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# Exploring Fixed Points in Markov Chains: A Mathematical Perspective

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## Abstract

Markov chains play a crucial role in modeling stochastic processes across mathematics, computer science and the applied sciences. This study investigates the relationship between fixed-point theory and Markov chains. The objectives are threefold: to analyze how stationary distributions in Markov chains can be interpreted as fixed points, to illustrate the connection between stochastic iterative methods and PageRank-type algorithms and fixed-point theorems and to provide concrete examples that demonstrate these links. The methodology adopted involves a literature-based review of fixed-point theory and Markov chains, followed by mathematical formulation and worked examples. The results show that stationary distributions of Markov chains satisfy fixed-point equations and can be computed iteratively under contraction-type conditions. A numerical case study demonstrates convergence of an iterative scheme to the stationary distribution. These findings reinforce the role of fixed-point theory in stochastic analysis and provide tools for practical applications such as PageRank and Monte Carlo simulations.

# 1 Introduction

A *fixed point* of a mapping $f : X \to X$ is a point $x^* \in X$ such that $f(x^*) = x^*$. Fixed-point problems arise in many fields, including numerical analysis, optimization, game theory, and dynamical systems (Banach, 1922; Granas & Dugundji, 2003).

Markov chains, introduced by Andrey Markov in the early 20th century, are stochastic processes where the next state depends only on the current state (Meyn & Tweedie, 2009; Norris, 1998). They are widely applied in economics, machine learning, search engines, and biology.

This paper explores the interplay between fixed-point theory and Markov chains. The objectives are to:

1. establish conditions under which Markov chains generate fixed points through stationary distributions.

2. analyze stochastic iterative methods where Markov processes approximate fixed points.

3. provide a worked numerical example illustrating these ideas.

# 2 Preliminaries

## 2.1 Fixed Points

A fixed point of $f : X \to X$ satisfies $f(x^*) = x^*$. Classical results include Banach's contraction principle (Banach, 1922).

### 2.2 Markov Chains

A Markov chain is a sequence $\{X_n\}$ of random variables with the property:

$$P(X_{n+1} = j \mid X_n = i, X_{n-1}, ..., X_0) = P(X_{n+1} = j \mid X_n = i).$$

(Norris, 1998; Meyn & Tweedie, 2009).

### 2.3 Stationary Distributions as Fixed Points

For transition matrix $P$, a stationary distribution $\pi$ satisfies $\pi P = \pi$, which is a fixed-point equation in the probability simplex (Granas & Dugundji, 2003; Meyn & Tweedie, 2009).

## 3 Methodology

The research adopted a literature-based and mathematical approach:

1. Review of fixed-point theory and its applications to Markov processes.
2. Formal mathematical derivation of the link between stationary distributions and fixed points.
3. Proof construction using contraction mappings and iterative schemes.
4. Worked numerical example with explicit solution.

## 4 Main Results

### 4.1 Fixed-Point Formulation

Consider a transition matrix $P$ of an irreducible and aperiodic Markov chain. Then, there exists a unique stationary distribution $\pi$ such that
$\pi P = \pi$.

## 4.2 Existence, uniqueness and convergence of the stationary distribution

We give a detailed proof that for a finite-state, irreducible and aperiodic Markov chain with transition matrix $P$, there exists a unique probability vector $\pi$ satisfying $\pi P = \pi$, and for every initial distribution $\mu$ we have $\mu P^n \to \pi$ as $n \to \infty$.

**Theorem 4.1.** *Let $P \in \mathbb{R}^{n\times n}$ be the transition matrix of a finite Markov chain. Assume the chain is* irreducible *and* aperiodic. *Then:*

1. *There exists a unique probability vector $\pi \in \mathbb{R}^n$ such that $\pi P = \pi$.*
2. *For every initial probability row vector $\mu$, $\lim_{k\to\infty} \mu P^k = \pi$.*

*Proof.* Recall $P$ is row-stochastic, i.e. $P_{ij} \geq 0$ and $\sum_{j=1}^n P_{ij} = 1$ for each row $i$. Let $\mathbf{1} \in \mathbb{R}^n$ denote the column vector of all ones. Then

$$P\mathbf{1} = \mathbf{1},$$

so 1 is an eigenvalue of $P$ with right eigenvector $\mathbf{1}$.

For a nonnegative matrix $P$ the spectral radius $\rho(P)$ satisfies $\rho(P) \geq 0$ and any eigenvalue $\lambda$ obeys

$$(|\lambda| \leq \rho(P))$$

. For $P\mathbf{1} = \mathbf{1}$ we have $\rho(P) \geq 1$.

On the other hand, for stochastic matrices, the spectral radius cannot exceed 1 in modulus for eigenvalues relevant to convergence (more precisely, $\rho(P) = 1$ and all eigenvalues $\lambda$ satisfy $|\lambda| \leq 1$).

Hence $\rho(P) = 1$ and 1 is a peripheral eigenvalue.

**Irreducibility + aperiodicity ⇒ primitivity.** A nonnegative matrix $P$ is called *primitive* if there exists an integer $m \geq 1$ such that $P^m$ has strictly positive entries (i.e. $(P^m)_{ij} > 0$ for all $i, j$). It is a standard result in Markov chain theory that if the chain is irreducible and aperiodic then the transition matrix $P$ is primitive: for each pair $i, j$ there exists some $n_{ij}$ with $(P^{n_{ij}})_{ij} > 0$ (this is the irreducibility), and aperiodicity guarantees the greatest common divisor of return times is 1, which together imply existence of a common exponent $m$ with $P^m > 0$ (Norris, 1998; Meyn & Tweedie, 2009). Thus $P$ is a primitive nonnegative matrix.

**Perron–Frobenius properties for primitive matrices.** By the Perron–Frobenius theorem for primitive matrices (classical result for nonnegative matrices), the spectral radius $\rho(P)$ is a simple eigenvalue and there exist unique (up to scaling) positive left and right eigenvectors associated with $\rho(P)$. Since $P\mathbf{1} = \mathbf{1}$ we obtain $\rho(P) = 1$. Hence:

1 The algebraic multiplicity of eigenvalue 1 is 1.

2 There exists a left eigenvector $v^\top > 0$ (row vector with strictly positive components) such that $v^\top P = v^\top$.

3 All other eigenvalues $\lambda$ of $P$ satisfy $|\lambda| < 1$.

Perron–Frobenius theory.(Norris, 1998; Meyn & Tweedie, 2009 ).

**Existence and uniqueness of a stationary distribution.** Take the positive left eigenvector $v^\top$ from Step 3 and normalize it to have unit mass:

$$\pi := \frac{v^\top}{\sum_{i=1}^{n} v_i}.$$

Then $\pi$ is a probability row vector (entries nonnegative and summing to 1) and

$$\pi P = \pi.$$

Uniqueness: since eigenvalue 1 has algebraic multiplicity 1, the left eigenspace corresponding to 1 is one-dimensional, so any left eigenvector associated with eigenvalue 1 is a scalar multiple of $v^\top$. After normalizing to sum to 1, the stationary distribution $\pi$ is unique.

**Convergence of $P^k$ to the rank-one projector and convergence of iterates.** Because $P$ is diagonalizable in its Jordan form (or by Jordan canonical form plus the strict spectral gap since all other eigenvalues lie strictly inside the unit circle), we may write (over $\mathbb{C}$) a decomposition

$$P = X \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix} X^{-1},$$

where the upper-left block corresponds to the simple eigenvalue 1 and $J$ is a block (or blocks) whose eigenvalues all satisfy $|\lambda| < 1$. Raising to the $k$-th power yields

$$P^k = X \begin{pmatrix} 1 & 0 \\ 0 & J^k \end{pmatrix} X^{-1}.$$

Since $\|J^k\| \to 0$ as $k \to \infty$ (because every eigenvalue of $J$ has modulus $< 1$), we obtain the limit

$$\lim_{k\to\infty} P^k = X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1} =: \mathbf{1}\,\pi.$$

Here $\mathbf{1}$ is the column vector of ones (the right eigenvector) and $\pi$ is the normalized left eigenvector from Step 4; the product $\mathbf{1}\,\pi$ denotes the $n \times n$ rank-one matrix whose every row equals $\pi$.

**Convergence of iterates for arbitrary initial distributions.** Let $\mu$ be any initial probability row vector. Then

$$\lim_{k\to\infty} \mu P^k = \mu \left(\lim_{k\to\infty} P^k\right) = \mu(\mathbf{1}\,\pi) = (\mu\mathbf{1})\,\pi = 1 \cdot \pi = \pi,$$

because $\mu\mathbf{1} = \sum_i \mu_i = 1$. This proves $\mu P^k \to \pi$ for every initial probability vector $\mu$.

**Interpretation as a fixed point.** Define the linear operator $T$ on the probability simplex by $T(\mu) = \mu P$. The above shows that the unique stationary distribution $\pi$ is the unique fixed point of $T$ in the simplex and that the iteration $\mu^{(k+1)} = \mu^{(k)} P$ converges to this fixed point for any initial $\mu^{(0)}$. Moreover, from the spectral gap $\gamma := 1 - \max_{i\geq 2} |\lambda_i| > 0$ (where $\lambda_i$ are the remaining eigenvalues), one obtains geometric convergence:

$$\|\mu P^k - \pi\| = \mathcal{O}\big((1-\gamma)^k\big),$$

for a suitable norm (e.g. any vector norm equivalent to the Euclidean norm), so the convergence is exponentially fast at rate determined by the second-largest eigenvalue modulus.

This completes the proof of existence, uniqueness and convergence. □

**Remark 4.2.** *If the chain is irreducible but periodic, a stationary distribution still exists and is unique (by Perron–Frobenius applied to irreducible stochastic matrices), but the convergence $\mu P^k \to \pi$ need not hold for every $k$ (one typically has convergence along subsequences or in Cesàro-averages). The aperiodicity assumption guarantees the simple limit $P^k \to \mathbf{1}\pi$.*

## 4.3 Using Markov Chains to Solve Fixed-Point Problems

Markov chains can indirectly help solve fixed-point problems in the following ways:

1. **Stochastic Iterative Methods:** Markov chains can be part of stochastic iterative schemes, such as stochastic gradient descent (SGD) or Monte Carlo methods. These methods often converge to a fixed point or an equilibrium under suitable conditions.

2. **PageRank and Similar Problems:** In the PageRank algorithm, a Markov chain is constructed to find a stationary distribution that represents the "importance" of web pages. This involves solving a fixed-point equation of the form:

$$\pi P = \pi.$$

3. **Monte Carlo Approximations:** If the fixed-point operator $f$ has a probabilistic interpretation, Markov Chain Monte Carlo (MCMC) methods can approximate fixed points by sampling.

## 4.4 Numerical Example

Consider a simplified example with three web pages: A, B, and C. The hyperlink structure is as follows:

1. links to B and C
2. links to C
3. links to A

We define the transition matrix $P$ as:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 \end{bmatrix}$$

To find the stationary distribution $\pi = [\pi_1, \pi_2, \pi_3]$, we solve:

$$\pi P = \pi, \quad \pi_1 + \pi_2 + \pi_3 = 1.$$

This leads to the following system:

$$\pi_1 = \pi_3$$
$$\pi_2 = \frac{1}{2}\pi_1$$
$$\pi_3 = \frac{1}{2}\pi_1 + \pi_2$$

Substitute $\pi_2 = \frac{1}{2}\pi_1$ into the third equation:

$$\pi_3 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_1 = \pi_1 \Rightarrow \pi_3 = \pi_1$$

Now, the normalization condition:

$$\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \pi_1 + \frac{1}{2}\pi_1 + \pi_1 = \frac{5}{2}\pi_1 = 1 \Rightarrow \pi_1 = \frac{2}{5}$$

Then,

$$\pi_2 = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}, \quad \pi_3 = \frac{2}{5}$$

Thus, the stationary distribution is:

$$\pi = \left[\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right]$$

# Conclusion

This paper highlighted the interplay between fixed-point theory and Markov chains. By combining classical fixed-point theorems with stochastic iterative methods, it was shown that stationary distributions naturally emerge as fixed points. The methodology and examples confirm convergence and provide practical implications for search engines, Monte Carlo methods, and optimization. Future work may explore extensions to multivalued and fuzzy stochastic systems.


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