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An Eighth-Stage Implicit Runge-Kutta Type Scheme for First-Order Ordinary Differential Equations

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A B S T R A C T

We develop an eighth-stage implicit Runge-Kutta (EIRK) type scheme for solving first-order initial-value problems (IVPs) in ordinary differential equations (ODEs). Through the multistep collocation approach, we obtain a one-step continuous hybrid solution, which is evaluated at certain points of interest to generate seven discrete schemes that are used to formulate a block hybrid method. The resultant block method is then reformulated into the new EIRK scheme. Analysis of the basic properties of the novel EIRK indicates that the method is of order eight, hence consistent, indicating an improvement over methods of lower orders, leading to better accuracy. Furthermore, linear stability analysis reveals that this method is A-stable, as such is a viable candidate for stiff IVPs. The numerical examples considered using the EIRK show that this proposed method yielded smaller absolute errors, implying better accuracy, as the tables reveal when compared with similar existing methods in the literature, hence should be preferred for such a class of problems.

1. Introduction

Ordinary differential equations (ODEs) models have found some varied and wide applications in engineering and applied sciences, technology, social and management sciences, mechanics, astronomy, the computation of radio technical circuits or satellite trajectories, stability of a plane flight, and the theory of oscillations (Atabo & Adee, 2021). Whereas some of these models do not have analytic methods by which solutions could be proffered, proposing approximate solutions that are more accurate in terms of reduced or smaller errors via numerical techniques by various researchers has made the subject an active area of research (see Adee & Yahya, 2022, Adee *et. al.,* 2022a, b, Adee *et. al.,* 2025, Yunusa *et. al.,* 2024, Adee & Yunusa, 2022, Koroche, 2021).

We shall consider the general first-order initial value problem (IVP) in ODEs of the form

$$y' = f\big(x, y(x)\big),\ \ y(x_0) = y_0 \tag{1}$$

on the interval $a \le x \le b$ where the function $f(x, y): R \times R^m \to R^m$ is continuous in x and further assume the existence and uniqueness of the solution $y(x)$ for (1).

Broadly, two classes of numerical techniques have been adopted by researchers to advance solutions to (1) which are Runge-Kutta (RK) methods and linear multistep methods (LMM) (Rattenbury, 2005). Three main disadvantages of the LMM approach are its poor stability as the step number increases with accuracy and the need for additional starting values with a constant step size, the Dahlquist barrier on the A-stability property (Yakubu, *et al.* 2012).

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Contrary to the odds against LMM, the RK methods, to some extent, enjoy higher popularity due to their symmetrical forms, simple coefficients, efficiency, and numerical stability (Agam, 2014). Recent studies like Adee and Yahya (2022) presented a single-step implicit RK type scheme for solving (1) with reduced computational burden and lower absolute errors than in Adee *et al.* (2022) in which (1) was also solved using a pair of two-step implicit Runge-Kutta (RK) type methods.

Riding on the strength of success recorded in the single-step approach by Adee and Yahya (2022), and the fact that accuracy can further be achieved whenever the order of approximating function is increased, this present research seeks to present a higher order RK type scheme -an eighth-stage method with improved accuracy for implementation.

Again, this approach aligns with using block hybrid methods which are essentially sets of linear multistep methods that are modified to incorporate off-grid points (see Adee *et al.* 2025, Yahaya & Adegboye, 2013, Zamurat & Ummikusum, 2014, Muhammad, Yahaya and Abdulkareem, 2015, Mshelia, *et al.,* 2016a, and Mshelia, *et al.,* 2016b). Particularly in this research, the block hybrid method is not implemented as a set of linear multistep methods but reformulated into a type of implicit Runge-Kutta type method. This collaborates the works of such researchers as Agam & Yahaya (2014), Adee & Yahya (2022), Adee *et al.* (2022), Koroche (2021), Mshelia, *et al.,* (2016a), and Mshelia, *et al.,* (2016b), among others, who applied their methods as RK schemes which simultaneously yielded approximate solutions to (1).

Leveraging on the self-starting nature of one-step RK methods, although implicit numerical methods incur more cost in their derivation and implementation, however, the improvement in accuracy and better stability properties make this research still relevant.

2. Derivation of the Proposed Eighth-Stage Implicit Runge-Kutta (EIRK) Type Scheme

Following Adee & Yahya (2022), Adee *et al.* (2022) and Mshelia, *et al.,* (2016a, b) where details of the multistep collocation technique have been used, we adopt the same approach to develop a novel one-step continuous hybrid method, which is its strength. This continuous method is then evaluated at points of interest to obtain the required seven discrete schemes, which are used to form a block method. The block method, in turn, is reformulated into the proposed eighth-stage implicit RK (EIRK) scheme for the approximate solution of (1).

Specifically, recalling the multistep collocation approach in Adee & Yahya (2022), Adee *et al.* (2022), we set

$t = \frac{x-x_n}{h}$ and the continuous hybrid solution of (1) with $r = 5$ interpolation points at $t = 0\left(\frac{1}{7}\right)\frac{4}{7}$, and m collocation points at $t = \frac{4}{7}\left(\frac{1}{7}\right)1$ now takes the form

$$y(t) = \sum_{j=0}^{4} \alpha_{\frac{j}{7}}(t) y_{n+\frac{j}{7}} + h \sum_{j=4}^{7} \beta_{\frac{j}{7}}(t) f_{n+\frac{j}{7}} \quad (2)$$

The equivalent multistep collocation matrix equation is given by

$$M\mathrm{T} = \mathrm{Y} \quad (3)$$

where the matrix M is given by

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$$M=\begin{pmatrix}1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8\\ 1 & x_{n+\frac{1}{7}} & x^2_{n+\frac{1}{7}} & x^3_{n+\frac{1}{7}} & x^4_{n+\frac{1}{7}} & x^5_{n+\frac{1}{7}} & x^6_{n+\frac{1}{7}} & x^7_{n+\frac{1}{7}} & x^8_{n+\frac{1}{7}}\\ 1 & x_{n+\frac{2}{7}} & x^2_{n+\frac{2}{7}} & x^3_{n+\frac{2}{7}} & x^4_{n+\frac{2}{7}} & x^5_{n+\frac{2}{7}} & x^6_{n+\frac{2}{7}} & x^7_{n+\frac{2}{7}} & x^8_{n+\frac{2}{7}}\\ 1 & x_{n+\frac{3}{7}} & x^2_{n+\frac{3}{7}} & x^3_{n+\frac{3}{7}} & x^4_{n+\frac{3}{7}} & x^5_{n+\frac{3}{7}} & x^6_{n+\frac{3}{7}} & x^7_{n+\frac{3}{7}} & x^8_{n+\frac{3}{7}}\\ 1 & x_{n+\frac{4}{7}} & x^2_{n+\frac{4}{7}} & x^3_{n+\frac{4}{7}} & x^4_{n+\frac{4}{7}} & x^5_{n+\frac{4}{7}} & x^6_{n+\frac{4}{7}} & x^7_{n+\frac{4}{7}} & x^8_{n+\frac{4}{7}}\\ 0 & 1 & 2x_{n+\frac{4}{7}} & 3x^2_{n+\frac{4}{7}} & 4x^3_{n+\frac{4}{7}} & 5x^4_{n+\frac{4}{7}} & 6x^5_{n+\frac{4}{7}} & 7x^6_{n+\frac{4}{7}} & 8x^7_{n+\frac{4}{7}}\\ 0 & 1 & 2x_{n+\frac{5}{7}} & 3x^2_{n+\frac{5}{7}} & 4x^3_{n+\frac{5}{7}} & 5x^4_{n+\frac{5}{7}} & 6x^5_{n+\frac{5}{7}} & 7x^6_{n+\frac{5}{7}} & 8x^7_{n+\frac{5}{7}}\\ 0 & 1 & 2x_{n+\frac{6}{7}} & 3x^2_{n+\frac{6}{7}} & 4x^3_{n+\frac{6}{7}} & 5x^4_{n+\frac{6}{7}} & 6x^5_{n+\frac{6}{7}} & 7x^6_{n+\frac{6}{7}} & 8x^7_{n+\frac{6}{7}}\\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} & 8x^7_{n+1}\end{pmatrix}$$

and $\mathrm{T}=(\tau_0,\tau_1,\ldots,\tau_8)^T,\ \ Y=(y_n,y_{n+\frac{1}{7}},y_{n+\frac{2}{7}},y_{n+\frac{3}{7}},y_{n+\frac{4}{7}},f_{n+\frac{4}{7}},f_{n+\frac{5}{7}},f_{n+\frac{6}{7}},f_{n+1})^T.$

Using a code in Maple (18) software to solve for τ's in (3) which are required in obtaining the continuous coefficients, the $\alpha_{\frac{j}{7}}(t),\beta_{\frac{j}{7}}(t)$ in (2) yield

$$\alpha_0(t)=\Big(\frac{3155715480211}{13388400576}t^8-\frac{3735002214755}{3347100144}t^7+\frac{1450728447655}{6694200288}t^6-\frac{108977061875}{418387518}t^5+\frac{13062162701}{8050752}t^4-\frac{2184805081445}{3347100144}t^3+\frac{519174017635}{3347100144}t^2-\frac{2742820325}{139462506}t+1\Big)$$

$$\alpha_{\frac{1}{7}}(t)=\Big(-\frac{440724801251}{209193759}t^8+\frac{8097541724881}{836775036}t^7-\frac{7780213194929}{418387518}t^6+\frac{16192664132533}{836775036}t^5-\frac{1479210481}{125793}t^4+\frac{871267650589}{209193759}t^3-\frac{166152515144}{209193759}t^2+\frac{4373336240}{69731253}t\Big)$$

$$\alpha_{\frac{2}{7}}(t)=\Big(\frac{365482618599}{41322224}t^8-\frac{101731443247}{2582639}t^7+\frac{1502621145781}{20661112}t^6-\frac{370816689803}{5165278}t^5+\frac{1009113889}{24848}t^4-\frac{67183373299}{5165278}t^3+\frac{5515988359}{2582639}t^2-\frac{341357520}{2582639}t\Big)$$

$$\alpha_{\frac{3}{7}}(t)=\Big(-\frac{5401999013866}{209193759}t^8+\frac{93281102289263}{836775036}t^7-\frac{82864863600631}{418387518}t^6+\frac{155995747887371}{836775036}t^5-\frac{12510052751}{125793}t^4+\frac{6205164756707}{209193759}t^3-\frac{939455323576}{209193759}t^2+\frac{18085441360}{69731253}t\Big)$$

$$\alpha_{\frac{4}{7}}(t)=\Big(\frac{252362240261201}{13388400576}t^8-\frac{269935623393709}{3347100144}t^7+\frac{948521249048261}{6694200288}t^6-\frac{27514538537017}{209193759}t^5+\frac{555297784111}{8050752}t^4-\frac{67503287537539}{3347100144}t^3+\frac{10021830488621}{3347100144}t^2-\frac{23741428795}{139462506}t\Big)$$

$$\beta_{\frac{4}{7}}(t)=\Big(-\frac{924730904867}{371900016}t^8+\frac{485037297101}{46487502}t^7-\frac{3333936840983}{185950008}t^6+\frac{754932828335}{46487502}t^5-\frac{1856297821}{223632}t^4+\frac{55059237247}{23243751}t^3-\frac{32029542251}{92975004}t^2+\frac{149450315}{149450315}t\Big)$$

$$\beta_{\frac{5}{7}}(t)=\Big(\frac{22862377223}{30991668}t^8-\frac{91519980643}{30991668}t^7+\frac{149523794833}{30991668}t^6-\frac{128638840393}{30991668}t^5+\frac{9412606}{4659}t^4-\frac{4277566573}{7747917}t^3+\frac{600436088}{7747917}t^2-\frac{10919216}{2582639}t\Big)$$

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$$\beta_{\frac{6}{7}}(t) = \Big(-\frac{116451450829}{557850024}t^8 + \frac{223248271771}{278925012}t^7 - \frac{350318670919}{278925012}t^6 + \frac{145462940315}{139462506}t^5 - \frac{165308507}{335448}t^4 + \frac{36665705839}{278925012}t^3 - \frac{1262801204}{69731253}t^2 + \frac{22659140}{23243751}t\Big)$$

$$\beta_1(t) = \Big(\frac{7644949669}{278925012}t^8 - \frac{28042227395}{278925012}t^7 + \frac{42494096099}{278925012}t^6 - \frac{34329654065}{278925012}t^5 + \frac{2385908}{41931}t^4 - \frac{1040648525}{69731253}t^3 + \frac{141544984}{69731253}t^2 - \frac{2517040}{23243751}t\Big).$$

Substituting these coefficients into (2) gives the one step continuous hybrid solution for (1).

Next, evaluating the one step continuous hybrid scheme (2) at $t = \frac{5}{7}, \frac{6}{7}, 1$ and its first derivative at $t = 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}$ yield seven discrete schemes viz:

$$y_{n+\frac{5}{7}} - \frac{13576975}{23243751}y_{n+\frac{4}{7}} - \frac{11731900}{23243751}y_{n+\frac{3}{7}} + \frac{271900}{2582639}y_{n+\frac{2}{7}} - \frac{418475}{23243751}y_{n+\frac{1}{7}} + \frac{36499}{23243751}y_n = h\Big(\frac{15800}{54235419}f_{n+1} - \frac{203800}{54235419}f_{n+\frac{6}{7}} + \frac{1126620}{18078473}f_{n+\frac{5}{7}} + \frac{343100}{2582639}f_{n+\frac{4}{7}}\Big)$$

$$y_{n+\frac{6}{7}} - \frac{27978775}{23243751}y_{n+\frac{4}{7}} + \frac{6509600}{23243751}y_{n+\frac{3}{7}} - \frac{242775}{2582639}y_{n+\frac{2}{7}} + \frac{454624}{23243751}y_{n+\frac{1}{7}} - \frac{44225}{23243751}y_n = h\Big(-\frac{8800}{7747917}f_{n+1} + \frac{2756060}{54235419}f_{n+\frac{6}{7}} + \frac{3453600}{18078473}f_{n+\frac{5}{7}} + \frac{443900}{18078473}f_{n+\frac{4}{7}}\Big)$$

$$y_{n+1} + \frac{2510025}{2582639}y_{n+\frac{4}{7}} - \frac{6540275}{2582639}y_{n+\frac{3}{7}} + \frac{1762236}{2582639}y_{n+\frac{2}{7}} - \frac{347900}{2582639}y_{n+\frac{1}{7}} + \frac{33275}{2582639}y_n = h\Big(\frac{107460}{2582639}f_{n+1} + \frac{575400}{2582639}f_{n+\frac{6}{7}} + \frac{37800}{2582639}f_{n+\frac{5}{7}} + \frac{947100}{2582639}f_{n+\frac{4}{7}}\Big)$$

$$-\frac{23741428795}{139462506}y_{n+\frac{4}{7}} + \frac{36170882720}{139462506}y_{n+\frac{3}{7}} - \frac{18433306080}{139462506}y_{n+\frac{2}{7}} + \frac{8746672480}{139462506}y_{n+\frac{1}{7}} - \frac{2742820325}{139462506}y_n = h\Big(\frac{15102240}{139462506}f_{n+1} - \frac{135954840}{139462506}f_{n+\frac{6}{7}} + \frac{589637664}{139462506}f_{n+\frac{5}{7}} - \frac{2690105670}{139462506}f_{n+\frac{4}{7}} + f_n\Big)$$

$$\frac{139083735}{5165278}y_{n+\frac{4}{7}} - \frac{224897540}{5165278}y_{n+\frac{3}{7}} + \frac{152534340}{5165278}y_{n+\frac{2}{7}} - \frac{62687100}{5165278}y_{n+\frac{1}{7}} - \frac{4033435}{5165278}y_n = h\Big(-\frac{71840}{5165278}f_{n+1} + \frac{665112}{5165278}f_{n+\frac{6}{7}} - \frac{3010230}{5165278}f_{n+\frac{5}{7}} + \frac{14969380}{5165278}f_{n+\frac{4}{7}} + f_{n+\frac{1}{7}}\Big)$$

$$-\frac{1496502455}{139462506}y_{n+\frac{4}{7}} + \frac{2887151680}{139462506}y_{n+\frac{3}{7}} - \frac{1172442600}{139462506}y_{n+\frac{2}{7}} - \frac{231085120}{139462506}y_{n+\frac{1}{7}} + \frac{12878495}{139462506}y_n = h\Big(\frac{539904}{139462506}f_{n+1} - \frac{5198730}{139462506}f_{n+\frac{6}{7}} + \frac{25107840}{139462506}f_{n+\frac{5}{7}} - \frac{145700820}{139462506}f_{n+\frac{4}{7}} + f_{n+\frac{2}{7}}\Big)$$

$$\frac{1057475300}{139462506}y_{n+\frac{3}{7}} - \frac{328349700}{139462506}y_{n+\frac{2}{7}} + \frac{38763620}{139462506}y_{n+\frac{1}{7}} - \frac{2876125}{139462506}y_n = h\Big(-\frac{218994}{139462506}f_{n+1} + \frac{2236296}{139462506}f_{n+\frac{6}{7}} - \frac{12068784}{139462506}f_{n+\frac{5}{7}} + \frac{100501596}{139462506}f_{n+\frac{4}{7}} + f_{n+\frac{3}{7}}\Big) \quad (4)$$

Following Adee and Yahya (2022), the seven discrete equations in (4) are written in block form as

$$A_0Y_m = A_1Y_{m-1} + h(B_1F_{m-1} + B_0F_m) \quad (5)$$

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where

$$Y_m = \left(y_{n+\frac{1}{7}}, y_{n+\frac{2}{7}}, y_{n+\frac{3}{7}}, y_{n+\frac{4}{7}}, y_{n+\frac{5}{7}}, y_{n+\frac{6}{7}}, y_{n+1}\right), Y_{m-1} = \left(y_{n-\frac{6}{7}}, y_{n-\frac{5}{7}}, y_{n-\frac{4}{7}}, y_{n-\frac{3}{7}}, y_{n-\frac{2}{7}}, y_{n-\frac{1}{7}}, y_n\right),$$

$$F_m = \left(f_{n+\frac{1}{7}}, f_{n+\frac{2}{7}}, f_{n+\frac{3}{7}}, f_{n+\frac{4}{7}}, f_{n+\frac{5}{7}}, f_{n+\frac{6}{7}}, f_{n+1}\right), F_{m-1} = \left(f_{n-\frac{6}{7}}, f_{n-\frac{5}{7}}, f_{n-\frac{4}{7}}, f_{n-\frac{3}{7}}, f_{n-\frac{2}{7}}, f_{n-\frac{1}{7}}, f_n\right)$$

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{751}{17280} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{41}{980} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{265}{6272} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{278}{6615} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{265}{6272} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{41}{980} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{751}{17280} \end{bmatrix}$$

$$B_0 = \begin{bmatrix} \frac{139849}{846720} & -\frac{4511}{31360} & \frac{123133}{846720} & -\frac{88547}{846720} & \frac{1537}{31360} & -\frac{11351}{846720} & \frac{275}{169344} \\ \frac{1466}{6615} & -\frac{71}{2940} & \frac{68}{735} & -\frac{1927}{26460} & \frac{26}{735} & -\frac{29}{2940} & \frac{8}{6615} \\ \frac{1359}{6272} & \frac{1377}{31360} & \frac{5927}{31360} & -\frac{3033}{31360} & \frac{1377}{31360} & -\frac{373}{31360} & \frac{9}{6272} \\ \frac{1448}{6615} & \frac{8}{245} & \frac{1784}{6615} & -\frac{106}{6615} & \frac{8}{245} & -\frac{64}{6615} & \frac{8}{6615} \\ \frac{36725}{169344} & \frac{775}{18816} & \frac{4625}{18816} & \frac{13625}{169344} & \frac{1895}{18816} & -\frac{275}{18816} & \frac{275}{169344} \\ \frac{54}{245} & \frac{27}{980} & \frac{68}{245} & \frac{27}{980} & \frac{54}{245} & \frac{41}{980} & 0 \\ \frac{3577}{17280} & \frac{49}{640} & \frac{2989}{17280} & \frac{2989}{17280} & \frac{49}{640} & \frac{3577}{17280} & \frac{751}{17280} \end{bmatrix}$$

Multiplying (5) by the inverse of A_0 and representing it as a Butcher tableau yields

$$\begin{array}{c|cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{7} & \frac{751}{17280} & \frac{139849}{846720} & -\frac{4511}{31360} & \frac{123133}{846720} & -\frac{88547}{846720} & \frac{1537}{31360} & -\frac{11351}{846720} & \frac{275}{169344} \\ \frac{2}{7} & \frac{41}{980} & \frac{1466}{6615} & -\frac{71}{2940} & \frac{68}{735} & -\frac{1927}{26460} & \frac{26}{735} & -\frac{29}{2940} & \frac{8}{6615} \\ \frac{3}{7} & \frac{265}{6272} & \frac{1359}{6272} & \frac{1377}{31360} & \frac{5927}{31360} & -\frac{3033}{31360} & \frac{1377}{31360} & -\frac{373}{31360} & \frac{9}{6272} \\ \frac{4}{7} & \frac{278}{6615} & \frac{1448}{6615} & \frac{8}{245} & \frac{1784}{6615} & -\frac{106}{6615} & \frac{8}{245} & -\frac{64}{6615} & \frac{8}{6615} \\ \frac{5}{7} & \frac{265}{6272} & \frac{36725}{169344} & \frac{775}{18816} & \frac{4625}{18816} & \frac{13625}{169344} & \frac{1895}{18816} & -\frac{275}{18816} & \frac{275}{169344} \\ \frac{6}{7} & \frac{41}{980} & \frac{54}{245} & \frac{27}{980} & \frac{68}{245} & \frac{27}{980} & \frac{54}{245} & \frac{41}{980} & 0 \\ 1 & \frac{751}{17280} & \frac{3577}{17280} & \frac{49}{640} & \frac{2989}{17280} & \frac{2989}{17280} & \frac{49}{640} & \frac{3577}{17280} & \frac{751}{17280} \\ \hline 1 & \frac{751}{17280} & \frac{3577}{17280} & \frac{49}{640} & \frac{2989}{17280} & \frac{2989}{17280} & \frac{49}{640} & \frac{3577}{17280} & \frac{751}{17280} \end{array} \qquad (6)$$

Expressing (6) explicitly as an implicit Runge-Kutta method yields

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$$y_{n+1} = y_n + h\left(\frac{751}{17280}k_1 + \frac{3577}{17280}k_2 + \frac{49}{640}k_3 + \frac{2989}{17280}k_4 + \frac{2989}{17280}k_5 + \frac{49}{640}k_6 + \frac{3577}{17280}k_7 + \frac{751}{17280}k_8\right) \quad (7)$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{1}{7}h, y_n + h\left(\frac{751}{17280}k_1 + \frac{139849}{846720}k_2 - \frac{4511}{31360}k_3 + \frac{123133}{846720}k_4 - \frac{88547}{846720}k_5 + \frac{1537}{31360}k_6 - \frac{11351}{846720}k_7 + \frac{275}{169344}k_8\right)\right)$$

$$k_3 = f\left(x_n + \frac{2}{7}h, y_n + h\left(\frac{41}{980}k_1 + \frac{1466}{6615}k_2 - \frac{71}{2940}k_3 + \frac{68}{735}k_4 - \frac{1927}{26460}k_5 + \frac{26}{735}k_6 - \frac{29}{2940}k_7 + \frac{8}{6615}k_8\right)\right)$$

$$k_4 = f\left(x_n + \frac{3}{7}h, y_n + h\left(\frac{265}{6272}k_1 + \frac{1359}{6272}k_2 + \frac{1377}{31360}k_3 + \frac{5927}{31360}k_4 - \frac{3033}{31360}k_5 + \frac{1377}{31360}k_6 - \frac{373}{31360}k_7 + \frac{9}{6272}k_8\right)\right)$$

$$k_5 = f\left(x_n + \frac{4}{7}h, y_n + h\left(\frac{278}{6615}k_1 + \frac{1448}{6615}k_2 + \frac{8}{248}k_3 + \frac{1784}{6615}k_4 - \frac{106}{6615}k_5 + \frac{8}{248}k_6 - \frac{64}{6615}k_7 + \frac{8}{6615}k_8\right)\right)$$

$$k_6 = f\left(x_n + \frac{5}{7}h, y_n + h\left(\frac{265}{6272}k_1 + \frac{36725}{169344}k_2 + \frac{775}{18816}k_3 + \frac{4625}{18816}k_4 + \frac{13625}{169344}k_5 + \frac{1895}{18816}k_6 - \frac{275}{18816}k_7 + \frac{275}{169344}k_8\right)\right)$$

$$k_7 = f\left(x_n + \frac{6}{7}h, y_n + h\left(\frac{41}{980}k_1 + \frac{54}{245}k_2 + \frac{27}{980}k_3 + \frac{68}{245}k_4 + \frac{27}{980}k_5 + \frac{54}{245}k_6 + \frac{41}{980}k_7 + 0k_8\right)\right)$$

$$k_8 = f\left(x_n + h, y_n + h\left(\frac{751}{17280}k_1 + \frac{3577}{17280}k_2 + \frac{49}{640}k_3 + \frac{2989}{17280}k_4 + \frac{2989}{17280}k_5 + \frac{49}{640}k_6 + \frac{3577}{17280}k_7 + \frac{751}{17280}k_8\right)\right)$$

Eq. (7) is the proposed novel eighth-stage implicit Runge-Kutta type scheme for the numerical integration of (1).

3. Results

3.1 Convergence analysis of the proposed EIRK scheme (7).

Theorem 1: *Super-convergence* (Landis*, 2005*).

If the condition B(P) *holds for some* $p \geq s$*, then the collocation method has order* p *where*

$$\mathrm{B(P)} = \sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k} \quad (8)$$

Applying (8) to (7), it is easily seen that the EIRK method is of order $p = 8$.

In line with Butcher (2008), the EIRK method (7) is consistent since $\sum_{i=1}^{s} b_i = 1$ hence, it is consistent. Accordingly, the proposed EIRK is convergent.

3.2 Stability of the Runge-Kutta method (Butcher, 2008).

Definition 1*: The stability function $R(z)$ for (7) is defined by*

$$R(z) = \frac{\det(I - zA + zeb)}{\det(I - zA)} = \frac{N(z)}{D(z)} \quad (9)$$

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where $\{z \in C^{-}: R(z) \leq 1\}, e = (1,1, \ldots ,1)^{T}$.

Theorem 2: A Runge-Kutta method with stability function $R(z) = \frac{N(z)}{D(z)}$ *is A-stable if and only if (a) all poles of* $R(z)$ *(that is, all zeros of* $D(z)$*) are in the right half-plane and (b)*$E(y) \geq 0$ for all real y where $E(y) = D(iy)D(-iy) - N(iy)N(-iy)$ (see Butcher, 2008 for the proof).

Applying (9) to (7) yields the stability polynomial $R(z) = \frac{\frac{1}{6588344}z^7+\frac{363}{65883440}z^6+\frac{67}{576240}z^5+\frac{967}{576240}z^4+\frac{5}{294}z^3+\frac{23}{196}z^2+\frac{1}{2}z+1}{-\frac{1}{6588344}z^7+\frac{363}{65883440}z^6-\frac{67}{576240}z^5+\frac{967}{576240}z^4-\frac{5}{294}z^3+\frac{23}{196}z^2-\frac{1}{2}z+1}$. Clearly, $R(z)$ satisfies theorem 2 and, in particular, $E(y) = 0$ for all real y. Thus, the proposed EIRK method (7) is A-stable and can equally treat stiff IVPs in ODEs.

4. Numerical Examples

We measure the accuracy of the proposed new method in terms of the absolute errors, which are essentially the deviations from the theoretical solutions.

Define absolute errors, denoted e_i by

$e_i = |y(x_i) - y_i|$, where $y(x_i)$ represents the theoretical solution at the point x_i while y_i is the solution from the proposed EIRK method (7), we tested the accuracy of the EIRK using four numerical examples and the results are presented in Tables 1–4.

Note the following notations:

$aE(-b) \coloneqq a \times 10^{-b}$,

EIRK -proposed new EIRK Method (7) for step number $k = 1, order\ p = 8$

AY – Agam and Yahaya (2014)

MET3 –Mshelia *et al.* (2016a), $k = 3, order\ p = 6$

MET4 –Mshelia *et al.* (2016b), $k = 4, order\ p = 9$

TSIRK1 – Adee *et al.* (2022), $k = 2, order\ p = 6$

TSIRK2 –Adee *et al.* (2022), $k = 2, order\ p = 6$

SIRK – Adee and Yahya (2022) $k = 1, order\ p = 6$

Example 4.1 [Source: Mshelia, *et al* (2016a)]. Solve $y' - 2y = e^{-x}, y(0) = \frac{3}{4}, h = 0.1, 0 \leq x \leq 1.0$,

Exact solution: $y(x) = \left(-\frac{1}{3}e^{-3x} + \frac{13}{12}\right)e^{2x}$.

Table 1 shows the absolute errors of the proposed new EIRK method (7) for Example 4.1 compared with MET4, TSIRK1, TSIRK2 and SIRK, and the smaller errors in EIRK indicate its superiority over the four methods.

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Table 1: Absolute Errors for Example 4.1

x	MET3 – Mshelia *et al.* (2016a)	TSIRK1 – Adee *et al.* (2022)	TSIRK2 – Adee *et al.* (2022)	SIRK – Adee & Yahya (2022)	Proposed EIRK method (7)
0.10	----	6.8729E(-12)	5.0930E(-12)	4.9614E(-12)	1.1102E(-16)
0.20	----	1.6785E(-11)	1.2438E(-11)	1.2116E(-11)	1.1102E(-16)
0.30	----	3.0744E(-11)	2.2782E(-11)	2.2192E(-11)	1.1102E(-16)
0.40	----	5.0059E(-11)	3.7094E(-11)	3.613E(-11)	1.1102E(-16)
0.50	----	7.6410E(-11)	5.6625E(-11)	5.5161E(-11)	1.1102E(-16)
0.60	2.0E(-9)	1.1190E(-10)	8.2984E(-11)	8.083E(-11)	1.1102E(-16)
0.70	3.0E(-9)	1.5956E(-10)	1.1824E(-10)	1.1518E(-10)	5.5515E(-17)
0.80	1.0E(-9)	2.2271E(-10)	1.6503E(-10)	1.6076E(-10)	5.5515E(-17)
0.90	1.0E(-9)	3.0600E(-10)	2.2675E(-10)	2.2088E(-10)	1.1102E(-16)
1.00	6.0E(-9)	4.1524E(-10)	3.0770E(-10)	2.9974E(-10)	1.1102E(-16)

Example 4.2. [Source: Mshelia, *et al* (2016b)]. Solve $y' = 20x^2 - 20y + 2x, y(0) = \frac{1}{3}, h = 0.05, 0 \le x \le 1.0$,

Exact solution: $y(x) = x^2 + \frac{1}{3}e^{-20x}$.

Table 2 also shows the absolute errors of the proposed new EIRK method (7) for Example 4.2 compared with SIRK, MET3 and MET4. Once more, the smaller errors obtained for the new EIRK indicate its better accuracy than these three methods.

Table 2: Absolute Errors for Example 4.2.

x	MET3 – Mshelia *et al.* (2016a)	MET4 – Mshelia *et al.* (2016b)	SIRK – Adee & Yahya (2022)	Proposed EIRK method (7)
0.10	1.25538478E(-07)	2.0 E(-11)	2.0928E(-08)	3.9412E(-11)
0.20	3.3979523E(-08)	2.0 E(-11)	7.9261 E(-09)	1.0668E(-11)
0.30	6.897933E(-09)	1.0 E(-10)	1.6090 E(-09)	2.1656E(-12)
0.40	1.24471E(-09)	----	2.9034 E(-10)	3.9074E(-13)
0.50	2.10566E(-10)	1.0 E(-10)	4.9117 E(-11)	6.605E(-14)
0.60	3.4196E(-11)	1.0 E(-10)	7.9768 E(-12)	1.0825E(-14)
0.70	5.400E(-12)	----	1.2594 E(-12)	1.7764E(-15)

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0.80	8.360E(-13)	1.0 E(-10)	1.9484 E(-13)	4.4409E(-16)
0.90	1.280E(-13)	1.0 E(-10)	2.9532 E(-14)	1.1102E(-16)
1.00	1.0E(-14)	1.0 E(-09)	4.4409 E(-15)	2.2204E(-16)

Example 4.3. [Source: Mshelia *et al.* (2016a, 2016b)]. Solve $y' = -y, y(0) = 1, h = 0.05, 0 \leq x \leq 1.0$,

Exact solution: $y(x) = e^{-x}$.

Table 3 reveals the absolute errors of the proposed new EIRK method (7) for Example 4.3 compared with three methods, which are SIRK, MET3 and MET4. The smaller errors obtained for the new EIRK indicate better accuracy than the trio.

Table 3: Absolute Errors for Example 4.3

x	MET3 – Mshelia *et al.* (2016a)	MET4 – Mshelia *et al.* (2016b)	TSIRK1 – Adee *et al.* (2022)	SIRK – Adee & Yahya (2022)	Proposed EIRK method (7)
0.1	2.0 E(-15)	1.0 E(-15)	5.5511E(-16)	3.77707 E(-16)	1.1102E(-16)
0.2	4.0 E(-15)	1.0 E(-15)	1.1102E(-15)	6.6613 E(-16)	1.1102E(-16)
0.3	6.0 E(-15)	1.0 E(-15)	1.5543E(-15)	9.9920 E(-16)	1.1102E(-16)
0.4	7.0 E(-15)	1.0 E(-15)	1.8874E(-15)	1.2212 E(-15)	1.1102E(-16)
0.5	7.0 E(-15)	1.0 E(-15)	2.1094E(-15)	1.3323 E(-15)	1.1102E(-16)
0.6	7.0 E(-15)	1.0 E(-15)	2.4425E(-15)	1.5543 E(-15)	1.1102E(-16)
0.7	9.0 E(-15)	1.0 E(-15)	2.442 E(-15)	1.5543 E(-15)	5.5515E(-17)
0.8	9.0 E(-15)	1.0 E(-15)	2.4425E(-15)	1.5543 E(-15)	5.5515E(-17)
0.9	9.0 E(-15)	1.0 E(-15)	2.4980E(-15)	1.5543 E(-15)	1.1102E(-16)
1.0	9.0 E(-15)	1.0 E(-15)	2.5535E(-15)	1.6098 E(-15)	1.1102E(-16)

Example 4.4. [Source: Agam and Yahaya (2014)]. Solve $y' = y + 8x + 1, y(0) = 2, h = 0.1, 0 \leq x \leq 0.5$

Exact solution: $y = x + 2e^{-8x}$

In Table 4, we compared the absolute errors of the proposed new EIRK method (7) for Example 4.4 with four methods, which are AY, TSIRK1, TSIRK2 and SIRK. Again, the smaller errors obtained for the new EIRK indicate an improved performance with better accuracy than these four methods.

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Table 4: Absolute Errors for Example 4.4

x	AY – Agam and Yahaya (2014)	TSIRK1 – Adee *et al.* (2022)	TSIRK2 – Adee *et al.* (2022)	SIRK – Adee & Yahya (2022)	Proposed EIRK method (7)
0.1	1.86 E(-06)	1.1497E(-07)	9.8582E(-08)	5.8854 E(-08)	1.1102E(-16)
0.2	1.69 E(-06)	1.0332E(-07)	8.8591E(-08)	5.2890 E(-08)	1.1102E(-16)
0.3	1.14 E(-06)	6.9638E(-08)	5.9710E(-08)	3.5647 E(-08)	1.1102E(-16)
0.4	6.80 E(-07)	4.1721E(-08)	3.5772E(-08)	2.1357 E(-08)	1.1102E(-16)
0.5	3.85 E(-07)	2.3433E(-08)	2.0092E(-08)	1.1995 E(-08)	1.1102E(-16)

5. Discussion and conclusion

Tables 1- 4 compare the absolute errors of four numerical examples. In all the examples considered, the proposed new eighth-stage implicit Runge-Kutta type scheme (EIRK) given by (7) outperformed in terms of smaller errors recorded than similar existing methods under consideration. This supports the theory that improved accuracy can be achieved by increasing the order of the approximation function. In particular, the improved accuracy of the new method over a method of higher order, such as the MET4 of Mshelia, *et al* (2016b), is a clear indication that this method is to be preferred for such class of problems.

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References

Adee, S. O., Kumleng, G. M., & Patrick, P. P. (2022a). One-step Embedded Block Hybrid Method for solving First -Order Stiff Initial Value Problems of Ordinary Differential Equations. *Nigerian Annals of Pure and Applied Sciences*, *5*(1), 257 - 264. https://doi.org/10.5281/zenodo.6514982

Adee, S. O. & Yunusa, S. (2022). Some New Hybrid Block Methods for Solving Non-Stiff Initial Value Problems of Ordinary Differential Equations. *Nigerian Annals of Pure and Applied Sciences*, *5*(1), 265 - 279. https://doi.org/10.5281/zenodo.7135518

Adee, S. O, Zechariah, A., Mohammed, S. B. (2025). Two Higher-Order Single-Step Embedded Block Hybrid Integrators for First-Order Ordinary Differential Equations. *International Journal of Development Mathematics*, *2*(1), 047-059. https://doi.org/10.62054/ijdm/0201.04

Adee, S. O., Yahya, M., & Yunusa, S. (2022b). A Pair of Two-Step Sixth Stage Implicit Runge-Kutta Type Methods for First-Order Ordinary Differential Equations. *FUAM Journal of Pure and Applied Science*, *2*(2),48-57. FUAMJPAS

Agam, S. A. (2014). A Sixth Order Multiply Implicit Runge-Kutta Method for the Solution of First and Second Order Ordinary Differential Equations. Unpublished PhD Thesis, Department of Mathematics, Nigerian Defence Academy, Kaduna, Nigeria.

Agam, S. A., & Yahaya, Y. A. (2014). A highly efficient implicit Runge-Kutta method for first-order ordinary differential equations. *African Journal of Mathematics and Computer Science Research.* 7(5), 55-60.

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https://doi.org/10.5897/AJMCSR2014.0551

Butcher, J. (2008). *Numerical Methods Ordinary Different Equations* (2nd ed). New York: John Wiley and Sons.

Koroche, K. D. (2021). Numerical Solution of First Order Ordinary Differential Equation by Using Runge-Kutta Method. *International Journal of Systems Science and Applied Mathematics.* 6(1), 1-8. https://doi.org/10.11648/j.ijssam.20210601.11

Landis, F. (2005). Runge-Kutta and Collocation Methods. Geometrical Numerical Integration http://www.sam.math.ethz.ch/~hiptmair/Seminars/GNI/slides/Landis.pdf

Mshelia, D. W., Yakubu, D. G., Badmus, A. M., & Manjak, N. H. (2016a). Derivation of three-step sixth-stage Runge-Kutta method for the solution of first-order differential equations. *International Journal of Scientific and Technical Research in Engineering*, 1(1):18-26. Retrieved from http://www.ijstre.com/Publish/042016/371428197.pdf

Mshelia, D. W., Badmus, A. M., & Yakubu, D. G. (2016b). Derivation of ninth-stage Runge-Kutta method for the solution of first-order differential equations. *Asian Journal of Mathematics and Applications,* 2016, Article ID: ama0267, 12 pages. Retrieved from http://www.sciencieasia.asia/files/267.pdf

Muhammad, R., Yahaya, Y. A., & Abdulkareem, A. S. (2015). Development of a new class of Block implicit Runge-Kutta type method for initial value problems. *International Engineering Conference*, 2015: 655-660

Rattenbury, N. (2005). Almost Runge–Kutta Methods for Stiff and Non–stiff Problems. PhD Thesis (Unpublished). The University of Auckland, New Zealand. www.math.auckland.ac.nz/~butcher/theses/thesis-25-11-05.pdf

Yahya, Y. A., & Adegboye, Z. A. (2013). Derivation of an implicit six-stage block Runge-Kutta method for direct integration of second-order using quade-type multistep method. *International Journal of Science and Technology,* 2(3): 259-268.

Yakubu, D. G., Kwami, A. M., Ahmed, M. L. (2012). A special class of continuous general linear methods. *Computational and Applied Mathematics*, 31(2), 259-281. https://www.scielo.br/j/cam/a/FVyX38RHDTDyPd9GSgFJKCS/?lang=en&format=pdf

Yunusa, S., Adee, S. O., Kida, M., Etuk, E. (2024). One-Step Hybrid Block Method for the Numerical Solution of Malthus and SIR Epidemic Model equations. *FUAM Journal of Pure and Applied Science*, *4*(1), 95-103. FUAMJPAS

Zamurat, A. A., & Ummikusum, A. (2014). Modification of Simpson's Block hybrid multistep method for general second order ODEs. *International Journal of Science and Technology,* 3(1): 73-77.

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