturbed collocation hybrid method to this problem and the results obtained are shown in Table 2 below, while the graphic surface curves are displayed in Figure 2. The comparison here is done based on method of the same order of convergence. As can be seen the new method is more accurate than that of Yakubu et al. (2004).

Table 2: Absolute errors in the numerical integration of example 1

x		Yakubu et al. (2004) $\lvert u(x)-u_n(x)\rvert$	Method (6) $\lvert u(x)-u_n(x)\rvert$
	$u(x)$	0	0
2	$v(x)$	$1.110223024625157 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	$u(x)$	$8.881784197001252 \times 10^{-16}$	$4.440892098500626 \times 10^{-16}$
50	$v(x)$	$2.220446049250313 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	$u(x)$	$1.332267629550188 \times 10^{-15}$	$4.440892098500626 \times 10^{-16}$
150	$v(x)$	$2.220446049250313 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$
	$u(x)$	$8.881784197001252 \times 10^{-16}$	$8.881784197001252 \times 10^{-16}$
250	$v(x)$	$2.220446049250313 \times 10^{-16}$	$3.330669073875470 \times 10^{-16}$
	$u(x)$	$2.220446049250313 \times 10^{-15}$	$1.332267629550188 \times 10^{-15}$
500	$v(x)$	$1.110223024625157 \times 10^{-16}$	$5.551115123125783 \times 10^{-16}$



(a) (b)

Figure 2: Graphic surface curves for example 1 using the new method with nfe =500

Example 2: This problem is a nonlinear stiff problem composed of three equations together with the initial conditions

$$\begin{cases} u' = u, & u(0) = 1, \\ v' = 2u^2, & v(0) = 1, \\ w' = 3uv, & w(0) = 0, \end{cases}$$

with exact solution

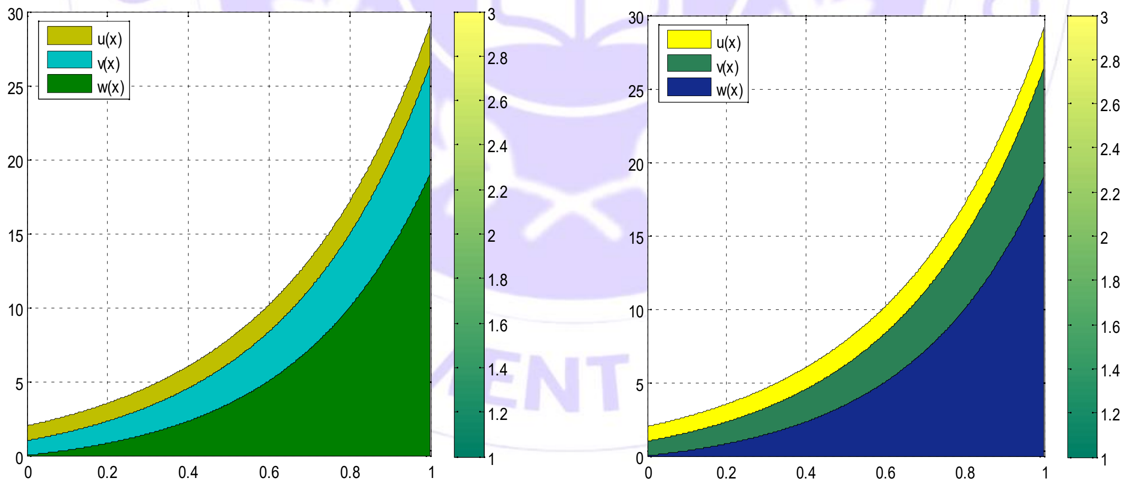
$$\begin{cases} u(x) = e^{x}, \\ v(x) = e^{2x}, \\ w(x) = e^{3x} - 1. \end{cases}$$

We solve this problem using the newly derived perturbed collocation hybrid methods and the results of the computation are shown in Table 3, while the graphic surface outputs are displayed in Figure 3. The two perturbed collocation hybrid methods are vying over accuracy of results (performance) as shown in the Table of values.

Table 3: Absolute errors in the numerical integration of example 2

x	Method (6) $\lvert u(x)-u_n(x)\rvert$	Method (7) $\lvert u(x)-u_n(x)\rvert$
$u(x)$	$1.110223024625157 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$

2	$v(x)$	$1.110223024625157 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	$w(x)$	$8.470329472543003 \times 10^{-22}$	$4.235164736271502 \times 10^{-22}$
	$u(x)$	$4.440892098500626 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
50	$v(x)$	$4.440892098500626 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
	$w(x)$	$6.776263578034403 \times 10^{-20}$	$3.388131789017201 \times 10^{-21}$
	$u(x)$	$3.330669073875470 \times 10^{-16}$	0
250	$v(x)$	$3.330669073875470 \times 10^{-16}$	0
	$w(x)$	$9.757819552369540 \times 10^{-19}$	$6.098637220230962 \times 10^{-20}$
	$u(x)$	$1.110223024625157 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$
500	$v(x)$	$1.110223024625157 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$
	$w(x)$	$8.673617379884036 \times 10^{-19}$	$7.995991022080595 \times 10^{-19}$



(a) (b)

Figure 3: Graphic surface curves for example 2 using the new methods with nfe =500

Example 3: This problem is a linear stiff problem composed of three equations together with the initial conditions,

$$\begin{cases} u' = -0.1u - 49.9v, & u(0) = 2, \\ v' = -50v, & v(0) = 1, \\ w' = 70v - 120w, & w(0) = 2, \end{cases}$$

with exact solution

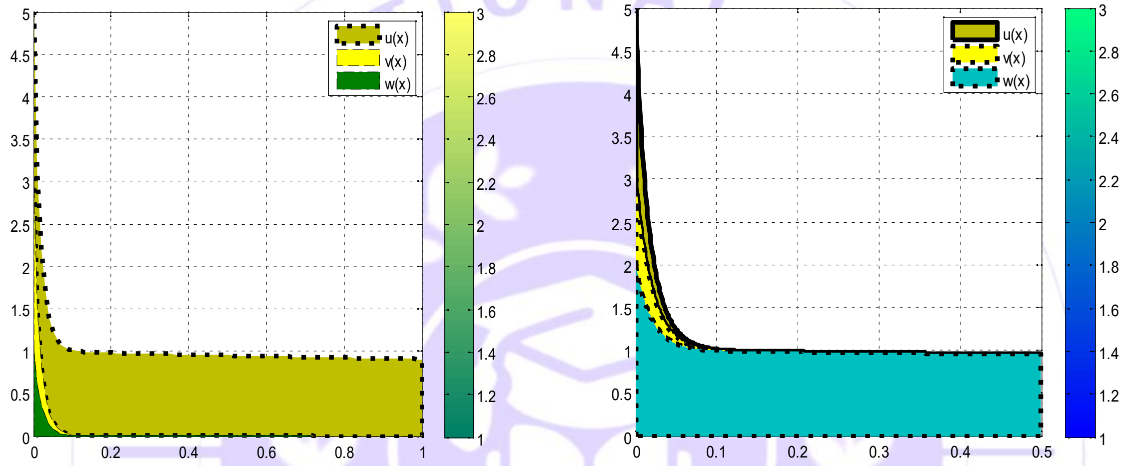
$$\begin{cases} u(x) = e^{-0.1x} + e^{-50x}, \\ v(x) = e^{-50x}, \\ w(x) = e^{-50x} + e^{-120x}. \end{cases}$$

The eigenvalues of the Jacobian of the system are $\lambda_1 = -0.1$, $\lambda_2 = -50$ and $\lambda_3 = -120$. We solve this problem using the new derived perturbed collocation hybrid method and the results of the calculation are shown in Table 4, while the graphic surface outputs are displayed in Figure 4.

Table 4: Absolute errors in the numerical integration of example 3

x		Yakubu et al. (2023) $\lvert u(x)-u_n(x)\rvert$	Method (6) $\lvert u(x)-u_n(x)\rvert$
2	$u(x)$	$9.174737192196858 \times 10^{-08}$	$4.760929428471172 \times 10^{-09}$
	$v(x)$	$9.174737192196858 \times 10^{-08}$	$4.760929650515777 \times 10^{-09}$
	$w(x)$	$5.874793216564811 \times 10^{-06}$	$3.348232144873009 \times 10^{-07}$
50	$u(x)$	$3.664355230359462 \times 10^{-08}$	$1.901502177936720 \times 10^{-09}$
	$v(x)$	$3.664355161057259 \times 10^{-08}$	$1.901502060842886 \times 10^{-09}$
	$w(x)$	$3.939256860291396 \times 10^{-08}$	$2.058429081699953 \times 10^{-09}$
250	$u(x)$	$9.992007221626409 \times 10^{-16}$	$3.330669073875470 \times 10^{-16}$
	$v(x)$	$3.687229691572919 \times 10^{-16}$	$1.913392498531147 \times 10^{-17}$
	$w(x)$	$3.687229691540607 \times 10^{-16}$	$1.913392498531147 \times 10^{-17}$

	$u(x)$	$3.330669073875470 \times 10^{-16}$	$4.440892098500626 \times 10^{-16}$
500	$v(x)$	$9.760579364148147 \times 10^{-27}$	$5.065067799297552 \times 10^{-28}$
	$w(x)$	$9.760579364195167 \times 10^{-27}$	$5.065067799767750 \times 10^{-28}$



(a) (b)

Figure 4: Graphic surface curves for example 3 using the new method with nfe =500

Example 4: Consider the system of nonlinear differential equation together with the initial value given by

$$\begin{cases} u' = 2v^2, & u(0) = 1, \\ v' = e^{-x}u, & v(0) = 1, \\ w' = v + w, & w(0) = 0, \end{cases}$$

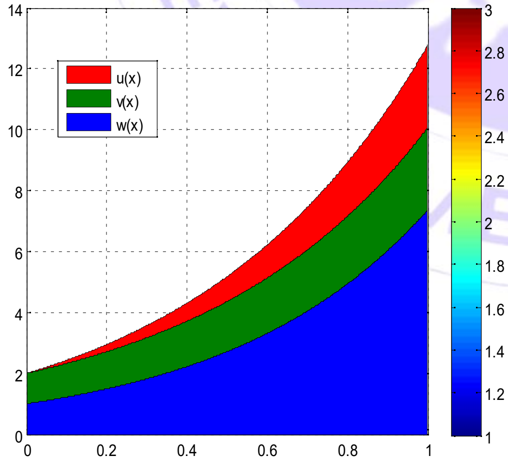
with exact solution

$$\begin{cases} u(x) = e^{2x}, \\ v(x) = e^{x}, \\ w(x) = xe^{x}. \end{cases}$$

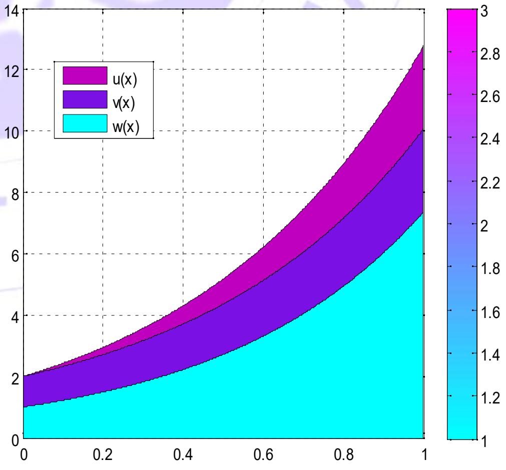
The results of using the newly derived perturbed collocation hybrid methods and some existing method of the same order of convergence for the solution of this problem on the interval [0, 1] are presented in Table 5, while graphical surface phase plots outputs are shown in Figure 5. In this particular problem the new perturbed collocation hybrid method performs well compared to the other method of the same order of convergence as indicated in both the Tabular form and the graphical plots.

Table 5: Absolute errors in the numerical integration of example 4

x		Kwami et al. (2016) $\lvert u(x)-u_n(x)\rvert$	Method (7) $\lvert u(x)-u_n(x)\rvert$
2	$u(x)$	$6.249667450219931 \times 10^{-12}$	$7.371880883511039 \times 10^{-14}$
	$v(x)$	$3.553379812615276 \times 10^{-12}$	$2.353672812205332 \times 10^{-14}$
	$w(x)$	$3.124902451404542 \times 10^{-12}$	$3.681377512326352 \times 10^{-14}$
50	$u(x)$	$3.062483600047017 \times 10^{-10}$	$3.607780740821909 \times 10^{-12}$
	$v(x)$	$1.741178312641978 \times 10^{-10}$	$1.147970607462412 \times 10^{-12}$
	$w(x)$	$1.531228987040819 \times 10^{-10}$	$1.803902198379030 \times 10^{-12}$
250	$u(x)$	$1.556368367516825 \times 10^{-09}$	$1.833377893944999 \times 10^{-11}$
	$v(x)$	$8.848521915183483 \times 10^{-10}$	$5.833333815985498 \times 10^{-12}$
	$w(x)$	$7.781710394943607 \times 10^{-10}$	$9.167344489047994 \times 10^{-12}$
500	$u(x)$	$3.119319247346652 \times 10^{-09}$	$3.674549553522866 \times 10^{-11}$
	$v(x)$	$1.773380553515835 \times 10^{-09}$	$1.169198071693245 \times 10^{-11}$
	$w(x)$	$1.559609352647377 \times 10^{-09}$	$1.837294913106281 \times 10^{-11}$



(a)



(b)

Figure 5: Graphic surface curves for example 4 using the new method with nfe =500

Example 5: The inverse-square law attraction in two body (Kepler Problem) Butcher (2003)

Here, we write u as a vector or as an ensemble of scalars and we see that the equation can be written again in one of the several equivalent forms. Therefore, it is interesting to compare the absolute errors generated in the very first step, for the given value of h that we have used. In the given problem, as you can see u_1 and u_2 are rectangular coordinates centered of the heavy body (sun). The negative sign in the last two equations are the inward direction of the acceleration. We describe the equations of motion in a simplified form as follows:

$$u''(x) = \frac{1}{\|u(x)\|^{\frac{3}{2}}}, \text{ where } \|u\| = \sqrt{u_1^2 + u_2^2}, \text{ such that we have;}$$

$$u_1' = u_3, \qquad u_1(0) = 1,$$

$$u_2' = u_4, \qquad u_2(0) = 0,$$

$$u_3' = -\frac{u_1}{\left(u_1^2 + u_2^2\right)^{3/2}}, \qquad u_3(0) = 0,$$

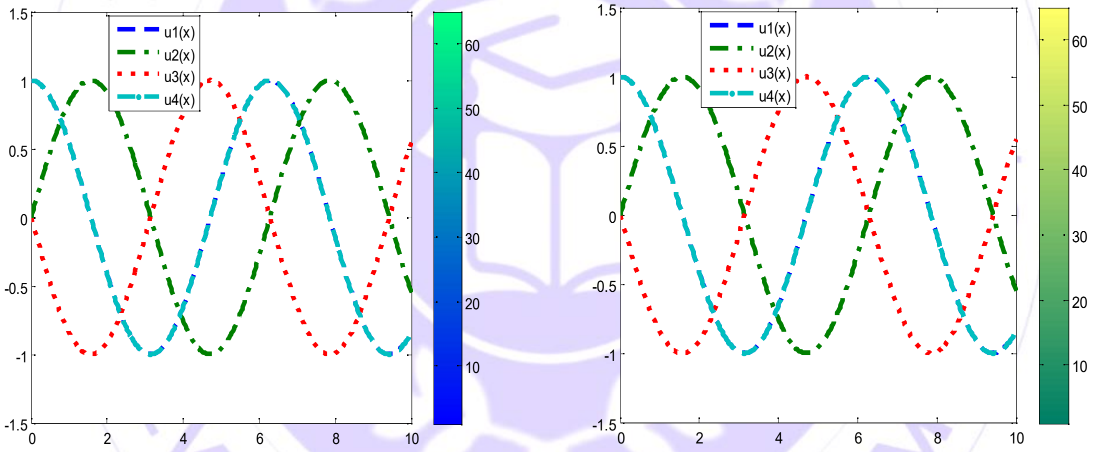
$$u_4' = -\frac{u_2}{\left(u_1^2 + u_2^2\right)^{3/2}}, \qquad u_4(0) = 1\ .$$

Based on the results obtained, we have shown exactly that the solutions of this system are known to be conic sections, that is, ellipses, parabolas or hyperbolas, displayed in Figure 6.

Table 6: Absolute errors in the numerical integration of example 5

x	$u_i(x)$	Method (7) $\lvert u(x) - u_n(x) \rvert$	Method (6) $\lvert u(x) - u_n(x) \rvert$
2	$u_1(x)$	$1.842970220877760 \times 10^{-14}$	$1.562416862554983 \times 10^{-12}$
	$u_2(x)$	$6.617444900424221 \times 10^{-22}$	$2.684300349408081 \times 10^{-19}$
	$u_3(x)$	$2.752857078576476 \times 10^{-21}$	$5.400099736542182 \times 10^{-19}$
	$u_4(x)$	$1.842970220877760 \times 10^{-14}$	$1.562416862554983 \times 10^{-12}$
50	$u_1(x)$	$9.019451852054772 \times 10^{-13}$	$7.656020262203356 \times 10^{-11}$
	$u_2(x)$	$8.705804631879699 \times 10^{-18}$	$7.496156879905833 \times 10^{-16}$
	$u_3(x)$	$4.202977484275838 \times 10^{-18}$	$3.946733077072467 \times 10^{-16}$
	$u_4(x)$	$9.019451852054772 \times 10^{-13}$	$7.656020262203356 \times 10^{-11}$

250	$u_1(x)$	$4.583666779467421 \times 10^{-12}$	$3.890505695380853 \times 10^{-10}$
	$u_2(x)$	$2.279399542379212 \times 10^{-16}$	$1.940249583177664 \times 10^{-14}$
	$u_3(x)$	$1.132720219704231 \times 10^{-16}$	$9.802217631332821 \times 10^{-15}$
	$u_4(x)$	$4.583666779467421 \times 10^{-12}$	$3.890505695380853 \times 10^{-10}$
500	$u_1(x)$	$9.184986105026383 \times 10^{-12}$	$7.796627787826083 \times 10^{-10}$
	$u_2(x)$	$9.170453025425518 \times 10^{-16}$	$7.794422930435668 \times 10^{-14}$
	$u_3(x)$	$4.570860833927326 \times 10^{-16}$	$3.917447421140918 \times 10^{-14}$
	$u_4(x)$	$9.184986105026383 \times 10^{-12}$	$7.796627787826083 \times 10^{-10}$



(a) (b)

Figure 6: Solution curves for example 5 using the new methods with nfe =500

6. Concluding Remarks

In the present study, effective perturbed collocation block hybrid methods to deal with linear and nonlinear stiff and highly oscillatory problems have been proposed. The proposed perturbed collocation methods provide efficient technique for obtaining accurate approximate solutions to linear and nonlinear stiff and highly oscillatory differential equations. Analysis of these methods shows that they exhibit convergence when certain stability and consistency conditions are met. The performance in the preliminary numerical experiments confirm that the perturbed collocation methods offer better accuracy and convergence compared to other derived methods for stiff and oscillatory problems in literature. These are illustrated in the Table of values as well as graphical outputs in Figures.

In our next research paper, we will apply some new derived methods to modeled problems of differential equations, found especially, in drug delivery using magnetic nanoparticles (tri-nanofluid) transport.

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