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On Geometric Properties of Certain Class of Analytic Functions Connected with a Multiplier Transformation

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ABSTRACT

A major interest in geometric function theory is to provide insight into the behaviour of analytic functions. In this study therefore, the authors defined a subclass of univalent functions by means of a generalised multiplier transformation and obtained some geometric properties related to the bounds of the coefficient for a certain class of analytic function. The results obtained in the study extend some existing results in literature.

1. Introduction

In Investigation of classes of univalent functions of complex variables becomes important due to the applications of analytic and univalent functions. Researchers in geometric function theory have therefore continued to explore and investigate this area of study as it bridges the gap between analysis and applied mathematics. For example, the following authors Alenazi & Mehrez (2023), Alsarari *et al.* (2023), Amini *et al.* (2022), Amourah & Darus (2016), Darus & Ibrahim (2009), Darus & Ibrahim (2010), Salalgean (1983), Oluwayemi & Fadipe-Joseph (2022) have successfully defined and investigated various classes of univalent functions and their properties.

Suppose $\mathbb{C}$ is a set of complex numbers. A function f defined on $\mathbb{C}$ is a rule that assigns to each $z \in \mathbb{C}$. That is, z in $\mathbb{C}$, a complex number ω. The number ω is called the value of f at z and is denoted by $f(z)$ such that $\omega = f(z) = u(x, y) + iv(x, y)$. The set $\mathbb{C}$ is called the domain of definition of f. It is important to know that both a domain of definition and a rule are necessary in order for a function to be well defined. In Geometric Function Theory (GFT) which is the focus of this work, the domain is the unit disc.

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition. These multiple-valued functions occur in the theory of functions of a complex variable, just as they do in the case of a real variable. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function. Contrary to the properties of a real-valued function of a real variable that are often exhibited by the graph of the function, functions of a complex variable with $\omega = f(z)$ where z and ω are complex, do not have such convenient graphical representation of the function f because each of the numbers z and ω is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs of corresponding points $z = (x, y)$ and $\omega = (u, v)$. This will involve drawing the z plane and ω plane separately. This process is referred to as a mapping or transformation. The image of a point z in the domain of definition $\mathbb{C}$ is the point $\omega = f(z)$, and the set of

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images of all points in a set T that is contained in $\mathbb{C}$ is called the image of T. The image of the entire domain of definition $\mathbb{C}$ is called the range of $f(z)$. The inverse image of a point ω is the set of all points z in the domain of definition of $f(z)$ that have ω as their image. The inverse image of a point may contain just one point, many points, or none at all (Brown & Churchill, 2009).

Analytic function is a complex function $f(z)$ that is differentiable at every point in some open subset of the complex plane $\mathbb{C}$. In other words, a function $f(z)$ is analytic at a point z_0 if it is differentiable in a neighborhood around z_0 and analytic on a domain if it is differentiable at every point in that domain. Now, every analytic function $f(z)$ has a Taylor series representation which is represented as a convergent power series:

$$f(z) = \sum_{k=2}^{\infty} a_k (z - z_0)^k.$$

The above series is a convergent power series, infinitely differentiable and satisfies the Cauchy-Riemann equations ($U_x = V_y$ and $U_y = -V_x$) as the necessary for differentiability in the complex plane. Without loss of generality, if the point z_0 is considered as the origin so that $z_0 = 0$, we have a convergence power series:

$$f(z) = \sum_{k=2}^{\infty} a_k z^k.$$

This is widely applied in fluid dynamics as the used to model 2D potential flows, Electromagnetic field theory to solve Laplace's equation in 2D, Signal processing through the analytic signal (Hilbert transform) and Complex integration as fundamental to evaluating complex integrals using Cauchy's Integral Theorem and Residue Theorem among other applications.

A function $f(z)$ is said to be univalent if it is one-to-one or injective analytic function on a domain. In other words, if $f(z_1) = f(z_2)$ implies that $z_1 = z_2$ or if $f(z_1) \neq f(z_2)$ implies that $z_1 \neq z_2$. This property is fundamental in Geometric function theory and for to understanding conformal mappings.

A unit disc is the open disc centered at the origin and of radius one. This plays an important role in GFT and defined mathematically as follows $U = \{z \in \mathbb{C}: \ |z| < 1\}$ (Krantz, 2006).

Now, let S be the class of functions analytic in U satisfying the conditions $f(0) = 0$ and $f'(0) = 1$ so that $z \epsilon U$ is used throughout the study. It is important to note that the results based of the unit disc is generalized based on the Riemman Mapping Theory (RMP) and its consequence.

Then each function $f \in A$ has the Taylor expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \qquad (1)$$

The class of all univalent functions normalized by $f(0) = 0$ and $f'(0) = 1$ is denoted by the class S**.** Similarly, T is the class of functions analytic in U with conditions $f(0) = 0$ and $f'(0) = 1$ and with the Taylor expansion

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 2. \qquad (2)$$

Silverman (1975) introduced a class of function with each function $f \in T$ and given by (2) above.

A differential operator is an operator defined as a function of the differentiation operator $\frac{d}{dz}$.

In Geometric Function Theory (GFT) which involves the studies of the geometric properties of analytic and univalent functions, differential operators are used to generate new functions, study function classes, and examine univalence, convexity, starlikeness, among other properties. Some foremost operators (differential and integral) are presented below for reference.

Differential Operators and their applications in Geometric Function Theory: There numerous differential operators in GFT being applied by authors to explore various classes of univalent functions. Primarily, we have $D[f] = f'(z) = \frac{df}{dz}$ is defined as the first derivative of the function $f(z)$ and has a wide application in GFT. On the other hand, the Salagean differential operator $D^n[f]$ is a recursive derivative that can be used to define other subclasses of univalent functions such as function of bounded turning, starlike subclass of functions and convex subclass of univalent functions with geometric representation of $Re\{f'(z)\} > 0$, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$ respectively.

Ruscheweyh (1975) introduced a differential operator $R^n[f]$ which was defined using Hadarmard product/convolution. This operator has been widely used to study starlike and convex functions in GFT.

The Libera integral $L[f] = \int_0^z \frac{f(\xi)}{\xi} d\xi$ has continued to gain attention follow the work Libera (1965). Several other integral operators have emerged extending and generalizing the Libera integral operator. However, the integral operator remains the reference point for other forms of integral operators. The Libera operator preserves univalence or convexity of functions.

The Ruscheweyh operator applied in Ruscheweyh (1975). New was the first differential operator in GFT after which Salagean (1983) introduced a recursive derivative which is became famous and widely used and extended by other authors. Recently too, researchers in GFT such as Shaba *et al.* (2024) have beginning to apply Ruscheweyh operator in q-derivatives. Also, Libera (1965) originated integral operator which preserves univalence property of functions. This operator has also been extended and applied by different authors. See Darus & Ibrahim (2009), Davids *et al.* (2025), Lasode & Opoola (2022), Oyekan & Awolere (2020), Gbolagade & Olatunji (2014), Hamzat & Olaleru (2022), Oyekan & Awolere (2020) and related studies for details.

Some applications of operators in geometric function theory include:

1. Classification of Function Classes: The starlike and convex subclasses of univalent functions involve inequalities with derivatives associated with the differential operators.
2. Coefficient Estimates: The differential operators help analyze Taylor series of analytic functions, e.g., estimating bounds for the coefficients $[a_k]$ in (1) above.
3. Establishing new univalent functions: Existing univalent functions can be used to produces new functions that may either belong to the same or related classes with the use differential operators such as the Salagean or any other known derivatives. See Darus & Ibrahim (2009), Davids *et al.* (2025), Amini *et al.* (2022), Amourah & Darus (2016), Darus & Ibrahim (2010), Lasode A. O. & Opoola (2022), Oyekan & Awolere (2020) for details.
4. Conformal Mappings: Differential operators help analyze how analytic functions behave under conformal mappings. This is very useful in physics and engineering.

Following the work of Salagean (1983), various authors have continued to extend and generalise the well-known Salagean operator. Amourah & Darus (2016) introduced the following operator:

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) = z - \sum_{k=2}^{\infty} \left\{1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right\}^m a_k z^k \tag{3}$$

for $f \in T$, $\alpha, \beta, \delta, \lambda, \epsilon, \omega \geq 0$; $\mu > 0$ and $\mu \neq \lambda$; $m \in N_0 = N \cup \{0\}$.

Various authors have also used linear combinations of two different differential operators to investigate classes of univalent functions. The study therefore aimed at introduction of a new class of univalent functions and investigate some geometric properties of the functions.

Motivated by Amourah & Darus (2016), the authors define the following class of univalent functions. The object of the study is to investigate a new class of univalent functions and study some of its properties.

2 Preliminaries

In this work, the authors introduced a new class of univalent functions defined as follows:

Definition 1.1 For $\alpha, \delta \geq 0$, $\beta, \lambda, \mu > 0$, $\alpha \neq \lambda$ and $m \in N_0$, let $0 < \xi \leq 1$, $\sigma \leq 1$ and $\varpi \in C - \{0\}$. Then, the function $f \in T$ is said to be in the class $T_m(\xi, \varpi, \sigma)$ if

$$\left| \frac{1}{\omega} \left(\frac{z\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)^1}{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)} - \sigma \right) \right| < \xi, \quad z \in U. \tag{4}$$

Furthermore, Maximum Modulus Theorem (MMT) that will be used in the proof of the necessary and sufficient conditions for functions belonging to the class $T_m(\xi, \varpi, \sigma)$ so define above, is also presented as follows:

Maximum Modulus Theorem: Let f be a non-constant analytic function on a connected open set $D \subset \mathbb{C}$. Then $|f(z)|$ the maximum of cannot occur in the interior of D, unless f is constant.

More precisely, if f is continuous on the closure $\overline{D}$ and analytic on D, then:

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)|.$$

That is, the maximum modulus of f occurs on the boundary ∂D, not in the interior of D, unless f is constant. See Bak & Newman (2010).

Riemman Mapping Theory [Krantz (2006)]: Let D be the unit disk. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain that is not the entire complex plane. Then there is a one-to-one, onto holomorphic mapping

$$\Phi: D \to \Omega.$$

More precisely, The Riemann mapping theorem (RMT) states that if $\Omega \subseteq \mathbb{C}$ is a simply connected domain, not all of $\mathbb{C}$, then there is a conformal mapping $\Phi: D \to \Omega$. where *D* is the unit disk. This result has had a profound influence over the way that we study complex function theory. It has no analogue in the complex analysis of several variables. The RMT was profoundly generalized by Koebe and Poincar´ e with their uniformization theorem. A deeper understanding of the Riemann mapping, or of the uniformization mapping, rather naturally raises the question of whether the mapping extends to the boundary in a nice way [Krantz (2006)].

The RMT guarantees the use of unit disc as the domain of investigation in GFT.

3. Methodology

In this study, the author applied a generalised differential operator introduced by the authors in Amourah & Darus (2016) and used the operator to establish a new class of univalent functions denoted as $T_m(\xi, \varpi, \sigma)$, defined by (4) of definition 1.1. Some geometric properties of the functions explored.

4. Main Results

In this section we state and prove the main results of this paper.

We begin by proving the necessary and sufficient condition for a function to belong to the class $T_m(\xi, \varpi, \sigma)$.

2.1 Necessary and sufficient condition for the class $T_m(\xi, \varpi, \sigma)$

Theorem 2.1 Let f(z) be defined by (2), then $f \in T_m(\xi, \varpi, \sigma)$ if and only if

$$\sum_{k=2}^{\infty} [k-\sigma+\xi|\varpi|]\left(1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right)^m a_k \leq [\,\xi|\varpi|+\sigma-1] \tag{5}$$

and $\xi|\varpi| \geq 1$ for all values of ξ and ϖ.

Proof:

Let $f \in T_m(\xi, \varpi, \sigma)$. Then,

$$\Re\left\{\frac{z\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}-\sigma\right\} > \; -\xi|\varpi| \;\; (6)$$

$$\left|\frac{z\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)^1}{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}-\sigma\right| = \left|\frac{(1-\sigma)-\sum_{k=2}^{\infty}\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m a_k z^{k-1}}\right| \quad > \;\; -\xi|\varpi|.$$

We choose values of z on the real axis so that $\frac{z\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}-\sigma$is real. Furthermore, we simplify (6) letting $z \to 1^-$ through the real axis to obtain (5) as required.

Hence, by using the maximum modulus Theorem and (6), $f \in T_m(\xi, \varpi, \sigma)$.

Conversely, assume that

$$\left|\frac{z\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)^1}{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}-\sigma\right| = \left|\frac{(\sigma-1)+\sum_{k=2}^{\infty}(k-1)\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m a_k z^{k-1}}\right|$$

$$\leq \frac{(\sigma-1)+\sum_{k=2}^{\infty}(k-1)\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m |a_k|}{\left(1-\sum_{k=2}^{\infty}(k-1)\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m\right)|a_k|} \leq \xi|\varpi|, \qquad z\epsilon\, U.$$

The above inequality is obtained by letting $z \to 1^-$ through the real axis and since $\Re(z) \leq |z|$ for all $z \in U$, we have

$$\Re\left\{\left|\frac{z\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)^1}{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}-\sigma\right| = \left|\frac{(\sigma-1)+\sum_{k=2}^{\infty}(k-1)\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m a_k z^{k-1}}\right|\right\} < \; \xi|\varpi|. \tag{7}$$

By choosing values of z on the real axis so that $A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)$ is real and letting $z \to$ real values, we obtain the desired inequality (5).

$$\sum_{k=2}^{\infty} [k-\sigma+\xi|\varpi|]\left(1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right)^m a_k \leq \; \xi|\varpi|+\sigma-1.$$

Theorem 2.2 Let $\mathrm{f} \in \mathrm{T_m}(\xi, \varpi, \sigma)$, then

$$\mathrm{r} - \mathrm{r}^2 \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}} \leq |\mathrm{f}(\mathrm{z})| \leq \mathrm{r} + \mathrm{r}^2 \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}} \tag{8}$$

and

$$1 - 2\mathrm{r} \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}} \leq |\mathrm{f}(\mathrm{z})| \leq 1 + 2\mathrm{r} \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}} \tag{9}$$

The inequalities (7) and (8) are sharp for the function

$$\mathrm{f}(\mathrm{z}) = \mathrm{z} - \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}} \mathrm{z}^2.$$

Proof:

From Theorem 2.1, for any function $\mathrm{f} \in \mathrm{T_m}(\xi, \varpi, \sigma)$, we have that

$$\sum_{\mathrm{k}=2}^{\infty} \mathrm{a_k} \leq \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}} \mathrm{z}^{\mathrm{k}} \qquad (\mathrm{k} = 2,3, \ldots).$$

Thus,

$$|\mathrm{f}(\mathrm{z})| \leq \mathrm{r} + \sum_{\mathrm{k}=2}^{\infty} |\mathrm{a_k}|\mathrm{r}^{\mathrm{k}} \leq \mathrm{r} + \mathrm{r}^2 \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}}$$

and

$$|\mathrm{f}(\mathrm{z})| \leq \mathrm{r} - \sum_{\mathrm{k}=2}^{\infty} |\mathrm{a_k}|\mathrm{r}^{\mathrm{k}} \leq \mathrm{r} - \mathrm{r}^2 \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}}.$$

Hence, (7) holds. Furthermore,

$$|\mathrm{f}'(\mathrm{z})| \leq 1 + \sum_{\mathrm{k}=2}^{\infty} |\mathrm{a_k}|\mathrm{k}\mathrm{r}^{\mathrm{k}-1} \leq 1 + 2\mathrm{r} \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}}$$

and

$$|\mathrm{f}'(\mathrm{z})| \leq 1 - \sum_{\mathrm{k}=2}^{\infty} |\mathrm{a_k}|\mathrm{k}\mathrm{r}^{\mathrm{k}-1} \leq 1 - 2\mathrm{r} \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}}$$

which completes the proof.

Theorem 2.3 Let

$$\mathrm{f_1}(\mathrm{z}) = \mathrm{z} \text{ and } \mathrm{f_k}(\mathrm{z}) = \mathrm{z} - \frac{\xi|\varpi|+\sigma-1}{[\mathrm{k}-\sigma+\xi|\varpi|]\left(1+\frac{(\mathrm{k}-1)[(\lambda-\alpha)\beta+\mathrm{k}\delta]}{\mu+\lambda}\right)^{\mathrm{m}}} \mathrm{z}^{\mathrm{k}}, \quad (\mathrm{k} \geq 2).$$

Then $\mathrm{f} \in \mathrm{T_m}(\xi, \varpi, \sigma)$ if and only if it can be represented in the form

$$\mathrm{f}(\mathrm{z}) = \sum_{\mathrm{k}=1}^{\infty} \varsigma_{\mathrm{k}} \mathrm{f_k}(\mathrm{z}), \qquad \left(\varsigma_{\mathrm{k}} \geq 0,\ \sum_{\mathrm{k}=1}^{\infty} \varsigma_{\mathrm{k}} = 1\right). \tag{10}$$

Proof: Suppose $\mathrm{f}(\mathrm{z})$ can be expressed as in (10), then

$$f(z) = \sum_{k=1}^{\infty} \varsigma_k f_k(z)$$

$$\varsigma_1 f_1(z) + \sum_{k=2}^{\infty} \varsigma_k f_k(z)$$

$$\varsigma_1 z + \sum_{k=2}^{\infty} \varsigma_k \left(\frac{\xi|\varpi|+\sigma-1}{[k-\sigma+\xi|\varpi|]\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m} z^k \right)$$

$$z - \sum_{k=2}^{\infty} \varsigma_k \frac{\xi|\varpi|+\sigma-1}{[k-\sigma+\xi|\varpi|]\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m} z^k.$$

Therefore,

$$\sum_{k=2}^{\infty} \frac{\xi|\varpi|+\sigma-1}{[k-\sigma+\xi|\varpi|]\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m} \frac{[k-\sigma+\xi|\varpi|]\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m}{\xi|\varpi|+\sigma-1} \varsigma_k$$

$$= \sum_{k=2}^{\infty} \varsigma_k = 1 - \varsigma_1 \leq 1.$$

Hence by Theorem 2.1, $f \in T_m(\xi, \varpi, \sigma)$.

Conversely, we suppose $f \in T_m(\xi, \varpi, \sigma)$. Since

$$a_k \leq \frac{\xi|\varpi|+\sigma-1}{[k-\sigma+\xi|\varpi|]\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m}, \qquad k \geq 2.$$

We take

$$\varsigma_k = \frac{[k-\sigma+\xi|\varpi|]\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m}{\xi|\varpi|+\sigma-1} a_k; \quad k \geq 2 \text{ and } \varsigma_1 = 1 - \sum_{k=2}^{\infty} \varsigma_k.$$

Then we have,

$$f(z) = \sum_{k=1}^{\infty} \varsigma_k f_k(z)$$

$$\varsigma_1 f_1(z) + \sum_{k=2}^{\infty} \varsigma_k f_k(z)$$

which completes the proof.

Corollary 2.1 The extreme points of $T_m(\xi, \varpi, \sigma)$ are the functions $f_1(z) = z$ and

$$f_k(z) = \frac{\xi|\varpi|+\sigma-1}{[k-\sigma+\xi|\varpi|]\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^m}, \qquad (k \geq 2).$$

Theorem 2.4

The set $T_m(\xi, \varpi, \sigma)$ is the convex set

Proof :

Let $f_i(z) = z - \sum_{k=1}^{\infty} a_{k,i} z^k (i = 1,2)$ belong to $T_m(\xi, \varpi, \sigma)$ and let $h(z) = \zeta_1 f_1(z) + \zeta_2 f_1(z)$ with ζ_1 and ζ_2 non-negative and $\zeta_1 + \zeta_2 = 1$, we thus have that

$$h(z) = z - \sum_{k=1}^{\infty} (\zeta_1 a_{k,1} + \zeta_1 a_{k,2}) z^k.$$

We need to show that $h(z) = T_m(\xi, \varpi, \sigma)$ which means

$$\sum_{k=1}^{\infty} \left[(k-\sigma) + \xi|\varpi| \left\{1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right\}^m\right] \left(\zeta_1 a_{k,1} + \zeta_2 a_{k,2}\right)$$

$$= \zeta_1 \sum_{k=1}^{\infty} \left[(k-\sigma) + \xi|\varpi| \left\{1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right\}^m\right] a_{k,1}$$

$$+\zeta_2 \sum_{k=1}^{\infty} \left[(k-\sigma) + \xi|\varpi| \left\{1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right\}^m\right] a_{k,2}$$

$$\leq \zeta_1(\xi|\varpi| + \sigma - 1) + \zeta_2(1 - \sigma\varpi - \xi\varpi)$$

$$[\zeta_1 + \zeta_2](\xi|\varpi| + \sigma - 1)$$

$$\Rightarrow \sum_{k=1}^{\infty} \left[(k-\sigma) + \xi|\varpi| \left\{1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right\}^m\right] \leq \xi|\varpi| + \sigma - 1.$$

Hence, $h \in T_m(\xi, \varpi, \sigma)$ from Theorem 2.1 as required.

2.2 Weighted Mean, Arithmetic Mean and Linear Combination

Definition 2.1 Let $g \in T_m(\xi, \varpi, \sigma)$. Then the weighted mean w_{fg} of f and g is defined as

$$w_{fg} = \frac{1}{2}[(1-t)f(z) + (1+t)g(z)], 0 < t < 1.$$

Definition 2.2 Let $f_i(z) = z - \sum_{k=2}^{\infty} a_{i,k} z^k, i = 1,2, \ldots n$ be in the functions in the class $T_m(\xi, \varpi, \sigma)$, then the arithmetic mean of $f_i(i = 1,2 \ldots)$ is defined by

$$g(z) = \frac{1}{n}\sum_{i=1}^{n} f_i(z).$$

Definition 2.3 Let $f_i(z) = z - \sum_{k=2}^{\infty} a_{i,k} z^k, i = 1,2, \ldots n$ be in the functions in the class $T_m(\xi, \varpi, \sigma)$, then the linear combination of $f_i(i = 1,2 \ldots n)$ is defined by

$$G(z) = \sum_{i=1}^{n} k_i f_i(z); \quad \text{where} \sum_{i=1}^{n} k_i = 1.$$

Theorem 2.5 Let $f, g \in T_m(\xi, \varpi, \sigma)$. Then the weighted mean w_{fg} is also in the class $T_m(\xi, \varpi, \sigma)$.

Proof: Suppose $g \in T_m(\xi, \varpi, \sigma)$. Then, from Definition 2.1,

$$w_{fg} = \frac{1}{2}[(1-t)f(z) + (1+t)g(z)], \qquad 0 < t < 1.$$

Hence,

$$w_{fg} = \frac{1}{2}\left[(1-t)\left(z - \sum_{k=2}^{\infty} a_k z^k\right) + (1+t)\left(z - \sum_{k=2}^{\infty} b_k z^k\right)\right]$$

$$w_{fg} = z - \sum_{k=2}^{\infty} \frac{1}{2}\{(1-t)a_k + (1+t)b_k\}z^k. \tag{11}$$

But $f, g \in T_m(\xi, \varpi, \sigma)$. Hence, by Theorem 2.1,

$$a_k \leq \frac{\xi|\varpi| + \sigma -}{(k-\sigma) + \xi|\varpi|\left\{1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right\}^m} z^k, \qquad k \geq 2. \text{ Thus,}$$

$$\sum_{k=2}^{\infty} [k - \sigma + \xi|\varpi|] \left(1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right)^m a_k \leq \xi|\varpi| + \sigma - 1$$

and

$$\sum_{k=2}^{\infty} (k-\sigma)+\xi|\varpi|\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^{m} b_k \ \le\ \xi|\varpi|+\sigma-1.$$

So that

$$\sum_{k=2}^{\infty} \left([(k-\sigma)+\xi|\varpi|]\left|\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^{m}\right)\left[\frac{1}{2}(1-t)a_k+(1+t)b_k\right]\right.$$

$$=\frac{1}{2}(1-t)\sum_{k=2}^{\infty} (k-\sigma)+\xi|\varpi|\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^{m} a_k$$

$$+\frac{1}{2}(1+t)\sum_{k=2}^{\infty} (k-\sigma)+\xi|\varpi|\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^{m} b_k$$

$$\le\frac{1}{2}(1-t)[\xi|\varpi|+\sigma-1]+\frac{1}{2}(1+t)[\xi|\varpi|+\sigma-1]$$

$$=\xi|\varpi|+\sigma-1$$

and thus $w_{f,g}\in T_m(\xi,\varpi,\sigma)$ as required.

Theorem 2.6 Let $f_1(z)=z-\sum_{k=2}^{\infty} a_{i,k}z^k, i=1,2,\dots n$ be the functions $T_m(\xi,\varpi,\sigma)$. Then the arithmetic mean of $f_i(i=1,2,\dots n)$ defined by

$$g(z)=\frac{1}{n}\sum_{i=1}^{n} f_i(z)$$

is also in the class $T_m(\xi,\varpi,\sigma)$.

Proof: By Definition 2.2, since $f_1(z)=z-\sum_{k=2}^{\infty} a_{i,k}z^k, i=1,2,\dots n$. Then,

$$g(z)=\frac{1}{n}\sum_{i=1}^{n} f_i(z)=\frac{1}{n}\sum_{i=1}^{n}\left(z-\sum_{k=2}^{\infty} a_{i,k}z^k\right)$$

$$z-\sum_{k=2}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_{i,k}\right)z^k.$$

By Theorem 2.1, for any $f\in T_m(\xi,\varpi,\sigma)$,

$$\sum_{k=2}^{\infty} (k-\sigma)+\xi|\varpi|\left\{1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right\}^{m} a_k\le\xi|\varpi|+\sigma-1.$$

Now, since $f_1(z)=z-\sum_{k=2}^{\infty} a_{i,k}z^k, i=1,2,\dots n$ is also in the class $T_m(\xi,\varpi,\sigma)$. We need to show that

$$\sum_{k=2}^{\infty} (k-\sigma)+\xi|\varpi|\left(1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right)^{m}\left(\frac{1}{n}\sum_{i=1}^{n} a_{i,k}\right)$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=2}^{\infty} (k-\sigma)+\xi|\varpi|\left(1+\frac{(k-1)[(\lambda-\alpha)\beta+k\delta]}{\mu+\lambda}\right)^{m} a_{i,k}\right)$$

$$\le\frac{1}{n}\sum_{i=1}^{n}\ =\xi|\varpi|+\sigma-1$$

which completes the proof.

Theorem 2.7 Let $f_1(z)=z-\sum_{k=2}^{\infty} a_{i,k}z^k, i=1,2,\dots n$ be the functions $T_m(\xi,\varpi,\sigma)$. Then the linear combination of $f_i(i=1,2,\dots n)$ defined as

$$H(z) = \sum_{i=1}^{n} k_i f_i(z)$$

where $\sum_{i=1}^{n} k_i = k_1 + k_2 \ldots k_n = 1$ also belongs to the class $T_m(\xi, \varpi, \sigma)$.

Proof: Let $f_1(z) = z - \sum_{k=2}^{\infty} a_{i,k} z^k, i = 1,2, \ldots n$ be the functions in the class $T_m(\xi, \varpi, \sigma)$.

By Theorem 2.1, for any $f \in T_m(\xi, \varpi, \sigma)$,

$$\sum_{k=2}^{\infty} \left\{[k - \sigma + \xi|\varpi|] \left(1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right)^m\right\} a_k \leq \xi|\varpi| + \sigma - 1.$$

Thus,

$$\sum_{k=2}^{\infty} \left\{[k - \sigma + \xi|\varpi|] \left(1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right)^m\right\} a_k \leq \xi|\varpi| + \sigma - 1$$

$$H(z) = \sum_{i=1}^{n} k_i f_i(z)$$

$$\Rightarrow H(z) = \sum_{i=1}^{n} k_i \left(z - \sum_{k=2}^{\infty} a_{i,k} z^k\right)$$

$$H(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{n} k_i a_{i,k}\right) z^k.$$

By Theorem 2.1, we show that

$$\sum_{k=2}^{\infty} \left\{[k - \sigma + \xi|\varpi|] \left(1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right)^m\right\} \left(\sum_{k=2}^{\infty} k_i a_{i,k}\right) \leq \xi|\varpi| + \sigma - 1$$

$$\sum_{i=1}^{n} k_i \left[\sum_{k=2}^{\infty} \left\{[k - \sigma + \xi|\varpi|] \left(1 + \frac{(k-1)[(\lambda-\alpha)\beta + k\delta]}{\mu+\lambda}\right)^m\right\} a_{i,k}\right]$$

$$\leq \sum_{i=1}^{n} k_i [k - \sigma + \xi|\varpi|] = \xi|\varpi| + \sigma - 1$$

which completes the proof.

5. Conclusion

The authors introduced a new class of univalent functions using a multiplier transformation. Seven results on the class of functions were provided in the work such as Theorem 2.1 as the necessary and sufficient condition for the class $T_m(\xi, \varpi, \sigma)$, Theorem 2.2 as radius while Theorem 2.4 shows that $T_m(\xi, \varpi, \sigma)$ is the convex set. Also, Theorem 2.5 is the weighted Mean, Theorem 2.6 Arithmetic Mean and Theorem 2.7 the Linear Combination for the class of functions.

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