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Dynamic Response of Clamped-Clamped Uniform Bernoulli-Euler Beam Resting on a Pasternak Foundation Subjected to Concentrated Moving Load with Damping Term

Atiti, S. Ehidiame[a*], Momoh, S. Omeiza[a], Gyegwe, M. Jessica[a], and Edogbanya, O. Helen[a]

[a]Department of Mathematics, Federal University Lokoja, Kogi State, Nigeria.

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ABSTRACT

The dynamic response of a clamped-clamped uniform Bernoulli–Euler beam resting on a Pasternak foundation under the action of a concentrated moving load with a damping term was investigated in this paper. To solve the governing equation, the Dirac, delta function was expressed as a Fourier cosine series and the generalized finite integral transform, the Struble's asymptotic technique and the Laplace transform method were used. Thereafter, the clamped-clamped support condition was used to obtain the dynamic response of the beam. The study showed that, the moving force problem is structurally unsafe to approximate the moving mass problem in the design of the dynamical system. In addition, as the values of the damping term, axial force, foundation modulus and shear modulus increases, the deflection profiles of the beam decreases. This implies that the beam's safety and performance are ensured when the values of each parameter are increased. Finally, the results showed that the damping term had a far higher effect on the beam's deflection.

1. Introduction

The study of dynamic responses of structural members modelled as beams, like railway tracks, bridges, runways, roadways resting on elastic foundation subjected to moving loads like trains and vehicles is important by virtue of the effect these loads have on them. Usually, the movement of these loads induces vibrations which causes corrosion between contacting elements, impairs the function and life of the structure or its components, and their final failures. Therefore, the moving load problem has continued to attract researchers in the fields of applied mathematics, mechanical engineering and applied physics to provide insights into the design of dynamical structures that will ensure their safety, longevity and efficient performance. One of the earliest works in available literature on the dynamic response of beams to moving load is Stokes (1849), who obtained an approximate solution for the response of a beam by neglecting the mass of the beam. Reported thereafter was Willis, *et al.* (1851), who in their study, considered the mass of the beam to be much smaller than the mass of the moving load. Much later, Krylo (1905) investigated the dynamic response of a simply supported beam and assumed that the mass of the load was smaller than that

* Corresponding author. Tel.: +2348077026241
E-mail address: pastoratiti@gmail.com (Atiti, S. Ehidiame)
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of the beam. Others are Timoshenko (1921), Inglis (1934), Lee and Ng (1996), Oni and Awodola (2005), Omolofe, *et al.* (2009) and Misra (2012).

The different studies listed so far gave useful results but, they all rely on the one parameter Winkler foundation which however, has a known weakness. The Winkler soil model assumes that the displacement appears only in the loaded zone (Dobromir, 2012). Outside this, zone the deflections are zero. To overcome this weakness, a more realistic foundation model, known as the bi-parametric foundation model, in particular, Pasternak (Pasternak, 1954) improved on the Winkler model by adding a shear spring to simulate the interactions between separate springs in the Winkler model. For this model, in addition to the foundation stiffness$K$, a second foundation constant, the shear modulus G enters the analysis. Due to this advantage, Jitendra (2014), Jimoh and Ajoge (2020), Abbas, *et al.* (2021), Akhazhanov, *et al.* (2023) and Awodola, *et al.* (2024) used the Pasternak foundation to model their studies and got interesting results.

In the above works, researchers incorporated different parameters and conditions in the study of the dynamical system they considered. The terms and conditions they added were geared towards areas that captured their interest. The damping term in a dynamical system for instance, represents the energy dissipation mechanisms that occur within it. According to Oguntala and Sobamowo (2015), the mechanism of damping as a means of controlling undesirable effect of vibration and noise has received considerable research attention and can significantly affect the beam's dynamic behavior. In most researches however, the damping term was neglected [Oni and Awodola (2005); Mirzabeigy and Madoliat (2016); Abbas, *et al.* (2021); Olotu, *et al.* (2021); Akhazhanov, *et al.* (2023)]. Although fewer researchers considered the case of damping in their work, Adedowole and Adekunle (2018) and Jimoh and Ajoge (2020) incorporated damping. In this study, the damping term is considered.

In addition, some researchers did not consider the inertia effect of the moving load in modelling the dynamical system, but concentrated on the force effect of the moving load. Although the problem on this assumption has been greatly simplified and as such commonly treated in literature (Omolofe, 2010), the question is, how safe is a design based on this assumption? If it were a safe approximation, the more cumbersome problem of solving the moving mass would have been considered unnecessary. Thus, incorporating the inertia effect of the moving load is a more comprehensive representation of the dynamic system as it simultaneously considers the moving force and the more complex moving mass cases. Studies on the force effect alone of the moving load include Jitendra (2014), Ogunyebi, *et al.* (2015), Mustafa and Metin (2022), Wstawska, *et al.* (2022) and Beza (2023). However, researchers such as Ojih, *et al.* (2013), Oni and Ayankop-Andi (2017), Adeoye and Awodola (2018), Oni and Ogunyebi (2018) and Awodola, *et al.* (2024), considered the inertia effect of the moving load in their works and got interesting results which showed that the moving force solution is not an upper bound for an accurate solution of the moving mass problem for the dynamical system they considered. In this study, the inertia effect of the moving load is considered.

Whichever the cases considered in the studies above, where researchers added a damping term, the inertia effect of the moving load was not considered. In contrast, when the inertia effect was included, the damping term was not. However, studies that incorporate both the damping term and the inertia effect of the load are rare in literature, Jiya and Shaba (2018) and Sulaiman, *et al.* (2024) undertook such studies and got useful results by applying simply supported boundary conditions, while that of the clamped-clamped boundary condition was neglected.

From the works reported so far and to the authors' knowledge from available literature, the case of a clamped-clamped uniform Bernoulli-Euler resting on a Pasternak foundation under the action of a concentrated moving load with damping term, where the initial effects of the load is incorporated in its analysis has not been considered. In this study therefore, the case of the clamped-clamped condition is considered to provide valuable insights for designing and optimizing such dynamical systems. Thus, this paper investigated a clamped-clamped supported uniform Bernoulli-Euler beam resting on a Pasternak foundation subjected to a concentrated moving load with damping term, when the inertia effect of the moving load is considered.

2. Methods

2.1 Formulation of the Problem

The governing equation for a uniform Bernoulli Euler beam resting on constant Pasternak foundation subjected to concentrated moving load with damping term is given as:

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} + \bar{m}\frac{\partial^2 W(x,t)}{\partial t^2} + \alpha\frac{\partial W(x,t)}{\partial t} - N\frac{\partial^2 W(x,t)}{\partial x^2} + F_p(x,t) = P(x,t) \tag{1}$$

where x is the spatial coordinate, t is the time, $W(x,t)$ is the transverse displacement, E is the Young's modulus, I is the moment of inertia, EI is the flexural rigidity of the structure, $\bar{m}$ is the mass per unit length of the beam, α is the damping term coefficient, N is the axial force, $P(x,t)$is the transverse concentrated load and $F_p(x,t)$ is the foundation reaction. The boundary conditions of the structure are arbitrary while the initial condition is given as:

$$W(x,0) = 0 = \frac{\partial W(x,t)}{\partial t} \tag{2}$$

The foundation reaction was given by Fryba (1972) as:

$$F_p(x,t) = KW(x,t) - G\frac{\partial^2 W(x,t)}{\partial x^2} \tag{3}$$

where K is the foundation modulus and G is the shear modulus.

If the inertia effect of the moving load is considered, the load $P(x,t)$ takes the form:

$$P(x,t) = P_f(x,t)\left[1 - \frac{1}{g}\frac{d^2 W(x,t)}{dt^2}\right] \tag{4}$$

where the continuous moving force, $P_f(x,t)$ acting on the beam model is given as:

$$P_f(x,t) = mg\delta(x - ct) \tag{5}$$

where m and c are the mass and the speed of the moving load respectively, g is the acceleration due to gravity, $mg\delta(x - ct)$, is the continuos moving force acting on the beam and $\frac{d^2}{dt^2}$ is a convective acceleration given by Fryba (1972) as

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x \partial t} + c^2\frac{\partial^2}{\partial x^2} \tag{6}$$

when the operator, $\frac{d^2}{dt^2}$ acts as the transverse deflection $W(x,t)$ of the beam, the first term in the right-hand side of the equation (6), measures the effect of the acceleration on the deflection, the second term measures the effect of complementary acceleration (coriohs force) and the third term measures the effect of the path curvature (centripetal force).

Using equations (3), (4), (5) and (6) in equation (1), after simplifications, we obtain:

$$\frac{EI}{\bar{m}}\frac{\partial^4 W(x,t)}{\partial x^4} + \frac{\partial^2 W(x,t)}{\partial t^2} - \frac{N}{\bar{m}}\frac{\partial^2 W(x,t)}{\partial x^2} - \frac{G}{\bar{m}}\frac{\partial^2 W(x,t)}{\partial x^2} + \frac{\alpha}{\bar{m}}\frac{\partial W(x,t)}{\partial t} + \frac{k}{\bar{m}}W(x,t) +$$

$$\frac{m}{\bar{m}}\delta(x-ct)\left\{\frac{\partial^2 W(x,t)}{\partial t^2} + 2c\frac{\partial^2 W(x,t)}{\partial x \partial t} + c^2\frac{\partial^2 W(x,t)}{\partial x^2}\right\} = \frac{mg}{\bar{m}}\delta(x-ct) \tag{7}$$

where

$\delta(x-ct)$ represents the dira delta function defined as

$$\delta(x-ct) = \begin{cases} 0, x \neq ct \\ \infty, x = ct \end{cases} \tag{8}$$

with property

$$\int_0^L \delta\,(x-ct)f(x)dx = \begin{Bmatrix} 0; ct < 0 \\ f(x); 0 < ct < L \\ 0; ct > L \end{Bmatrix} \tag{9}$$

Equation (7) is the simplified governing equation for uniform Bernoulli Euler beam resting on constant Pasternak foundation subjected to concentrated moving load with damping term, when the initial effect of the moving load is put into consideration.

2.2. Solution to the Problem

To solve equation (7), a general approach is resorted to in order to, which involves expressing the Dirac delta function as a Fourier cosine series and then reducing the modified form of the 4$^{\text{th}}$ order PDE above using the generalized finite integral transform.

the generalized finite integral transform is defined by

$$W(m,t) = \int_0^L W(x,t)U_m(x)dx \tag{10}$$

with the inverse

$$W(x,t) = \sum_{m=1}^{\infty} \frac{\bar{m}}{W_m} W(m,t)U_m(x) \tag{11}$$

where $W_m = \int_0^L \bar{m}\ U_m^2(x)dx$ (12)

and $U_m(x)$ is any function chosen such that the pertinent boundary condition satisfied. Thus, the mth normal mode of vibration of a uniform beam

$$U_m(x) = \text{Sin}\frac{\lambda_m x}{L} + A_m Cos\frac{\lambda_m x}{L} + B_m Sinh\frac{\lambda_m x}{L} + C_m Cosh\frac{\lambda_m x}{L} \tag{13}$$

is chosen as a suitable kernel of the integral transform (10) where λ_m is the node frequency, $A_m, B_m\ and\ C_m$ are constants which are obtained by substituting the boundary condition into (13).

By applying the generalized finite integral transform (10) on equation (7), one obtains

$$\frac{EI}{\bar{m}}\int_o^L \frac{\partial^4 W(x,t)}{\partial x^4}U_m(x)dx + \int_o^L \frac{\partial^2 W(x,t)}{\partial t^2}U_m(x)dx - \frac{N}{\bar{m}}\int_o^L \frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx - \frac{G}{\bar{m}}\int_o^L \frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx + \frac{\alpha}{\bar{m}}\int_o^L \frac{\partial W(x,t)}{\partial t}U_m(x)dx + \frac{k}{\bar{m}}\int_o^L W(x,t)U_m(x)dx - \frac{M}{\bar{m}}\int_o^L \delta(x-ct)\frac{\partial^2 W(x,t)}{\partial t^2}U_m(x)dx - \frac{2Mc}{\bar{m}}\int_o^L \delta(x-ct)\frac{\partial^2 W(x,t)}{\partial x\partial t}U_m(x)dx + \frac{Mc^2}{\bar{m}}\int_o^L \delta(x-ct)\frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx = \frac{Mg}{\bar{m}}\int_o^l \delta(x-ct)U_m(x)dx \tag{14}$$

we use the property of the Dirac delta function as an even function to express the above term in Fourier cosine series namely

$$\delta(x-ct) = \frac{1}{L} + \frac{2}{L}\sum_{n-1}^{\infty}\cos\frac{n\pi ct}{L}\cos\frac{n\pi x}{l} \tag{15}$$

Using (15) on (14) and after some simplification and rearrangements, (14) becomes

$$W_{tt}(m,t) + \frac{\alpha}{\bar{m}}W_t(m,t) + \left[\frac{EI}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^4 + \frac{N}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^2 + \frac{G}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^2 + \frac{K}{\bar{m}}\right]W(m,t) + \frac{EI}{\bar{m}}E_1(x) -$$

$\left(\frac{N+G}{\bar{m}}\right)E_2(x)$ - $\frac{M}{\bar{m}L}U_1(m,t)W_{tt}(m,t) - \frac{M}{\bar{m}L}P_1(k,m,t)W_{tt}(k,t) - \frac{2Mc}{\bar{m}L}U_2(m,t)W_t(m,t) - \frac{2Mc}{\bar{m}L}P_2(k,m,t)W_t(k,t) - \frac{Mc^2}{\bar{m}L}U_3(m,t)W(m,t) - \frac{Mc^2}{\bar{m}L}P_3(k,m,t) = \frac{Mg}{\bar{m}}\left[\sin\frac{\lambda_m ct}{L} + A_m\cos\frac{\lambda_m ct}{L} + B_m\sinh\frac{\lambda_m ct}{L} + C_m\cosh\frac{\lambda_m ct}{L}\right]$ (16)

where

$$U_1(m,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[H_a(m,n) + 2H_c(m,n)\cos\frac{n\pi ct}{L}\right] \tag{17}$$

$$U_2(m,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[H_e(m,n) + 2H_g(m,n)\cos\frac{n\pi ct}{L}\right] \tag{18}$$

$$U_3(m,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[H_i(m,n) + 2H_k(m,n)\cos\frac{n\pi ct}{L}\right] \tag{19}$$

$$P_1(k,m,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[H_b(k,m,n) + 2H_d(k,m,n)\cos\frac{n\pi ct}{L}\right] \tag{20}$$

$$P_2(k,m,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[H_f(k,m,n) + 2H_L(k,m,n)\cos\frac{n\pi ct}{L}\right] \tag{21}$$

$$P_3(k,m,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[H_j(k,m,n) + 2H_L(k,m,n)\cos\frac{n\pi ct}{L}\right] \tag{22}$$

Rearranging (16)

$$W_{tt}(m,t) + \frac{\alpha}{\bar{m}}W_t(m,t) + \left[\frac{EI}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^4 + \left(\frac{N+G}{\bar{m}}\right)\left(\frac{\lambda_m}{L}\right)^4 + \frac{K}{\bar{m}}\right]W(m,t) + \frac{EI}{\bar{m}}E_1(x) - \left(\frac{N+G}{\bar{m}}\right)E_2(x) - \varepsilon_0[U_1(m,t)W_{tt}(m,t) + P_1(k,m,t)W_{tt}(k,t) + 2cU_2(m,t)W_t(m,t) + 2cP_2(k,m,t)W_t(k,t) + c^2U_3(m,t)W(m,t) + c^2P_3(k,m,t)W(k,t)]$$

$$= \frac{Mg}{\bar{m}}\left[\sin\frac{\lambda_m ct}{L} + A_m \cos\frac{\lambda_m ct}{L} + B_m \sinh\frac{\lambda_m ct}{L} + C_m \cosh\frac{\lambda_m ct}{L}\right] \tag{23}$$

where,

$$E_1(x) = E_1(x) = \left[\frac{\partial^3}{\partial x^3}W(x,t)U_m(x) - \frac{\partial^2}{\partial x^2}W(x,t)\frac{dU_m(x)}{dx} + \frac{\partial}{\partial x}w(x,t)\frac{d^2}{dx^2}U_m(x) - W(x,t)\frac{d^3}{dx^3}U_m(x)\right]_0^L \tag{24}$$

$$E_2(x) = \left[\frac{\partial}{\partial x}W(x,t)U_m(x) - W(x,t)\frac{d}{dx}U_m(x)\right]_0^L \tag{25}$$

$$\varepsilon_0 = \frac{M}{\bar{m}L} \tag{26}$$

Rearranging (23), it becomes

$$W_{tt}(m,t) + A_1W_t(m,t) + A_2W(m,t) + A_3(x) - \varepsilon_0[U_1(m,t)W_{tt}(m,t) + 2cU_2(m,t)W_t(m,t) + c^2U_3(m,t)w(m,t)] - \varepsilon_0[P_1(k,m,t)W_{tt}(k,t) + 2cP_2(k,m,t)W_t(k,t) + c^2P_3(k,m,t)W(k,t)] = \frac{Mg}{\bar{m}}\left[\sin\frac{\lambda_m ct}{L} + A_m\cos\frac{\lambda_m ct}{L} + B_m\sinh\frac{\lambda_m ct}{L} + C_m\cosh\frac{\lambda_m ct}{L}\right] \tag{27}$$

where $A_1 = \frac{\alpha}{\bar{m}}$ (28)

$$A_2 = \frac{EI}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^4 + \left(\frac{N+G}{\bar{m}}\right)\left(\frac{\lambda_m}{L}\right)^4 + \frac{k}{\bar{m}} \tag{29}$$

$$A_3(x) = \left[\frac{EI}{\bar{m}}E_1(x) - \left(\frac{N+G}{\bar{m}}\right)E_2(x)\right]_0^L \tag{30}$$

Equation (27) is the transformed equation governing the problem. This non-homogeneous 2nd order ODE holds for all variants of the classical Boundary Conditions.

Case 1: Uniform Bernoulli Euler Beam Traversed by Concentrated Moving Force

An approximate model of the differential equation describing the response of a uniform

Bernoulli Euler beam resting on elastic Pasternak foundation under the action of a concentrated moving force is obtained when we assume that the inertia effect of the moving mass is negligible.

Thus, $A_3(x) = 0$ and setting $\varepsilon_0 = 0$, (27) reduces to:

$$W_{tt}(m,t) + A_1 W_t(m,t) + A_2 W(m,t) =$$

$$\frac{Mg}{\bar{m}}\left[\sin\frac{\lambda_m ct}{L} + A_m \cos\frac{\lambda_m ct}{L} + B_m \sinh\frac{\lambda_m ct}{L} + \cosh\frac{\lambda_m ct}{L}\right] \tag{31}$$

In order to solve equation (31), the method of Laplace transform, Convolution theorem and substitution of (2) is resorted to. The following are some of the relations employed:

$$F(s) = \int_0^\infty e^{-st} f(t)dt \; ; s > 0 \tag{32}$$

where $'s'$'is the Laplace parameter in conjunction with equation (2)

The following representation were adopted to reduce the problem to finding the Laplace inversion

$$\left.\begin{aligned} F_f(s) &= \frac{\theta_m}{s^2+\theta_m^2} + \frac{sA_m}{s^2+\theta_m^2} + \frac{\theta_m B_m}{s^2-\theta_m^2} + \frac{s\theta_m}{s^2-\theta_m^2} \\ G_{1f}(s) &= \frac{1}{s+r_2}, \quad G_{2f}(s) = \frac{1}{s+r_1} \end{aligned}\right\} \tag{33}$$

Hence, we have

$$W(m,s) = Q\{F_f(s) * G_{1f}(s) - F_f(s) * G_{2f}(s)\} \tag{34}$$

$$\text{where } Q = \frac{R}{r_1 - r_2} \tag{35}$$

$$R = \frac{Mg}{\bar{m}} \tag{36}$$

$$\theta_m = \frac{\lambda_m c}{L} \tag{37}$$

Hence, finding Laplace inversion of (34)

$$\mathcal{L}^{-1}\{W(m,s)\} = Q\{\mathcal{L}^{-1}[F_f(s) * G_{1f}(s)] - \mathcal{L}^{-1}[F_f(s) * G_{2f}(s)]\} \tag{38}$$

To evaluate the right hand side of (38), one makes use of convolution integral defined as

$$F(t) * g(t) = \int_0^t f(u)g(t-u)du \tag{39}$$

equation (38) becomes

$$W(m,t) = Q\left\{F_f(t)g_{1f}(t) - F_f(t)g_{f(t)}\right\} \tag{40}$$

where $F_f(t) = \mathcal{L}^{-1}\left[F_f(s)\right]$

$$= S_m\theta_m t + A_m cos\theta_m t + B_m sinh\theta_m t + C_m cosh\theta_m t \tag{41}$$

$$g_{1f}(t) = \mathcal{L}^{-1}\left[G_{1f}(s)\right] \tag{42}$$

$$= e^{-r_2 t} \tag{43}$$

$$g_{2f(t)} = \mathcal{L}^{-1}\left[G_{2f}(s)\right] \tag{44}$$

$$= e^{-r_1 t} \tag{45}$$

Applying (39)

$$F_f(t) * g_{1f}(t) = \int_o^t F_f(u)g_{1f}(t-u)du \tag{46}$$

$$= \int_o^t [S_m\theta_m u + A_m cos\theta_m u + B_m sinh\theta_m u + C_m cosh\theta_m u]\left[e^{-r2(t-u)}\right]du$$

$$= \int_o^t e^{-r_2(t-u)}]S_m\theta_m du + A_m\int_o^t e^{-r_2(t-u)} cos\theta_m u\, du$$

$$+ B_m\int_o^t e^{-r_2(t-u)} sinh\theta_m u\, du + C_m\int_o^t e^{-r_2(t-u)} cosh\theta_m u\, du \tag{47}$$

$$F_f(t) * g_{2f}(t) = \int_o^t F_f(u)g_{2f}(t-u)du$$

$$= \int_o^t e^{-r_1(t-u)} sinh\theta_m u\, du + A_m\int_o^t e^{-r_1(t-u)} cos\theta_m u\, du$$

$$+ B_m\int_o^t e^{-r_1(t-u)} sinh\theta_m u\, du + C_m\int_o^t e^{-r_1(t-u)} cosh\theta_m u\, du \tag{48}$$

We then substitute (47) and (48) into (40), determine the values of the integral, make some simplifications and substituting into equation (11), we obtained

$$W(x,t) = \sum_{m=1}^{a} Q \left\{ \begin{array}{l} \left[\begin{array}{l} \frac{\theta_m}{{\theta_m}^2 + {r_2}^2}[e^{-r_2 t} - \cos\theta_m t + \frac{r_2}{\theta_m}\sin\theta_m t] \\ - \frac{\theta_m}{{\theta_m}^2 + {r_1}^2}[e^{-r_1 t} - \cos\theta_m t + \frac{r_1}{\theta_m}\sin\theta_m t] \end{array} \right] \\ +A_m \left[\begin{array}{l} \frac{\theta_m}{{\theta_m}^2 + {r_2}^2}[\sin\theta_m t + \frac{r_2}{\theta_m}(\cos\theta_m t - e^{-r_2 t})] \\ - \frac{\theta_m}{{\theta_m}^2 + {r_1}^2}[\sin\theta_m t + \frac{r_1}{\theta_m}(\cos\theta_m t - e^{-r_1 t})] \end{array} \right] \\ +B_m \left[\begin{array}{l} \frac{\theta_m}{{\theta_m}^2 - {r_2}^2}[\cosh\theta_m t - e^{-r_2 t} - \frac{r_2}{\theta_m}\sinh\theta_m t)] \\ - \frac{\theta_m}{{\theta_m}^2 - {r_1}^2}[\cosh\theta_m t - e^{-r_1 t} - \frac{r_1}{\theta_m}\sinh\theta_m t)] \end{array} \right] \\ +C_m \left[\begin{array}{l} \frac{\theta_m}{{\theta_m}^2 - {r_2}^2}[\sinh\theta_m t - \frac{r_2}{\theta_m}(e^{-r_2 t} - \cosh\theta_m t)] - \\ \frac{\theta_m}{{\theta_m}^2 - {r_1}^2}[\sinh\theta_m t - \frac{r_1}{\theta_m}(e^{-r_1 t} - \cosh\theta_m t)] \end{array} \right] \end{array} \right\} U_m(x) \tag{49}$$

Equation (49) above represents the transverse displacement response of the beam under the action of moving concentrated force.

Case 2: Uniform Bernoulli Euler Beam Traversed by Concentrated Moving Mass

When the mass of the moving load is considered, the inertia effect of the moving mass is not negligible. Thus, $\varepsilon_o \neq 0$ and the solution to the entire equation 27 is required. Setting $A_3(x) = 0$, retaining the inertia term ε_o in equation (27) and applying strubble asymptotic method with further arrangement, equation (27) becomes

$$W_{tt}(m,t) + A_1 W_t(m,t) + A_2 W(m,t) - \varepsilon_o[U_1(m,t)W_{tt}(m,t) + 2CU_2(m,t)W_t(m,t) + C^2 U_3(m,t)W(m,t)] - \varepsilon_o[P_1(k,m,t)W_{tt}(k,t) + 2CP_2(k,m,t)W_t(k,t) + C^2 P_3(k,m,t)W(k,t)]$$

$$=\frac{Mg}{m}\left[Sin\theta_m t + A_m cos\theta_m t + B_m sinh\theta_m t + C_m cosh\theta_m t\right] \quad (50)$$

where $\varepsilon_o = \frac{M}{\bar{m}L}$ (51a)

$$\theta_m = \frac{\lambda_m c}{L} \quad (51b)$$

Let us consider a parameter $\in < 1$ for any arbitrary mass ratio ε_o, defined as

$$\in = \frac{\varepsilon_o}{1+\varepsilon_o} \quad (52)$$

$$\varepsilon_o = \in (1-\in)^{-1} \quad (53)$$

From the theory of Binomial expansion of integer and truncating after the second term, one obtains

$$\varepsilon_o = \in + 0(\in^2) \quad (54)$$

After some simplification and rearrangements of (50), we have

$$W_{tt}(m,t) + \in \frac{(A_1 + R_2(t))}{R_1(t)} W_t(m,t) + \in \frac{(A_2 + R_3(t))}{R_1(t)} W(k,t)$$

$$-\frac{\in}{R_1(t)}\left[P_1 W_{tt}(k,t)\right] + 2CP_2\, W_t(k,t) + C^2 P_3 W(k,t)]$$

$$= \frac{Mg}{R_1(t)\bar{m}}\left[Sin\theta_m t + A_m cos\theta_m t + B_m sinh\theta_m t + C_m cosh\theta_m t\right] \quad (55)$$

where $R_1(t) = \varepsilon_o U_1(m,t)$ (56)

$$R_2(t) = \frac{2c}{\varepsilon_o} U_2(m,t) \quad (57)$$

$$R_3(t) = \frac{C^2}{\varepsilon_o} U_3(m,t) \quad (58)$$

$$\frac{1}{R_1(t)} = \frac{1}{1-\varepsilon_o U_1(m,t)} \quad (59)$$

$$\frac{1}{R_1(t)} = \left[1 - \in U_1(m,t)\right]^{-1})$$

(60)

Whenever $/R_1(t)/< 1$ (61)

Substituting (60) into (55) and after some rearrangements and only O (∈), we obtain

$$W_{tt}(m,t)+\in(1+\in U_1(m,t)(A_1+R_1(t)W_t(m,t)$$

$$+\in(1+\in U_1(m,t)(A_2+R_3(t)W_t(m,t)$$

$$-\in(1+\in U_1(m,t)(P_1W_{tt}(k,t)]+2CP_2\,W_t(m,t)+C^2P_3W(k,t)$$

$$=(1+\in U_1(m,t)\frac{Mg}{\bar{m}}[Sin\theta_m t+A_m cos\theta_m t+B_m sinh\theta_m t+C_m cosh\theta_m t \tag{62}$$

Setting $\in= 0$ in equation (62), a situation corresponding to the case in which the inertia effect is regarded as negligible is obtained, then the solution can be written as

$$W_{mf}(\mathrm{m,t})=\beta_{mf}\mathrm{Cos}(\alpha_{mf}-W_{mf}) \tag{63}$$

where β_{mf}, α_{mf} *and* W_{mf} are constants.

Since $\in < 1$, an asymptotic solution of the homogenous part of the expression (62) can be written as

$$\mathrm{W\,(m,t)}=\emptyset(m,t)\cos[\alpha_{mf}t-\Omega(m,t)]+\in W_1(m,t)+0(\in^2) \tag{64}$$

where $\beta(m,t)$ & $\Omega(m,t)$ are slowly time varying functions or equivalently

In view of 64 and Neglecting 0 ($\in^2$) part, we have

$$W_t(m,t)=\dot{\emptyset}(m,t)\cos[\alpha_{mf}t-\Omega(m,t)]-\emptyset(m,t)\,\alpha_{mf}\sin\left[\alpha_{mf}t-\Omega(m,t)\right]$$
$$+\emptyset(m,t)\,\dot{\Omega}(m,t)sin[\alpha_{mf}t-\Omega(m,t)]+\in\dot{W}_1(m,t) \tag{65}$$

$$W_{tt}(m,t)=-2\dot{\emptyset}(m,t)\,\alpha_{mf}\sin\left[\alpha_{mf}t-\Omega(m,t)\right]$$
$$+2\emptyset(m,t)\,\alpha_{mf}t\dot{\Omega}(m,t)\mathrm{Cos}[\alpha_{mf}t-\Omega(m,t)]-$$

$$\emptyset(m,t)\ \alpha_{mf}{}^2 \cos\left[\ \alpha_{mf}t - \Omega(m,t)\right] + \in \ddot{W}_1(m,t) \tag{66}$$

Substituting (64) (65) and (66) into the homogenous part of (62), and considering only 0 (∈), we have $-2\dot{\emptyset}(m,t)\ \alpha_{mf} \sin\left[\ \alpha_{mf}t - \Omega(m,t)\right]\ \ 2\emptyset(m,t)\ \alpha_{mf}t\dot{\Omega}(m,t)$ Cos $[\ \alpha_{mf}t - \Omega(m,t)] - \emptyset(m,t)\ \alpha_{mf}{}^2 \cos\left[\ \alpha_{mf}t - \Omega(m,t)\right] + \ \in \ddot{W}_1(m,t)\ \ -\in(A_1 + R_2(t))\ \emptyset(m,t)\ \alpha_{mf} \sin\left[\ \alpha_{mf}t - \Omega(m,t)\right]\ +\in (A_2 +\ R_3(t))\ [\ \emptyset(m,t)\cos[\ \alpha_{mf}t - \Omega(m,t)]\ \ +\in \{P_1[\emptyset(k,t)\ \alpha_{mf}{}^2 \cos\left[\ \alpha_{mf}t - \Omega(k,t)\right] +\ 2cP_2\ [\emptyset(k,t)\ \alpha_{mf} \sin\left[\ \alpha_{mf}t - \Omega(k,t)\right] - C^2P_3\left[\emptyset(k,t) \cos\left[\ \alpha_{mf}t - \Omega(k,t)\right]\right]\} = 0$ (67)

In order to obtain the modified frequency, we extract only the variational part of (67) that describe the behavior of $\emptyset(m,t)$ and $\Omega(m,t)$ during the motion of mass. That is, neglecting terms which do not have $sin[\ \alpha_{mf}t - \Omega(m,t)\ and\ cos\left[\ \alpha_{mf}t - \Omega(m,t)\right]$ and equating their coefficients, we have

$$[-2\dot{\emptyset}(m,t) -\in(A_1 + R_2(t))\emptyset(m,t)]\ \ \alpha_{mf} = 0 \tag{68}$$

$$[2\dot{\Omega}(m,t)\ \alpha_{mf} -\ \alpha_{mf}{}^2 +\in (A_2\ + R_3(t))][\ \emptyset(m,t) = 0 \tag{69}$$

From (68)

$$\emptyset(m,t) = Ae^{\frac{-\in(A_1+R_2(t))}{2}} \tag{70a}$$

where A= e^{C_k} is the constant on integration (70b)

From (69)

$$\Omega(m,t) = \frac{\alpha_{mf}{}^2 + C^2U_3(m,t) - \in A_2]\ t}{2\ \alpha_{mf}} + C_m \tag{70c}$$

Substituting (70a) and (70c into $W(m,t) = \emptyset(m,t)\cos[\ \alpha_{mf}t - \Omega(m,t)]$, one obtains

$$W(m,t) = Ae^{\frac{-\in(A_1+R_2)}{2}t} \cos\left[\ \alpha_{mf}t - \{\frac{\alpha_{mf}{}^2 - \in(A_2 - \frac{C^2U_3(m,t)}{\in})}{2\ \alpha_{mf}} t + C_m\}\right] \tag{71}$$

$$W(m,t) = Ae^{\frac{-\in(A_1+R_2)}{2}t} cos[\alpha_{mm}t + Cm\ ] \tag{72}$$

where $\alpha_{mm} = \alpha_{mf} - \left(\frac{\alpha_{mf}{}^2 + C^2 U_3(m,t) - \in A_2}{2\,\alpha_{mf}}\right)$ (73)

Equation (73) is called the modified natural frequency.

Thus, to solve the non-homogenous equation in (27), the differential operator which acts on (m, t) is replaced by the modified natural frequency α_{mm}. Hence, we have

$$W_{tt}(m,t) + A_1 W_t(m,t) + \alpha_{mm} W(m,t) = \frac{Mg}{\bar{m}}[Sin\theta_m t + A_m\, Cos\theta_m t + BSinh\theta_m t + CmCosh\theta_m t] \quad (74)$$

To solve equation (74), the method of Laplace transformation and convolution theorem are resorted to as in the case of moving force to obtain,

$$W(x,t) = \sum_{m=1}^{\infty} Q_1 \left\{ \begin{array}{l} \left[\begin{array}{l} \frac{\theta_m}{\theta_m{}^2 + w_2{}^2}[e^{-w_2 t} - Cos\theta_m t + \frac{w_2}{\theta_m} Sin\theta_m t] \\ -\frac{\theta_m}{\theta_m{}^2 + w_1{}^2}[e^{-w_1 t} - Cos\theta_m t + \frac{w_1}{\theta_m} Sin\theta_m t] \end{array} \right] \\ +A_m \left[\begin{array}{l} \frac{\theta_m}{\theta_m{}^2 + w_2{}^2}[Sin\theta_m t + \frac{w_2}{\theta_m}(Cos\theta_m t - e^{-w_2 t})] \\ -\frac{\theta_m}{\theta_m{}^2 + w_2{}^2}[Sin\theta_m t + \frac{w_1}{\theta_m}(Cos\theta_m t\, e^{-w_1 t})] \end{array} \right] \\ +B_m \left[\begin{array}{l} \frac{\theta_m}{\theta_m{}^2 - w_2{}^2}[Cosh\theta_m t - e^{-w_2 t} - \frac{w_2}{\theta_m} Sinh\theta_m t\,)] \\ -\frac{\theta_m}{\theta_m{}^2 - w_1{}^2}[Cosh\theta_m t - e^{-w_1 t} - \frac{w_1}{\theta_m} Sinh\theta_m t\,)] \end{array} \right] \\ +C_m \left[\begin{array}{l} \frac{\theta_m}{\theta_m{}^2 - w_2{}^2}[Sinh\theta_m t - \frac{w_2}{\theta_m}(e^{-w_2 t} - Cosh\theta_m t\,)] \\ -\frac{\theta_m}{\theta_m{}^2 - w_1{}^2}[Sinh\theta_m t - \frac{w_1}{\theta_m}(e^{-w_1 t} - Cosh\theta_m t\,)] \end{array} \right] \end{array} \right\} U_m(x) \quad (75)$$

Equation (75) above represents the transverse displacement response of the beam under the action of moving concentrated mass.

2.3. Application of the Clamped-Clamped Boundary Condition

The clamped-clamped boundary conditions are

$$W(0,t) = W(L,t) = 0, \frac{\partial W(0,t)}{\partial x^2} = 0 = \frac{\partial W(L,t)}{\partial x} \tag{76}$$

For normal modes,

$$U_m(0) = U_m(L) = 0, \frac{\partial U}{\partial x} = 0 = \frac{\partial U_m{}^{(L)}}{\partial x} \tag{77}$$

Also $u_m(o) = u_k(L) = 0, \frac{\partial u_k{}^{(o)}}{\partial x} = 0 = \frac{\partial u_k{}^{(L)}}{\partial x}$ (78)

Using the above Boundary Condition in equation (13), results to:

$$A_m = \frac{Sinh\lambda_m - Sin\lambda_m}{Cos\lambda_m - Cosh\lambda_m} = \frac{Cos\lambda_m - Cosh\lambda_m}{Sinh\lambda_m + Sin\lambda_m} = -C_m, B_m = -1 \tag{79}$$

And $A_k = \frac{Sinh\lambda_k - Sin\lambda_m}{Cos\lambda_k - Cosh\lambda_k} = \frac{Cos\lambda_k - Cosh\lambda_k}{Sin\lambda_k + Sin\lambda_k} = -C_k,\ B_k = -1$ (80)

The corresponding frequency is obtained as

$$Cos\lambda_m\ Cosh\lambda_m = 1 \tag{81}$$

and $Cos\lambda_k\ Cosh\lambda_k = 1$ (82)

It follows that:

$\lambda_1 = 4.73004, \quad \lambda_2 = 7.85320,\ \lambda_3 = 10.99561$

For m, k = 1, 2, 3.

Substituting the equations (79), (80), (81) and (82) into (49) and (75), one obtains respectively the equations of the moving force and the moving mass for the dynamic response of a clamped-clamped uniform Bernoulli Euler beam resting on a Pasternak foundation, subjected to a concentrated load with a damping term.

3. Results

3.1 Comments on Closed Form Solution

The deflections of the clamped-clamped uniform Bernoulli–Euler beam may experience a phenomenon referred to as a state of resonance which could cause the failure of the beam. Here, the deflection goes beyond bound and the speed of the load which brings about resonance effect in the system is termed the critical speed. Clearly, the clamped-clamped uniform Bernoulli– Euler beam resting on a Pasternak foundation and traversed by a moving concentrated force reaches a state of resonance whenever

$$\alpha_{mf} = \theta_m = \frac{\lambda_m c}{L}$$

Similarly, the clamped-clamped uniform Bernoulli– Euler beam resting on a Pasternak foundation and traversed by a moving concentrated mass reaches a state of resonance whenever

$$\alpha_{mm} = \theta_m = \frac{\lambda_m c}{L}$$

Thus, for the same critical speed, $\theta_m = \frac{\lambda_m c}{L}$, the moving force ($\alpha_{mf}$) and the moving mass ($\alpha_{mm}$) problem of the same beam will attain a state of resonance. However, from (73), α_{mf} and α_{mm} are related by,

$$\alpha_{mm} = \alpha_{mf} - \left(\frac{\alpha_{mf}{}^2 + C^2 U_3(m,t) - \in A_2}{2\,\alpha_{mf}}\right) \tag{73}$$

From (73), α_{mm} is smaller than α_{mf} which shows that resonance is reached earlier for the moving mass problem than for the moving force problem. Therefore, the moving force problem cannot be used as a safe approximation for the moving mass problem in the design of the dynamical system.

3.2. Numerical Analysis of Results

Numerical analysis for both moving concentrated force and moving concentrated mass problems were carried out for all the parameters considered. In order to obtain the effects of the interacting beam parameters, the analytic results were simulated using MATLAB. The analysis in this work were simulated using MATLAB and illustrated by considering a homogenous beam of modulus of elasticity $E = 2.02 \times 10^{11}\ \ N/m^2$, the moment of inertia $I = 2.87698 \times 10^{-3}m^4$, the beam span $L = 100m$, the mass per unit length of the beam $\overline{m} = 2758.291\ Kg/m$.

With respect to the parameters, the values of axial force N was varied between $0N\ and\ 20000000N$, the values of the shear modulus G varies between $0\ N/m^3 and\ 50000000\ N/m^3$, the values of k varies between $0\ N/m^3 and\ 50000000\ N/m^3$ and the values of the damping coefficient α varied between 0 and 6. The results were shown in the various graphs 1 to 9 below for the clamped-clamped supported boundary condition considered for varied and fixed values of the four parameters. The 9^{th}graph shows the comparison of the concentrated moving force and concentrated moving mass (load).

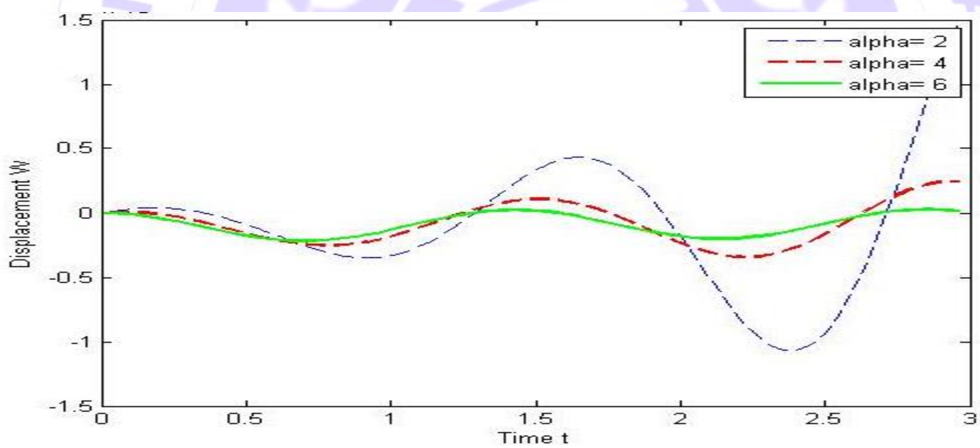

Figure 1: Transverse displacement of the beam for various values of damping coefficient and fixed values of other parameters traversed by a moving concentrated force

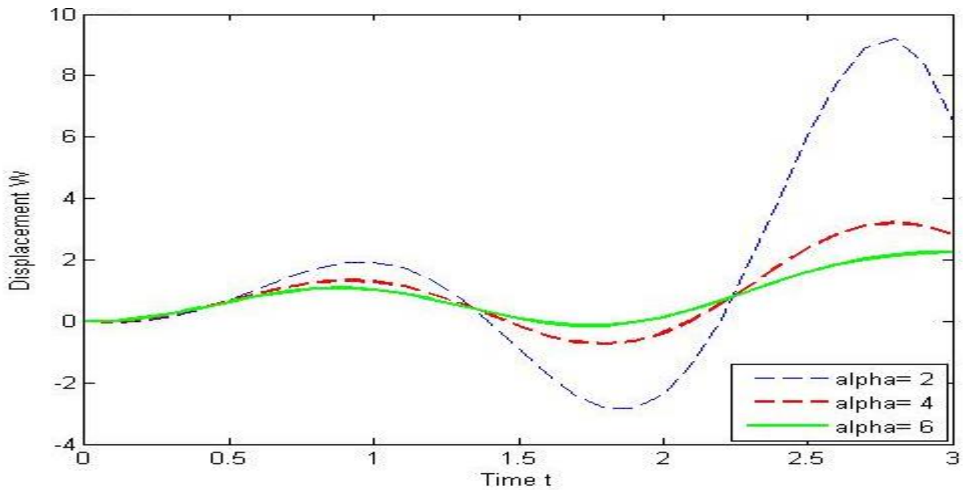

Figure 2: Transverse displacement of the beam for various values of damping coefficient and fixed values of other parameters traversed by a moving concentrated mass

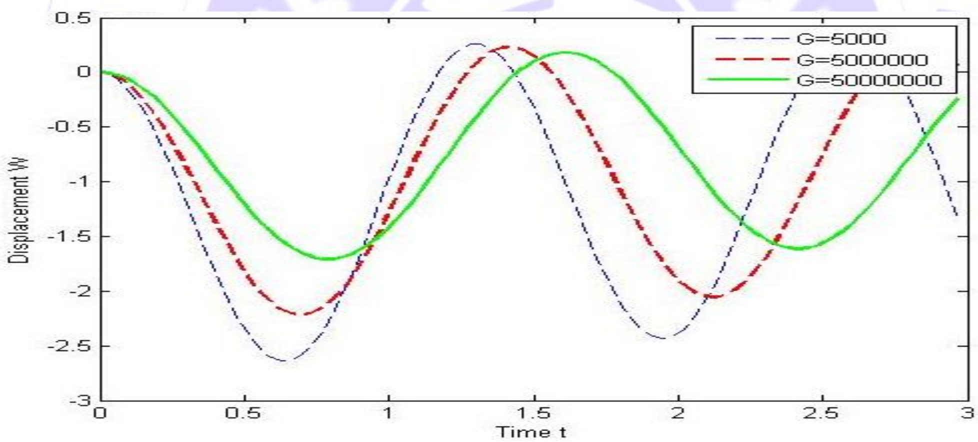

Figure 3: Transverse displacement of the beam for various values of shear modulus and fixed values of other parameters traversed by a moving concentrated force

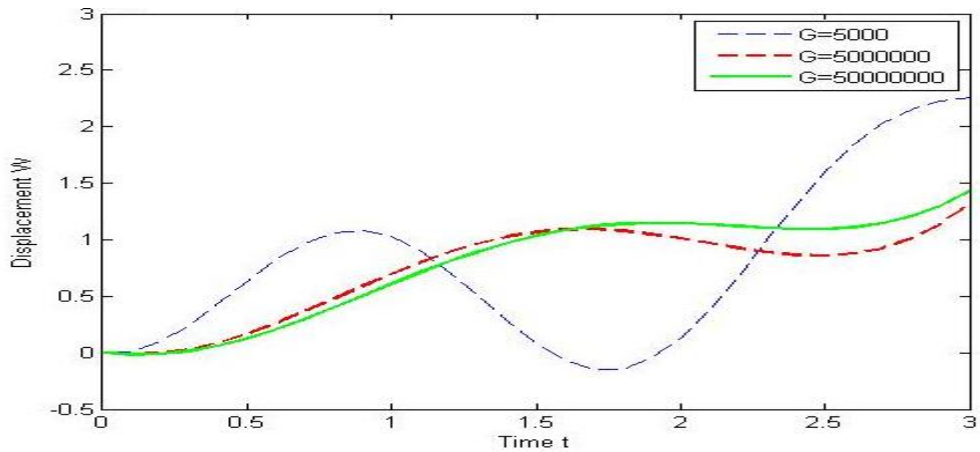

Figure 4: Transverse displacement of the beam for various values of shear modulus and fixed values of other parameters traversed by a moving concentrated mass

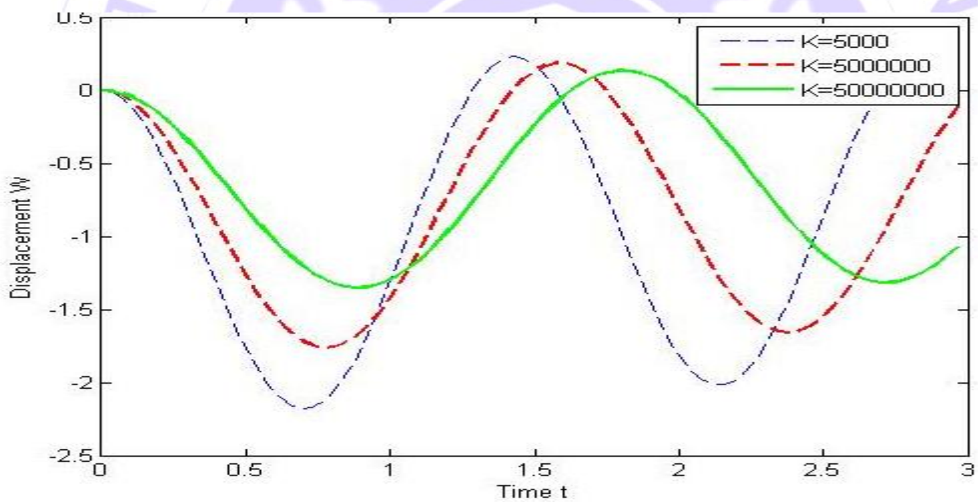

Figure 5: Transverse displacement of the beam for various values of foundation modulus and fixed values of other parameters traversed by a moving concentrated force

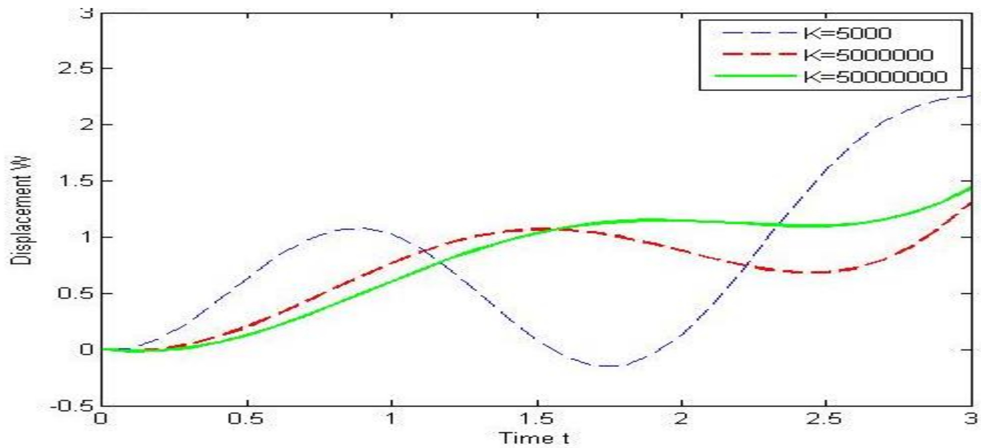

Figure 6: Transverse displacement of the beam for various values of foundation modulus and fixed values of other parameters traversed by a moving concentrated mass

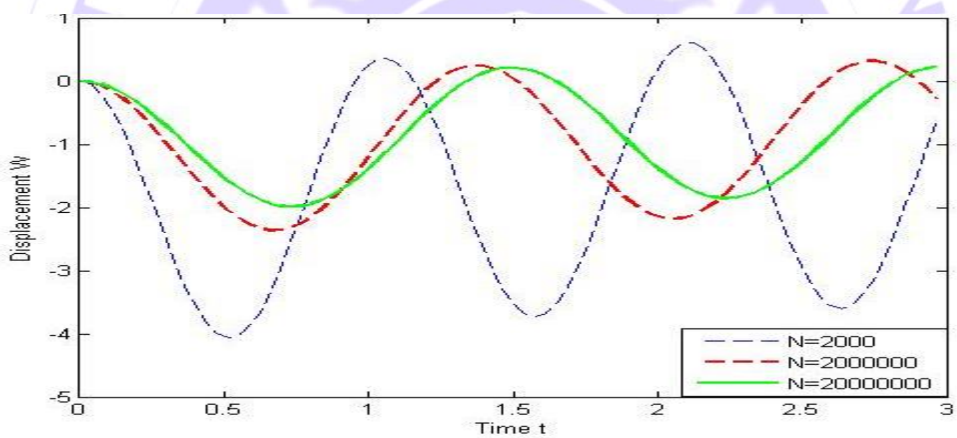

Figure 7: Transverse displacement of the beam for various values of axial force and fixed values of other parameters traversed by a moving concentrated force

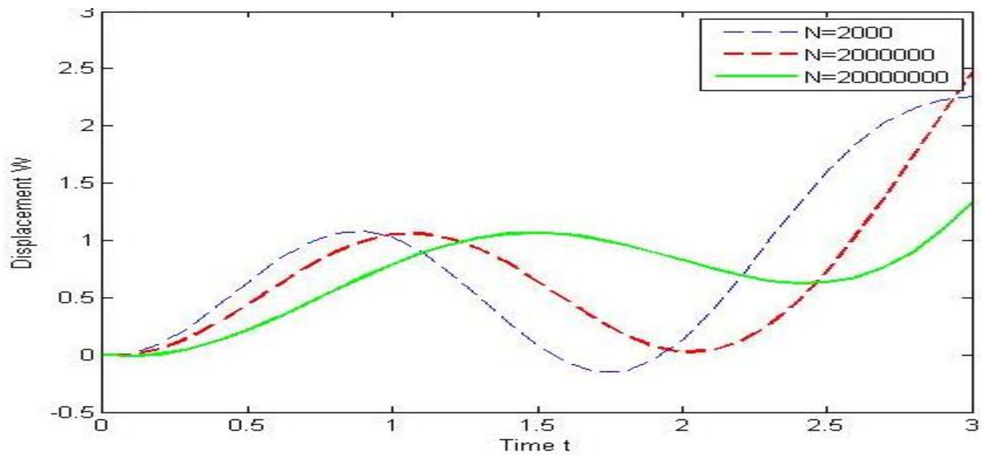

Figure 8: Transverse displacement of the beam for various values of axial force and fixed values of other parameters traversed by a moving concentrated mass

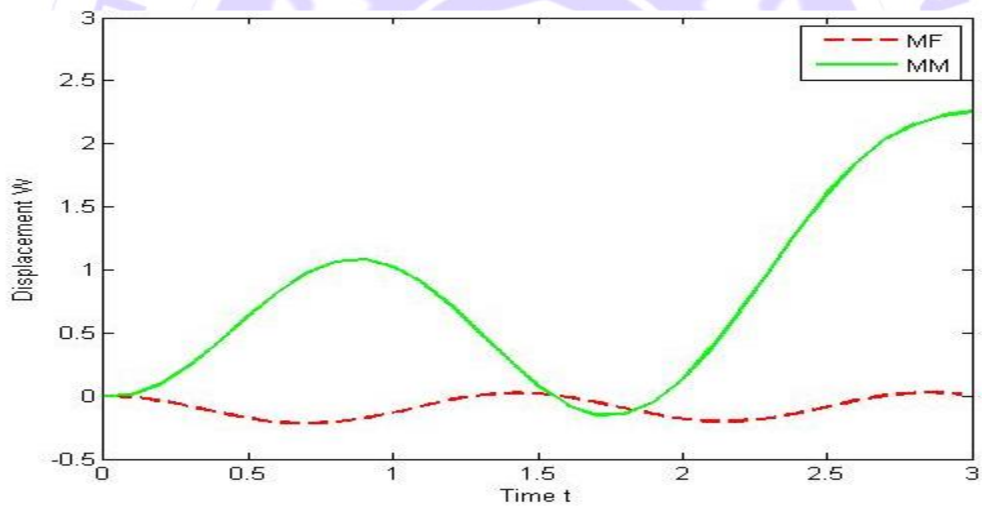

Figure 9: Comparison of the transverse displacement of the beam traversed by moving concentrated force and moving concentrated mass problem.

4. Discussion

In Figure 1, the deflection profiles of the beam under the action of moving concentrated force for various values of damping coefficient α, while the axial force N, foundation stiffness K and shear modulus, G fixed, were presented. The figure revealed that as α increases and other parameters fixed, the response amplitude of the uniform Bernoulli Euler beam decreases. From

Figure 2, similar results were found when the beam was subjected to a moving concentrated mass. In Figure 3, the deflection profile of the beam for various values of shear modulus *G,* while the values of damping coefficient α, axial force *N* and foundation modulus *K* were fixed traversed by a moving concentrated force was presented. The graph indicated that as G increases and other parameters fixed, the deflection profile of the uniform Bernoulli Euler beam decreases. Figure 4 also showed similar results, when the beam was subjected to a moving concentrated mass. In Figure 5, the deflection profile of the beam for various values of foundation modulus *K* and fixed damping coefficient α, axial force *N* and shear modulus *G* traversed by moving concentrated force was displayed. The graph showed that as K increases and other parameters fixed, the response amplitude of the uniform Bernoulli Euler beam decreases. Figure 6 also gave similar results when the beam was subjected to a moving concentrated mass. From Figure 7, the deflection profile of the beam for various values of axial force N and fixed values of damping coefficient α, shear modulus *G* and foundation modulus *K* traversed by moving concentrated force was displayed. The graph showed that as N increases and other parameters fixed, the response amplitude of the uniform Bernoulli Euler beam decreases. Figure 8 gave similar results, when the beam was subjected to a moving concentrated mass.

From the foregoing, it is evident that an increase in the values of the four parameters reduced the deflection profile of the beam. In effect, their increase ensured a safe passage of load and a prolonged beam's life. Again, it is clear from the values used to generate the above graphs that, of the four parameters considered, an increase in the damping coefficient had a very high significant effect on the response amplitude of the beam when its values are compared with other parameters. This suggests that, in the design of a dynamical system, the introduction of the damping term will most assure safety than other parameters. Again, from Figure 9, the moving mass problem had a higher deflection profile than the moving force problem for the dynamical system. This implies that, the moving force problem is safer than that of the moving mass. However, it cannot be used as a safe approximation of the moving mass problem as in the case of the comments on the closed form solution above. The findings of this study are in agreement with the works of Oni and Ayankop-Andi (2017), Adeoye and Awodola (2018), Oni and Ogunyebi (2018) and Awodola, *et* *al.* (2024).

5. Conclusion

The study showed that the moving concentrated force problem was not a safe approximation for the moving concentrated mass problem. It also revealed that, increases in the values of the vital parameters decreased the transverse displacement of the beam for both moving concentrated force and moving mass cases. Furthermore, it was observed that, small changes in the values of the damping coefficient lead to more noticeable effects on the deflection profiles of the beam compared to other parameters. This work contributes to existing knowledge on beam behavior by providing valuable insight for a clamped-clamped case when both the damping term and inertia effect of the moving load are considered. Its finding has implications for engineers in the design of such dynamical system. That is, increasing the values of these parameters can ensure safety, stability and functional performance of such dynamical system. In addition, the introduction of the damping term will most ensure safety than other parameters. Further research should explore cases of other classical boundary conditions. The results herein, were found to agree with those in literature.

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