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The Eigenspace of Stratified Deep Water Under Modified Gravity and Coriolis Effect

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A R T I C L E I N F O

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A B S T R A C T

Eigenspace is a mathematical concept that is used to describe the set of all possible eigenvalues and eigenvectors of a linear operator. Within the context of deep water stratification under gravity modification and Coriolis effect, eigenspace plays an important role in understanding the dynamics of the deep water regime. Eigenspace is the vector space spanned by all eigenvectors corresponding to a given eigenvalue of a linear operator. In stratified deep water, the density variations modify the effective gravitational force experienced by fluid parcels, producing a slightly reduced buoyancy effect compared to the standard gravitational acceleration. These factors can cause the water to move in complex ways, and eigenspace can be used to describe the different modes of motion that are possible. The eigenspace of the linearized equations of motion for stratified deep water was used to identify wave modes such as internal gravity waves and inertial (Coriolis) waves. Importantly, we studied the regime of deep water stability under gravity modification and Coriolis effect. By examining the eigenvalues of the linearized equations of motion, we determined whether the deep water is stable or unstable, and if it is unstable, we identify the modes of motion responsible for the instability. The model is based on the Navier-Stokes equations, which describe the motion of fluid in a velocity field. The equations are modified to account for the effects of modified gravity and the Coriolis effect, which can cause the fluid to move in a curved path. The model is then solved using a combination of analytical and numerical techniques. The results show that the stability of stratified deep water under gravity modification and Coriolis effect is determined by a number of factors, including the magnitude of the modified gravity, the magnitude of the Coriolis effect, and the depth of the water. The model also shows that the stability of deep water regime under modified gravity and Coriolis effect conditions can be significantly different from that of deep water under normal gravity. The model has a number of potential applications, including the study of ocean currents and the design of underwater vehicles. It can also be used to study the stability of other fluids, such as gases and plasmas, under similar conditions.

1. Introduction

Mathematical models have long been used to study deep water stability under gravity modification and Coriolis effect (Topman & Mbah, 2025). In recent years, there has been a growing interest in developing new mathematical models that can better capture the intricate dynamics systems of deep water stratification (Martin, 2021; Martin, 2022).

One of the recent developments in this field of oceanography and climate science is the use of the Navier-Stokes equations with a modified gravity term to study the stability of deep water under the influence of a

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rotating frame of reference (Abd-el-Malek *et al*., 2007). This approach takes into account the effects of gravity and Coriolis force on the fluid motion, and has been shown to be effective in capturing the dynamics of deep water systems in a variety of environments (Joyce *et al.*, 2020; Liu *et al.*, 2024).

Another recent development is the use of the primitive equations, which is a simplified version of the Navier-Stokes equations that assumes hydrostatic balance and geostrophic motion (De Boer *et al*., 2020). This approach has been shown to be effective in capturing the large-scale circulation patterns in deep water systems, and has been used to study a variety of phenomena, including ocean currents, atmospheric circulation, and climate dynamics (De Boer & Von der Heydt, 2020). The most important thing is that our model offers the first opportunity for deep water regime to be studied under modified gravity and coriolis effect (De Boer *et al*., 2020).

In addition to these traditional approaches, recent advances in computational power and numerical methods have made it possible to perform high-resolution simulations of deep water systems leading to stability predictions (Chae, 2020). These simulations can provide valuable insights into the behavior of fluids in different environments, and can be used to test and validate mathematical models (Charney, 1948; Chen *et al*., 2023; Mbah & Udogu, 2015).

Eigenspace is a fundamental mathematical concept used to characterize the behavior of a linear system. Within the context of deep water stratification under gravity modification and Coriolis effect, eigenspace can be used to describe the patterns of motion of the water column (Constantin, & Kartashova, 2009).

In general, the motion of stratified deep water could be described by a set of linear equations that govern the evolution of the water column over time (Dijkstra, 2010; Durran, 2010). These equations can be written in matrix form, where the matrix represents the linear operators that act on the water column (Waugh *et al*., 2020). The eigenvectors of this matrix are the patterns of motion that are most naturally associated with the system, and the corresponding eigenvalues represent the growth or decay rates of these patterns (Escher *et al*., 2011).

The eigenspace of the matrix was used to describe the different patterns of motion that are possible in the system (Waugh *et al*., 2020; Joyce *et al*., 2020; Fraccarollo *et al*., 2003). The eigenspace corresponding to the largest eigenvalue represents the pattern of motion that grows the fastest (Le Veque, 2004), while the eigenspace corresponding to the smallest eigenvalue represents the pattern of motion that decays the fastest ow . The eigenspace can also be used to describe the different motion patterns possible in the system (De

Boer *et al*., 2020; Martin, 2021). The eigenspace corresponding to the largest eigenvalue represents the motion pattern associated with the largest internal waves (Martin, 2021, 2022), whereas that of the smallest eigenvalue corresponds to the smallest waves (Martin, 2023). In conclusion, the model has provided a means of studying the stability of stratified deep water without considering the vertical advection and this was done by considering Coriolis effect and modified gravity (Martin, 2017; Topman & Mbah, 2024). The model has important applications in a variety of fields, including oceanography, meteorology, and climate science (Topman & Mbah, 2024; Topman *et al*., 2023).

2.0 The Model Assumptions

The next step is to formulate the fundamental mathematical assumptions underlying the model.

The assumptions for the model are:

i. Deep water is assume to be incompressible and stratified with continuous density

ii. Deep water is in a state of hydrostatic equilibrium, meaning that the pressure at a given depth is constant equal to the weight of the water above it.

iii. There is variation in gravity with time or space as the deep water is continuously stratified.

iv. The Coriolis effect varies across the strata with time.

v. An infinite depth for deep water flow, hence the vertical length scale (h) and the horizontal length scale (L) guarantee deep water regime when

vi. $\frac{L}{h} \ll 1$ ($deep\ water\ assumption$) But fails when $\frac{L}{h} \gg 1$

vii. The velocity components denoted by $u, v\ and\ w$ are in the directions of increasing $x, y\ and\ z$

viii. The depth-average velocity in the x direction is denoted as U=$u(x, y, t)$ and the depth-average velocity in the y- direction as $v = v(x; y; t)$. While the plane (z=0) can be chosen arbitrarily, it is usually positioned at mean water level.

ix. Let the horizontal plane be (x, y) which is parallel to the deep water surface and as h =(x, y, t)>0 the depth of the water at a given point

x. Equation $z = -\zeta(x, y)$ is the equation for the bottom surface at which the diameter of the orbital path is zero at any instant. The transition zone which is at thermocline, the point where the circular orbit of the deep water particles decrease at depth $z = -\zeta(x, y)$.

xi. We used the cartesian coordinates $x, y\ and\ z$, in Cartesian coordinates for deep water waves, the x-axis is the horizontal direction, the z-axis typically measures the vertical direction and the y-axis represents another horizontal direction.

xii. For deep water stratification, the motion of water particle becomes circular as it moves from the perturbed surface through the medium.

xiii. Partial derivative of velocity with respect to y is zero since velocity variation in direction of y is constant which justifies that the flow is predominantly in x-direction.

3. Methodology

The model formulation of stratified deep water under modified gravity and coriolis effect could be carried out mathematically through different method and principles, particularly by employing momentum conservation and continuity equation.

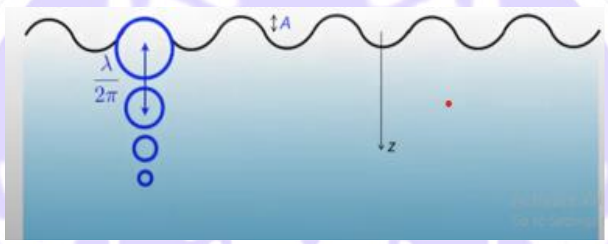

Figure 1

Figure 1. Shows the exponential decay of deep water waves due to density variation under modified gravity and Coriolis effect.

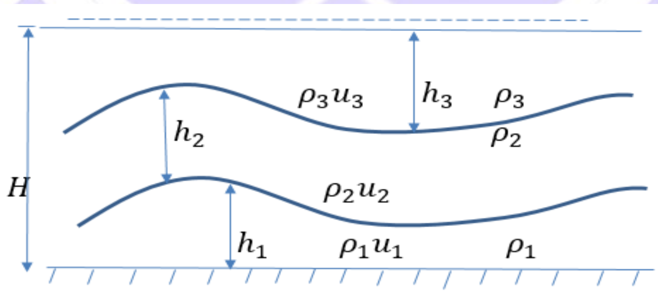

Figure 2. Stratification in three layers under modified gravity and Coriolis effect.

Consider Figure 2, three strata of deep water under kinematic and dynamic boundary conditions. Let the upper layer, intermediate and lower layers be represented by $h_3, h_2\ and\ h_1$, with respective densities $\rho_3, \rho_2\ and\ \rho_1$, velocities $u_3, u_2 and\ u_1$ and the thicknesses $h_3, h_2 and\ h_1$, with $H = h_3 + h_2 + h_1$, where the distance between the mean surface and the overall thermocline position is H. The variables of interest are $h_3, h_2, h_1, u_3, u_2, u_1, \rho_3, \rho_2, \rho_1$ and which are all functions of (x, t) at constant pressure.

The substantial derivative of stratified deep water property under gravity modification and Coriolis effect, $f(x, y, z, t)\ in\ a\ flow\ field\ with\ velocity\ vector$

$U = (u, v, w)^T$ is given by

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \quad (1)$$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + U.\nabla f \quad (2)$$

$$CurlF = \nabla \times F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\right)$$

The operator ∇ is a differential operator defined as

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \quad (3)$$

so that $\nabla f = \left(f_x, f_y, f_z\right)^T$; such that any arbitrary fluid property can be expressed as

$$f(x, y, z, t) = f(x(t), y(t), z(t), t) \quad (4)$$

Differentiating equation (1) with respect to t using chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}\frac{dt}{dt} \quad (5)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \quad (6)$$

Thus, in the circumstance of velocity components the material derivative is the Lagrangian acceleration which in most cases is the same meaning with Eulerian acceleration. For the x-component of velocity, the substantial derivative from equation (1), we have

$$\frac{Du}{Dt} = \frac{\partial(u)}{\partial t} + u\frac{\partial(u)}{\partial x} + v\frac{\partial(u)}{\partial y} + w\frac{\partial(u)}{\partial z} \quad (7)$$

$\frac{Du}{Dt}$ is the total acceleration;

$\frac{\partial u}{\partial t}$ is the local acceleration;

$u\frac{\partial(u)}{\partial x} + v\frac{\partial(u)}{\partial y} + w\frac{\partial(u)}{\partial z}$ is the convective or advective terms.

This represents the x-direction (Lagrangian, or total) acceleration, a_x, of a fluid parcel expressed in Eulerian reference frame. Then in Lagrangian frame the spartial changes in velocity is

$$U.\nabla u = \frac{\partial(u)}{\partial t} + u\frac{\partial(u)}{\partial x} + v\frac{\partial(u)}{\partial y} + w\frac{\partial(u)}{\partial z} \quad (8)$$

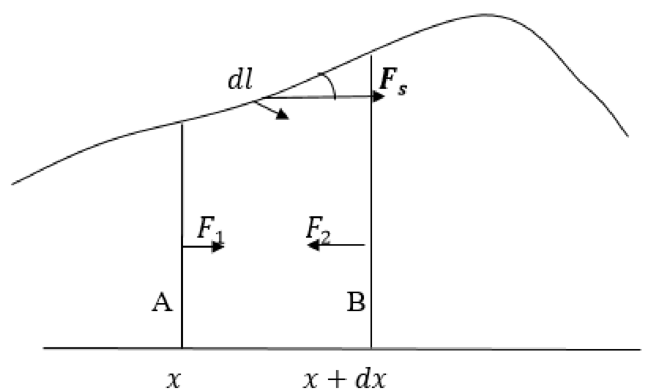

Figure 3: Cross sectional area pressure of stratified deep water under modified gravity and coriolis effect

Since the surface area is inclined, the product of pressure and its cross-sectional area sustain a non-zero component $p_0 dlsin \propto$ stand in the x-direction positively, $(\propto)$, is the interface angle. And, $dl = \frac{dx}{cos\propto}$, is influence on the force in x-direction, which is $F_s = p_0\frac{\partial h}{\partial x}dx$ (but $tan \propto \frac{\partial h}{\partial x}$). From this our net force on the volume, per unit length in y, is expressed as:

$$F = p_0\frac{\partial h}{\partial x}dx + \int_0^h p(x,z)dz - \int_0^h p(x+dx,z)dz \quad (9)$$

But, from (9),we have

$$\int_0^h pdz = \int_0^h p_0 dz + \rho g\int_0^h (h-z)dz = \rho_0 h + \frac{1}{2}\rho g h^2 \quad (10)$$

$$\int_{\mathbf{0}}^{\boldsymbol{h}} \boldsymbol{p}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{dz} - \int_{\mathbf{0}}^{h} p.(x+dx,z)dz = p_0 h(x+dx) + \frac{1}{2}.\rho g h^2(x) - \frac{1}{2}\rho g h^2.(x+dx)$$

$$\int_{\mathbf{0}}^{\boldsymbol{h}} \boldsymbol{p}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{dz} - \int_{\mathbf{0}}^{h} p.(x+dx,z)dz = -p_0\frac{\partial h}{\partial x}dx - \rho g h\frac{\partial h}{\partial x}dx \qquad (11)$$

Therefore, for stratified deep water the acceleration of the volume is given by

$$m\frac{du}{dt} = F = -\rho g h\frac{\partial h}{\partial x}dx \qquad (12)$$

Now, using our expression $m = \rho h dx$, the preceding equation gives us (cancelling the factors $ph\ dx$)

$$\frac{du}{dt} = -g\frac{\partial h}{\partial x}dx \qquad (13)$$

Our derivative $\frac{d}{dt}$ is the substantial derivative that tells us how the velocity of the deep water under modified gravity and coriolis effect changes volume as it travels around and get stratified. This has to be converted into form that tells how the velocity u changes in a stratified coordinate.

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \qquad (14)$$

And thus to write our equation of motion in final form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} \qquad (15)$$

Like equation (14), the equation (15) connects the local acceleration in two expressions:

i. The pressure slope expression and
ii. The momentum advection.

Two predictive equations with two unknown velocity and height expressed as $\mathrm{u(x,t)}$ and $\mathrm{h(x,t)}$ are obtained from the two equations (14) and (15) which further gives in principle all we need to know to determine how stratified is the deep water and its progress within given initial boundary conditions. The equations have complex properties in general and are always nonlinear even in the presence of its simplicity.

4. Mathematical Formulation of the Problem

The possibility of extending those equations in two dimensions(x,y) with vector velocity $U = (u,v)$ and in order to include rotation therefore we shall add coriolis effect on the equation of momentum, equation (15). Our set of equations will now in vector form become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + fv = -g\frac{\partial h}{\partial x} \quad (16)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} \quad (17)$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad (18)$$

where $fu\ and\ fv\ account\ for\ the\ rotation$ due to coriolis effect.

5. Model Equation

By incorporating the effect of rotation and modified gravity in equations (16), (17) and (18) we obtain our model equations, which are nine set of first order partial differential equations.

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$$\frac{\partial(h_1u_1)}{\partial t} + \frac{\partial\left(h_1u_1^2 + \frac{g\frac{\rho_1-\rho_0}{\rho_0}h_1^2}{2}\right)}{\partial x} = -g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial h_2}{\partial x} - g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x} + fu_1 \quad (19)$$

$$\frac{\partial(h_1u_1)}{\partial t} + = -g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x} - fu_1 \quad (20)$$

$$\frac{\partial h_1}{\partial t} + \frac{\partial(h_1u_1)}{\partial x} = 0 \quad (21)$$

$$\frac{\partial(h_2u_2)}{\partial t} + \frac{\partial\left(h_2u_2^2 + \frac{g\frac{\rho_2-\rho_1}{\rho_1}h_2^2}{2} + g\frac{\rho_2-\rho_1}{\rho_1}h_2h_1\right)}{\partial x} = -g\frac{\rho_2-\rho_1}{\rho_1}h_1\frac{\partial h_2}{\partial x} - g\frac{\rho_2-\rho_1}{\rho_1}h_2\frac{\partial(\xi)}{\partial x} + fu_2 \quad (22)$$

$$\frac{\partial(h_2u_2)}{\partial t} = -g\frac{\rho_2-\rho_1}{\rho_1}h_1\frac{\partial h_2}{\partial y} - g\frac{\rho_2-\rho_1}{\rho_1}h_2\frac{\partial(\xi)}{\partial y} - fu_2 \quad (23)$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial(h_2u_2)}{\partial x} = 0 \quad (24)$$

$$\frac{\partial(h_3u_3)}{\partial t} + \frac{\partial\left(h_3u_3^2 + \frac{g\frac{\rho_3-\rho_2}{\rho_3}h_2^2}{2} + g\frac{\rho_3-\rho_2}{\rho_3}h_3h_2h_1\right)}{\partial x} = -g\frac{\rho_3-\rho_2}{\rho_3}h_1\frac{\partial h_3}{\partial x} - g\frac{\rho_3-\rho_2}{\rho_3}h_3\frac{\partial(\xi)}{\partial x} + fu_3 \quad (25)$$

$$\frac{\partial h_3}{\partial t} + \frac{\partial(h_3u_3)}{\partial x} = 0 \quad (26)$$

$$\frac{\partial(h_3u_3)}{\partial t} = -g\frac{\rho_3-\rho_2}{\rho_2}h_2\frac{\partial h_3}{\partial y} - g\frac{\rho_3-\rho_2}{\rho_2}h_3\frac{\partial(\xi)}{\partial y} - fu_3 \quad (27)$$

Equations (19) – (27) are the model equations that describe the stratification in deep water under modified gravity and coriolis effect (Topman *et al*, 2024).

6. Eigenspace

To determine the eigenspace of the deep water system, we need to find the eigenvalues and eigenvectors of the associated linearized system. The process begins by linearizing the deep water equations about a steady-state solution. This involves taking the partial derivatives of the equations with respect to the variables and setting them equal to zero which we have already obtained. The resulting system of equations is then written in matrix form, with the coefficients of the variables forming the matrix.

Let us just consider stratification in a single layer:

$$\frac{\partial(h_1u_1)}{\partial t}+\frac{\partial(h_1u_1^2+gh_1/2)}{\partial x}=-g\frac{\partial(\xi)}{\partial x}+fu_1 \tag{28}$$

$$\frac{\partial(hv_1)}{\partial t}+\frac{\partial(h_1v_1^2+gh_1/2)}{\partial x}=-g\frac{\partial(\xi)}{\partial y}+fv_1 \tag{29}$$

$$\frac{\partial h_1}{\partial t}+\frac{\partial(h_1u_1)}{\partial x}=0 \tag{30}$$

In conservation form,

$$\frac{\partial q}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=\psi$$

According to LeVeque (2003), q, F and ψ can be further expressed as:

$$q=\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix}=\begin{bmatrix}h_1\\h_1u_1\\h_1v_1\end{bmatrix}, F=\begin{bmatrix}q_2\\\frac{q_2^2}{q_1}+\frac{g^{'}q_1^2}{2}\\\frac{q_3^2}{q_1}+\frac{g^{'}q_1^2}{2}\end{bmatrix}=\begin{bmatrix}h_1u_1\\hu_1^2+\frac{g^{'}h_1^2}{2}\\hv_1^2+\frac{g^{'}h_1^2}{2}\end{bmatrix}, \psi=\begin{bmatrix}0\\-g^{'}\xi_x+fu_1\\-g^{'}\xi_y+fv_1\end{bmatrix} \tag{31}$$

From equation (31), we have that

$$\frac{q_2^2}{q_1}+g^{'}\frac{q_1^2}{2}=\frac{(h_1u_1)^2}{h_1}+g^{'}\frac{h_1^2}{2} \tag{32}$$

$$\Rightarrow h_1u_1^2+g^{'}\frac{h_1^2}{2}$$

And

$$\frac{q_3^2}{q_1}+g^{'}\frac{q_1^2}{2}=\frac{(h_1v_1)^2}{h_1}+g^{'}\frac{h_1^2}{2} \tag{33}$$

$$\Rightarrow h_1v_1^2+g^{'}\frac{h_1^2}{2}$$

Equation (31) contains non-conservative terms which is from definition and that is transforms the equation to the equation of deep water because of density variation.

In vector form, the unknown is $q = [h_1; h_1u_1; h_2u_2, h_3u_3]^T$, which are the vectors of conserved variables. The right hand side vector of source terms ψ contains the effect of densities stratification at transition zone and the effects of Coriolis force.

To calculate the eigenvalues of the deep water equations, we defined the Jacobian matrix of differentiated coefficients of $F(q) = [f_1; f_2; f_3]^T$ as

$F'(q) = \frac{\partial f_1}{\partial q_1}$ for $i, j = 1,2,3$:

$$F'(q) = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \frac{\partial f_1}{\partial q_3} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \frac{\partial f_2}{\partial q_3} \\ \frac{\partial f_3}{\partial q_1} & \frac{\partial f_3}{\partial q_2} & \frac{\partial f_3}{\partial q_3} \end{bmatrix} \tag{34}$$

Where,

$$\frac{\partial f_1}{\partial q_1} = \frac{\partial(h_1u_1)}{\partial h_1} = u_1, \frac{\partial f_1}{\partial q_2} = \frac{\partial(h_1u_1)}{\partial(h_1u_1)} = 1, \frac{\partial f_1}{\partial q_3} = \frac{\partial(h_1u_1)}{\partial(h_1v_1)} = 0 \tag{35}$$

And

$$\frac{\partial f_2}{\partial q_1} = \frac{\partial\left(h_1u_1^2 + g^{'}\frac{h_1^2}{2}\right)}{\partial h_1} = u_1^2 + g^{'}h_1, \frac{\partial f_2}{\partial q_2} = \frac{\partial\left(h_1u_1^2 + g^{'}\frac{h_1^2}{2}\right)}{\partial h_1u_1} = 2u_1, \frac{\partial f_2}{\partial q_3} = \frac{\partial\left(h_1u_1^2 + g^{'}\frac{h_1^2}{2}\right)}{\partial h_1v_1} = 0 \tag{36}$$

Also

$$\frac{\partial f_3}{\partial q_1} = \frac{\partial\left(h_1v_1^2 + g^{'}\frac{h_1^2}{2}\right)}{\partial h_1} = v_1^2 + g^{'}h_1, \frac{\partial f_3}{\partial q_2} = \frac{\partial\left(h_1v_1^2 + g^{'}\frac{h_1^2}{2}\right)}{\partial h_1v_1} = v_1, \frac{\partial f_3}{\partial q_3} = \frac{\partial\left(h_1v_1^2 + g^{'}\frac{h_1^2}{2}\right)}{\partial h_1v_1} = 2v_1. \tag{37}$$

Hence, we have

$$F'(q) = \begin{bmatrix} u_1 & 1 & 0 \\ u_1^2 + g^{'}h_1 & u_1 & 0 \\ v_1^2 + g^{'}h_1 & 0 & v_1 \end{bmatrix} \tag{38}$$

Since the velocity in the y-direction in each of the strata or stratified layers is negligible as the movement is mainly in x-direction then the derivative partially will be zero. It is convenient to substitute the value of $v_1 = v_2 = v_3 = 0$

$$F'(q) = \begin{bmatrix} u_1 & 1 & 0 \\ u_1^2 + g^{'}h_1 & 2u_1 & 0 \\ g^{'}h_1 & 0 & 0 \end{bmatrix} \tag{39}$$

$$F'(q).x = \lambda x \tag{40}$$

That is;

$$(F'(q) - \lambda I)x = 0 \tag{41}$$

The characteristics determinant:

$$\left|F'(q) - \lambda I\right| = \begin{bmatrix} u_1 - \lambda & 1 & 0 \\ u_1^2 + g^{'}h_1 & u_1 - \lambda & 0 \\ g^{'}h_1 & 0 & -\lambda \end{bmatrix} \tag{42}$$

This will give us the characteristic equation:

$$\left|F'(q) - \lambda I\right| = \begin{bmatrix} u_1 - \lambda & 1 & 0 \\ u_1^2 + g^{'}h_1 & u_1 - \lambda & 0 \\ g^{'}h_1 & 0 & -\lambda \end{bmatrix} = 0 \tag{43}$$

$$(u_1 - \lambda)\begin{vmatrix} u_1 - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} - 1\begin{vmatrix} u_1^2 + g^{'}h_1 & 0 \\ g^{'}h_1 & -\lambda \end{vmatrix} + 0\begin{vmatrix} u_1^2 + g^{'}h_1 & u_1 - \lambda \\ g^{'}h_1 & 0 \end{vmatrix} = 0$$

$$(u_1 - \lambda)\{-\lambda(u_1 - \lambda) - 0\} - 1\{-\lambda(u_1^2 + g^{'}h_1) - 0\} + 0\{0 - g^{'}h_1(u_1 - \lambda)\} = 0$$

$$-\lambda\{(u_1 - \lambda)(u_1 - \lambda) - (u_1^2 + g^{'}h_1)\} = 0$$

And $(u_1 - \lambda)^2 - (u_1 + g^{'}h_1) = 0$

We now obtain the following eigenvalues;

$$\left.\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = u_1 + \sqrt{u_1 + g\left(\frac{\rho_1 - \rho_0}{\rho_0}\right)h_1} \\ \lambda_3 = u_1 - \sqrt{u_1 + g\left(\frac{\rho_1 - \rho_0}{\rho_0}\right)h_1} \end{matrix}\right\} \tag{44}$$

Similarly for the second layer, we have

$$\left.\begin{array}{c}\lambda_1 = 0\\ \lambda_2 = u_2 + \sqrt{u_2 + g\left(\frac{\rho_2-\rho_1}{\rho_1}\right)h_2}\\ \lambda_3 = u_2 - \sqrt{u_2 + g\left(\frac{\rho_2-\rho_1}{\rho_1}\right)h_2}\end{array}\right\} \quad (45)$$

And for the third layer, we have

$$\left.\begin{array}{c}\lambda_1 = 0\\ \lambda_2 = u_3 + \sqrt{u_3 + g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3}\\ \lambda_3 = u_3 - \sqrt{u_3 + g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3}\end{array}\right\} \quad (46)$$

Using the Mapple software, the following eigenvectors were produced;

$$\chi_1(q) = \begin{bmatrix} 0\\ \frac{u_1+\sqrt{u_1^2+\frac{g(\rho_1-\rho_0)h_1}{\rho_0}}}{\frac{g(\rho_1-\rho_0)h_1}{\rho_0}}\\ \frac{u_1+\sqrt{u_1^2+\frac{g(\rho_1-\rho_0)h_1}{\rho_0}}}{\frac{g(\rho_1-\rho_0)h_1}{\rho_0}} \end{bmatrix} \quad (47)$$

$$\chi_2(q) = \begin{bmatrix} 0\\ \frac{\left(u_1+\sqrt{u_1^2+\frac{g(\rho_1-\rho_0)h_1}{\rho_0}}\right)\left(\sqrt{u_1^2+\frac{g(\rho_1-\rho_0)h_1}{\rho_0}}\right)}{\frac{g(\rho_1-\rho_0)h_1}{\rho_0}}\\ -\frac{\left(u_1-\sqrt{u_1^2+\frac{g(\rho_1-\rho_0)h_1}{\rho_0}}\right)\left(\sqrt{u_1^2+\frac{g(\rho_1-\rho_0)h_1}{\rho_0}}\right)}{\frac{g(\rho_1-\rho_0)h_1}{\rho_0}} \end{bmatrix} \quad (48)$$

$$\chi_3(q) = \begin{bmatrix} 1\\ 1\\ 1 \end{bmatrix} \quad (49)$$

For the two layers, we have the eigenvectors as;

$$\chi_1(q) = \begin{bmatrix} 0\\ \frac{u_2+\sqrt{u_2^2+\frac{g(\rho_2-\rho_1)h_2}{\rho_1}}}{\frac{g(\rho_2-\rho_1)h_2}{\rho_1}}\\ \frac{u_2+\sqrt{u_2^2+\frac{g(\rho_2-\rho_1)h_2}{\rho_1}}}{\frac{g(\rho_2-\rho_1)h_2}{\rho_1}} \end{bmatrix} \quad (50)$$

$$\chi_2(q) = \begin{bmatrix} 0 \\ \dfrac{\left(u_2 + \sqrt{u_2^2 + \frac{g(\rho_2-\rho_1)h_2}{\rho_0}}\right)\left(\sqrt{u_2^2 + \frac{g(\rho_2-\rho_1)h_2}{\rho_1}}\right)}{\frac{g(\rho_2-\rho_1)h_2}{\rho_1}} \\ -\dfrac{\left(u_2 - \sqrt{u_2^2 + \frac{g(\rho_2-\rho_1)h_2}{\rho_0}}\right)\left(\sqrt{u_1^2 + \frac{g(\rho_2-\rho_1)h_2}{\rho_1}}\right)}{\frac{g(\rho_2-\rho_1)h_2}{\rho_1}} \end{bmatrix} \tag{51}$$

$$\chi_3(q) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{52}$$

And for the three layers, we have the eigenvectors as;

$$\chi_1(q) = \begin{bmatrix} 0 \\ \dfrac{u_3 + \sqrt{u_3^2 + \frac{g(\rho_3-\rho_2)h_3}{\rho_2}}}{\frac{g(\rho_3-\rho_2)h_3}{\rho_2}} \\ \dfrac{u_3 + \sqrt{u_3^2 + \frac{g(\rho_3-\rho_2)h_3}{\rho_2}}}{\frac{g(\rho_3-\rho_2)h_3}{\rho_2}} \end{bmatrix} \tag{53}$$

$$\chi_2(q) = \begin{bmatrix} 0 \\ \dfrac{\left(u_3 + \sqrt{u_3^2 + \frac{g(\rho_3-\rho_2)h_1}{\rho_2}}\right)\left(\sqrt{u_3^2 + \frac{g(\rho_3-\rho_2)h_3}{\rho_2}}\right)}{\frac{g(\rho_3-\rho_2)h_3}{\rho_2}} \\ -\dfrac{\left(u_3 - \sqrt{u_3^2 + \frac{g(\rho_3-\rho_2)h_3}{\rho_2}}\right)\left(\sqrt{u_3^2 + \frac{g(\rho_3-\rho_2)h_3}{\rho_2}}\right)}{\frac{g(\rho_3-\rho_2)h_3}{\rho_2}} \end{bmatrix} \tag{54}$$

$$\chi_3(q) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{55}$$

Since this system has two spatial dimensions, the waves (solutions) move in horizontal direction indicating that the vertical amplitude of the wave is equal to the horizontal amplitude for the orbital regime of deep water stratification which is given by the unit normal vector n where $n(n_x)$. According to Brufau & Garcia-Navarro (2000) and LeVeque (2003), the eigenvalues of the entire system are found by solving.

$$\left| n_x F'(q) - \lambda_i I \right| = 0 \; for \; i = 1,2,3 \tag{56}$$

The corresponding quasi-linear coefficient matrices obtained above in two dimensional equations can be extended in similar way. We now define the quantities;

$$q = \begin{bmatrix} h_1 \\ h_1u_1 \\ h_1v_1 \\ h_2 \\ h_2u_2 \\ h_2v_2 \\ h_3 \\ h_3u_3 \\ h_3v_3 \end{bmatrix} \qquad S(q) = \begin{bmatrix} 0 \\ -g^{'}\frac{\partial\xi}{\partial x} + fu_1 \\ -g^{'}\frac{\partial\xi}{\partial x} + fv_1 \\ 0 \\ -g^{'}\frac{\partial\xi}{\partial x} + fu_2 \\ -g^{'}\frac{\partial\xi}{\partial x} + fv_2 \\ 0 \\ -g^{'}\frac{\partial\xi}{\partial x} + fu_3 \\ -g^{'}\frac{\partial\xi}{\partial x} + fv_3 \end{bmatrix} \tag{57}$$

$$f(q) = \begin{bmatrix} h_1u_1 \\ h_1u_1^2 + \frac{g^{'}h_1^2}{2} \\ hv_1^2 + \frac{g^{'}h_1^2}{2} \\ h_2u_2 \\ h_2u_2^2 + \frac{g^{'}h_2^2}{2} \\ h_2v_2^2 + \frac{g^{'}h_2^2}{2} \\ h_3u_3 \\ h_3u_3^2 + \frac{g^{'}h_3^2}{2} \\ h_3v_3^2 + \frac{g^{'}h_3^2}{2} \end{bmatrix}$$

We can now calculate the flux Jacobians of $f(q)$ as;

$$f^{'}(q) = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \frac{\partial f_1}{\partial q_3} & \frac{\partial f_1}{\partial q_4} & \frac{\partial f_1}{\partial q_5} & \frac{\partial f_1}{\partial q_6} & \frac{\partial f_1}{\partial q_7} & \frac{\partial f_1}{\partial q_8} & \frac{\partial f_1}{\partial q_9} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \frac{\partial f_2}{\partial q_3} & \frac{\partial f_2}{\partial q_4} & \frac{\partial f_2}{\partial q_5} & \frac{\partial f_2}{\partial q_6} & \frac{\partial f_2}{\partial q_7} & \frac{\partial f_2}{\partial q_8} & \frac{\partial f_2}{\partial q_9} \\ \frac{\partial f_3}{\partial q_1} & \frac{\partial f_3}{\partial q_2} & \frac{\partial f_3}{\partial q_3} & \frac{\partial f_3}{\partial q_4} & \frac{\partial f_3}{\partial q_5} & \frac{\partial f_3}{\partial q_6} & \frac{\partial f_3}{\partial q_7} & \frac{\partial f_3}{\partial q_8} & \frac{\partial f_3}{\partial q_9} \\ \frac{\partial f_4}{\partial q_1} & \frac{\partial f_4}{\partial q_2} & \frac{\partial f_4}{\partial q_3} & \frac{\partial f_4}{\partial q_4} & \frac{\partial f_4}{\partial q_5} & \frac{\partial f_4}{\partial q_6} & \frac{\partial f_4}{\partial q_7} & \frac{\partial f_4}{\partial q_8} & \frac{\partial f_4}{\partial q_9} \\ \frac{\partial f_5}{\partial q_1} & \frac{\partial f_5}{\partial q_2} & \frac{\partial f_5}{\partial q_3} & \frac{\partial f_5}{\partial q_4} & \frac{\partial f_5}{\partial q_5} & \frac{\partial f_5}{\partial q_6} & \frac{\partial f_5}{\partial q_7} & \frac{\partial f_5}{\partial q_8} & \frac{\partial f_5}{\partial q_9} \\ \frac{\partial f_6}{\partial q_1} & \frac{\partial f_6}{\partial q_2} & \frac{\partial f_6}{\partial q_3} & \frac{\partial f_6}{\partial q_4} & \frac{\partial f_6}{\partial q_5} & \frac{\partial f_6}{\partial q_6} & \frac{\partial f_6}{\partial q_7} & \frac{\partial f_6}{\partial q_8} & \frac{\partial f_6}{\partial q_9} \\ \frac{\partial f_7}{\partial q_1} & \frac{\partial f_7}{\partial q_2} & \frac{\partial f_7}{\partial q_3} & \frac{\partial f_7}{\partial q_4} & \frac{\partial f_7}{\partial q_5} & \frac{\partial f_7}{\partial q_6} & \frac{\partial f_7}{\partial q_7} & \frac{\partial f_7}{\partial q_8} & \frac{\partial f_7}{\partial q_9} \\ \frac{\partial f_8}{\partial q_1} & \frac{\partial f_8}{\partial q_2} & \frac{\partial f_8}{\partial q_3} & \frac{\partial f_8}{\partial q_4} & \frac{\partial f_8}{\partial q_5} & \frac{\partial f_8}{\partial q_6} & \frac{\partial f_8}{\partial q_7} & \frac{\partial f_8}{\partial q_8} & \frac{\partial f_8}{\partial q_9} \\ \frac{\partial f_9}{\partial q_1} & \frac{\partial f_9}{\partial q_2} & \frac{\partial f_9}{\partial q_3} & \frac{\partial f_9}{\partial q_4} & \frac{\partial f_9}{\partial q_5} & \frac{\partial f_9}{\partial q_6} & \frac{\partial f_9}{\partial q_7} & \frac{\partial f_9}{\partial q_8} & \frac{\partial f_9}{\partial q_9} \end{bmatrix} \quad (58)$$

We can write the flux Jacobians of $f(q)$ as;

$$f'(q) = \begin{bmatrix} u_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1^2 + g^{'}h_1 & u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1^2 & 0 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2^2 + g^{'}h_2 & u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2^2 + g^{'}h_2 & 0 & v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_3^2 + g^{'}h_3 & u_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_3^2 + g^{'}h_3 & 0 & v_3 \end{bmatrix} \quad (59)$$

Where $v_1 = v_2 = v_3 = 0$, then we have;

$$f'(q) = \begin{bmatrix} u_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1^2 + g^{'}h_1 & u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2^2 + g^{'}h_2 & u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g^{'}h_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_3^2 + g^{'}h_3 & u_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g^{'}h_3 & 0 & 0 \end{bmatrix} \quad (60)$$

If we write the source term of the quasi-linear coefficient matrices of the original equations as

$$\frac{\partial q}{\partial t} + A(q)\frac{\partial q}{\partial x} + B(q)\frac{\partial q}{\partial y} = S(q)$$

But the variation of velocity along y-direction is neglected, therefore we have

$$\frac{\partial q}{\partial t} + A(q)\frac{\partial q}{\partial x} = S(q)$$

$$\begin{bmatrix} u_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1^2 + g^{'}h_1 & u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2^2 + g^{'}h_2 & u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g^{'}h_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_3^2 + g^{'}h_3 & u_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g^{'}h_3 & 0 & 0 \end{bmatrix}$$

And $$\boldsymbol{S}(q) = \begin{bmatrix} 0 \\ -g^{'}\frac{\partial \xi}{\partial x} + fu_1 \\ -g^{'}\frac{\partial \xi}{\partial x} + fv_1 \\ 0 \\ -g^{'}\frac{\partial \xi}{\partial x} + fu_2 \\ -g^{'}\frac{\partial \xi}{\partial x} + fv_2 \\ 0 \\ -g^{'}\frac{\partial \xi}{\partial x} + fu_3 \\ -g^{'}\frac{\partial \xi}{\partial x} + fv_3 \end{bmatrix} \quad (61)$$

Eigenvalues

$$\left.\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= u_3 + \sqrt{u_3^2 + g\left(\frac{\rho_3 - \rho_2}{\rho_2}\right) h_3} \\
\lambda_5 &= u_3 - \sqrt{u_3^2 + g\left(\frac{\rho_3 - \rho_2}{\rho_2}\right) h_3} \\
\lambda_6 &= u_2 + \sqrt{u_2^2 + g\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) h_2} \\
\lambda_7 &= u_2 - \sqrt{u_2^2 + g\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) h_2} \\
\lambda_8 &= u_1 + \sqrt{u_1^2 + g\left(\frac{\rho_1 - \rho_0}{\rho_0}\right) h_1} \\
\lambda_9 &= u_1 - \sqrt{u_1^2 + g\left(\frac{\rho_1 - \rho_0}{\rho_0}\right) h_1}
\end{aligned}\right\} \tag{62}$$

Our Corresponding Eigenvectors are;

$$\chi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{u_2 + \sqrt{u_2^2 + g\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) h_2}}{g\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) h_2} \\ \dfrac{\left(u_2 + \sqrt{u_2^2 + g\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) h_2}\right)\left(\sqrt{u_2^2 + g\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) h_2}\right)}{g\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) h_2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{63}$$

$$\chi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{u_2 - \sqrt{u_2^2 + g\left(\frac{\rho_2-\rho_1}{\rho_1}\right)h_2}}{g\left(\frac{\rho_2-\rho_1}{\rho_1}\right)h_2} \\ -\dfrac{\left(u_2 - \sqrt{u_2^2 + g\left(\frac{\rho_2-\rho_1}{\rho_1}\right)h_2}\right)\left(\sqrt{u_2^2 + g\left(\frac{\rho_2-\rho_1}{\rho_1}\right)h_2}\right)}{g\left(\frac{\rho_2-\rho_1}{\rho_1}\right)h_2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{64}$$

$$\chi_3 = \begin{bmatrix} \dfrac{1}{\sqrt{u_1^2 + g\left(\frac{\rho_1-\rho_0}{\rho_0}\right)h_1}} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \chi_4 = \begin{bmatrix} \dfrac{-1}{\sqrt{u_1^2 + g\left(\frac{\rho_1-\rho_0}{\rho_0}\right)h_1}} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \chi_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{65}$$

$$\chi_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \chi_7 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \chi_8 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{u_3 + \sqrt{u_3^2 + g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3}}{g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3} \\ \dfrac{\left(u_3 + \sqrt{u_3^2 + g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3}\right)\left(\sqrt{u_3^2 + g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3}\right)}{g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3} \\ 1 \end{bmatrix} \tag{66}$$

$$\chi_9 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{-\left(u_3+\sqrt{u_3^2+g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3}\right)\left(\sqrt{u_3^2+g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3}\right)}{g\left(\frac{\rho_3-\rho_2}{\rho_2}\right)h_3} \\ 1 \end{bmatrix} \tag{67}$$

The eigenvalues and eigenvectors obtained show the impacts of gravity modification and Coriolis effect on stratified deep water regime. Zero and one entry in eigenspace show the eigenvalues of the system's governing equations are stable. These eigenvalues represent the growth or decay rates of the system's modes of variability, such as waves or currents as shown in figure 1. Specifically, zero indicates a neutral mode as equally demonstrated in our numerical simulation, meaning the amplitude of the mode remains constant over time, while one indicates an unstable mode, where the amplitude of the mode grows exponentially. Equations (19 – 27) which define our model, demonstrate stability and validity for the study of stratified deep water under gravity modification and Coriolis effect, *(Topman & Mbah, 2024).*

7.0 Result and Conclusion.

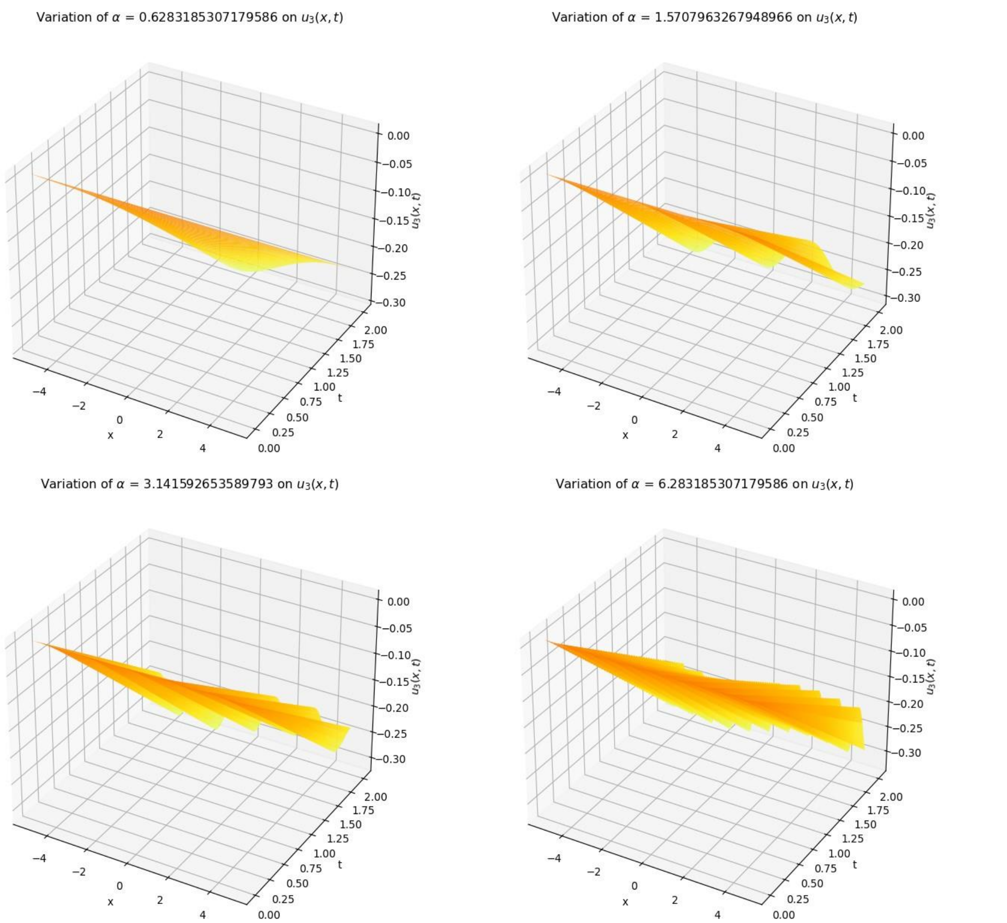

Figure 4. The strength of measurement and its variation on stratified flow

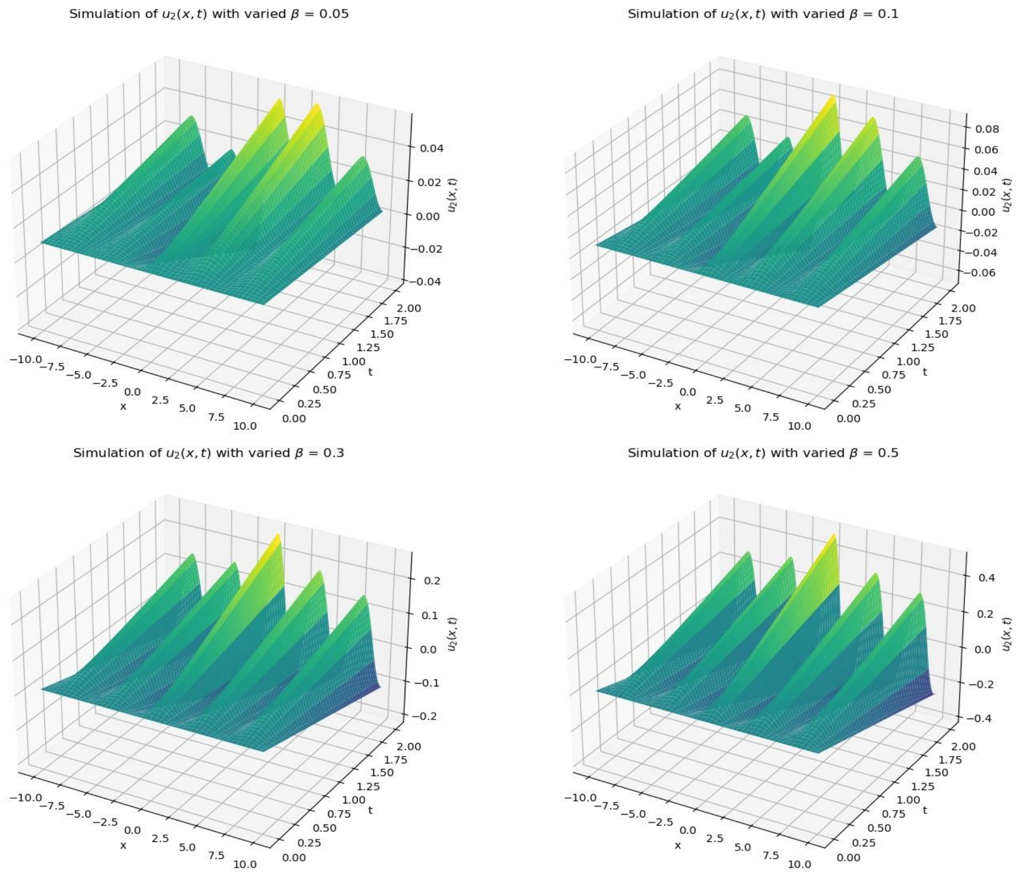

Figure 5. The simulation of speed at second layer with varied values of the measure of stability

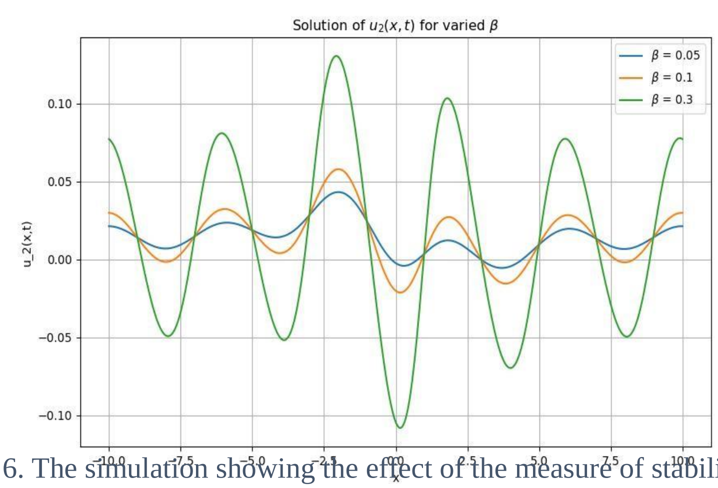

Figure 6. The simulation showing the effect of the measure of stability on amplitude.

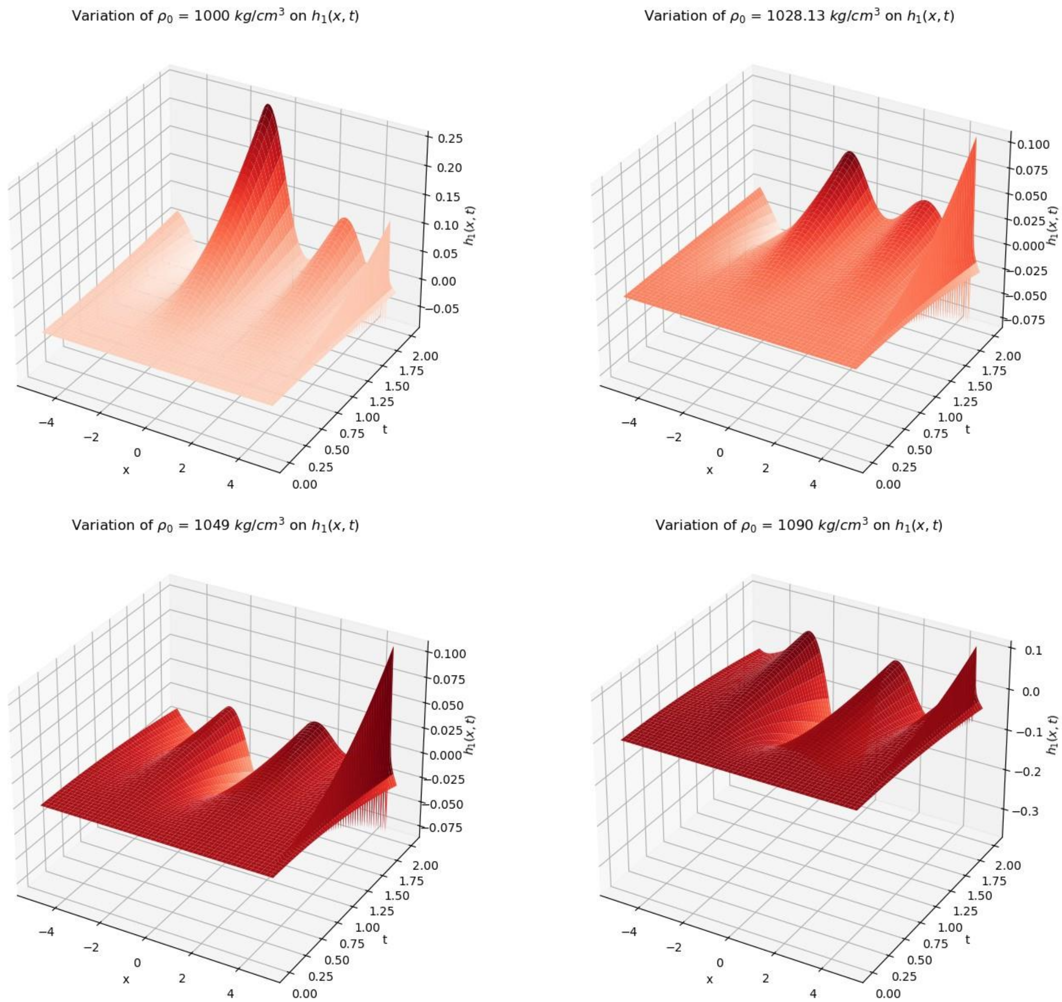

Figure 7. Variation in densities with height

8. Discussion of Results

In Figure 4, the measure of strength alpha (α) shows significant changes against the speed at various values. When $\alpha = 0.62831853071795\text{86}$ the velocity of the stratified flow was relatively uniform approximately 0.25m/s. when the α =1.5707963267948966, the speed is between 0.25m/s to 0.30m/s and there is more layering of deep water masses. At α=3.141592653589793, there is greater number of stratified columns indicating higher instability of deep water. When α=6.283185307179586, the number of stratified columns equally increased, which suggest that the measure of strength α contribute greatly in deep water stratification and stability.

In Figure 5, the stability parameter(β) has a significant effect on deep-water stability. When β=0.05, 0.1, 0.3 and 0.5 the stratification occurs at the speed of 0.04m/s, 0.06m/s, 0.2m/s and 0.4m/s.

The changes in speed lead to stratification which results to instability of the deep water system.

Figure 6, the effect of beta (β) which is the measure of the stability of the system shows the variation in amplitudes of deep water. At β=0.05, the system is at equilibrium position and the amplitude is relatively stable (0,0). At β=0.1, the phase angle is $\pi/_2$ $.When$ β=0.3, the amplitude of oscillation acquired a critical value which represent highest displacement.

Figure 7, shows how density plays crucial role in eigenspace of stratified flow under gravity modification and coriolis effect. Density of deep water varies with depth, and this variation creates a stratified structure in the water column. The density of the stratified flow affects eigenvalues of the eigenfunctions and that determines the growth or decay rate of the corresponding eigenmodes.

9. Conclusion

The eigenspace of stratified flow under gravity modification and Coriolis effect has the potential to provide significant advantages in understanding and predicting the stability of flow regime under gravity modification and coriolis effect. Some of the potential advantages of our model include:

The model can improve our understanding of ocean dynamics: By studying the behavior of stratified deep water under modified gravity and Coriolis effect, researchers can gain a better understanding of the complex dynamics of the ocean. This knowledge can be used to improve ocean models, which are essential for predicting ocean currents, waves, and other phenomena that can impact weather patterns, marine ecosystems, and human activities.

The model provides an enhance understanding of stratified deep water environments which are often rich in oil and gas resources. By studying the behavior of fluids in these environments under modified gravity and Coriolis effect, we can develop new techniques for oil and gas exploration and extraction, leading to more efficient and cost-effective methods for extracting these valuable resources.

The model provides a better understanding of the climate, and the behavior of stratified deep water under modified gravity and Coriolis effect captured in this model provides a significant insights into global oceanic patterns for more accurate predictions of future climate change and help policymakers develop strategies to mitigate its impacts.

The model helps in advancement of underwater robotics. The study of eigenspace of stratified deep water under modified gravity and Coriolis effect, lead to advancements in underwater robotics. Thereby enhancing our understanding in developing robots that can operate in these environments.

In conclusion, this model provides researchers with opportunities to explore and study the ocean in ways that were previously impossible, leading to new discoveries and insights into the ocean's depths and its stability.

List of Symbols

λ	Wavelength
$u = (u, v, w)$	The velocity vector
ρ	The density of flow
p	The constant pressure
g	The gravitational constant
g'	The gravity modification
f	The effect of coriolis
Ω	The angular rotation
u	Velocity in the x-direction
v	The constant velocity in the y-direction
L	The length scale
R_0	The Rossby number of the flow
ρ_0, T_0, p_0	Are reference values of density, temperature and Salinity respectively
h	Vertical length scale
H^*	Vertical height of deep water at thermocline
ς	Free surface elevation

x	x direction
y	y direction
z	z direction
t	The required time
$\frac{D}{Dt}$	The total material derivative
$h(x,y,t)$	The height of water surface from the same reference height
$\xi(x,y)$	Denotes the thermocline regime
H	Deep water dept
h	The water height above each stratified column
δx	Width in the $x-$ direction
δy	Width in the $y-$ direction
u_1	Velocity in the first layer in the $x-$ direction
u_2	Velocity in the second layer in the $x-$ direction
v_1	Velocity in the first layer in the $y-$ direction
v_2	Velocity in the second layer in the $y-$ direction
α	Measure of strength of the system
β	Measure of stability of the system
F	Sum of all forces
m	Mass
a	Acceleration of the block of water
K	Wave number

Conflict of Interest: The authors declare that there is no conflict of interest.

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