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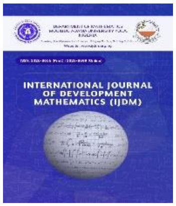

Simulating the Dynamics of Oscillating Differential Equations of Mass in Motion

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ABSTRACT

This research explores the practical implementation and simulation of oscillatory differential equations concerning objects in motion. The methodology incorporates power series polynomials, ensuring adherence to the fundamental properties of these functions. The novel approach is applied to various oscillatory differential equations, encompassing harmonic motion, spring motion, dynamic mass motion, Betiss and Stiefel equations, and nonlinear differential equations. The results demonstrate computational reliability, showcasing enhanced accuracy and quicker convergence compared to currently examined methods.

1. Introduction

Numerous unexplored and incompletely addressed physical problems persist within the realms of science, social science, and technology. While certain issues have garnered attention from researchers, there remains a vast expanse of uncharted territory. Oscillatory phenomena, often pivotal in these areas, find expression through the modelling capabilities of differential equations, (Blanka, 2019; Kusano & Naito, 1997).

Researchers leverage oscillatory differential equations to grapple with intricate systems involving multiple variables (Kusano & Naito, 1997). This field bears substantial significance for numerical analysts, facilitating the simulation of diverse phenomena across science, engineering, and social sciences (Agarwal, Grace, Li & Zhang, 2003; Bainov & Mishev, 1991; Agarwal, Bohner, Li & Zhang, 2013). Notably, it offers solutions for challenges in transportation, mass-spring systems, simple harmonic motion, and dynamic object systems, among others (Agarwal, Bohner, Li & Zhang, 2013; Triana & Fajardo, 2013). The simulation of these fields relies on oscillatory differential equations in various forms.

$$\frac{d^2u}{dv^2}=f\left(v,u,\frac{du}{dv}\right),\ u(0)=\delta_0,\ \frac{du}{dv}(0)=\delta_1 \tag{1}$$

Hence, (1) continues to hold great importance for numerical analysts in the fields of science and technology, as it is used to numerically simulate various laws, theorems, and physical relationships (Triana & Fajardo, 2013; Saker, 2010).

In their work, the authors, (French, 1965; Donald, Skwame, Sabo, & Ayinde, 2021; Fatunla, 1980; Skwame, Bakari & Sunday, 2017; Sabo, Kyagya & Vashawa, 2021; Omole & Ogunware, 2018) attempted to simulate second-order oscillatory differential equation (1). However, the accuracy of their methods in terms of error was found to be notably low and not particularly encouraging.

The force governing the motion is consistently directed towards the equilibrium position and is directly proportional to the distance from it. In other words,

$$F=-kv \tag{2}$$

In this framework, "F" signifies the force, "v" denotes the displacement, and "k" represents a constant—a relationship commonly known as Hooke's law. In simpler terms, a spring-mass system involves a block attached to the free end of a spring. Typically, this system is employed to ascertain the period of an object undergoing simple harmonic motion, (Triana & Fajardo, 2013; Saker, 2010). Moreover, it finds applications in a diverse array of scenarios.

For instance, a spring-mass system can be employed to model (1). One of the most challenging aspects in the numerical solution of differential equations pertains to handling highly oscillatory systems. The primary objective of this research is to formulate a method for simulating second-order oscillatory differential equations associated with objects in motion.

2. Methodology

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The power series polynomial was used to develop the new method, for simulation of (1).
Let the approximate solution of

$$\gamma(\tau) = \sum_{j=0}^{\psi+\varsigma-1} \varpi_j \, \tau^j \tag{3}$$

be the expected solution of (3) where $\tau \in [0, 1]$ and the number of interpolating and collocating points are ψ and ς. Differentiating (3) twice, yield

$$\gamma''(\tau) = \sum_{j=0}^{\psi+\varsigma-1} \tau(\tau-1)\varpi_j \, \tau^{j-2} \tag{4}$$

Substituting (3) into (1) yield

$$\sum_{j=0}^{\psi+\varsigma-1} \tau(\tau-1)\varpi_j \, \tau^{j-2} = f(\tau, \gamma, \gamma') \tag{5}$$

equation (3) is interpolated at $\psi = \frac{1}{6}, \frac{1}{4}$ while equation (5) is collocated at $\varsigma = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, 1$ which lead to

$$AX = U \tag{6}$$

where

$$A = \begin{bmatrix}
1 & \frac{1}{6} & \frac{1}{36} & \frac{1}{216} & \frac{1}{1296} & \frac{1}{7776} & \frac{1}{46656} & \frac{1}{279936} & \frac{1}{1679616} & \frac{1}{10077696} & \frac{1}{60466176} \\
1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{1024} & \frac{1}{4096} & \frac{1}{16384} & \frac{1}{65536} & \frac{1}{262144} & \frac{1}{1048576} \\
0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{3h^2} & \frac{5}{54h^2} & \frac{5}{216h^2} & \frac{7}{1296h^2} & \frac{7}{5832h^2} & \frac{1}{3888h^2} & \frac{5}{93312h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{3}{2h^2} & \frac{3}{4h^2} & \frac{5}{16h^2} & \frac{15}{128h^2} & \frac{21}{512h^2} & \frac{7}{512h^2} & \frac{9}{2048h^2} & \frac{45}{32768h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{3}{h^2} & \frac{4}{3h^2} & \frac{20}{27h^2} & \frac{10}{27h^2} & \frac{14}{81h^2} & \frac{56}{729h^2} & \frac{8}{243h^2} & \frac{10}{729h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{3}{h^2} & \frac{3}{h^2} & \frac{5}{2h^2} & \frac{15}{8h^2} & \frac{21}{16h^2} & \frac{7}{8h^2} & \frac{9}{16h^2} & \frac{45}{128h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{4}{h^2} & \frac{16}{3h^2} & \frac{160}{27h^2} & \frac{160}{27h^2} & \frac{448}{81h^2} & \frac{3584}{729h^2} & \frac{1024}{243h^2} & \frac{2560}{729h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{9}{2h^2} & \frac{27}{4h^2} & \frac{135}{16h^2} & \frac{1215}{128h^2} & \frac{5103}{512h^2} & \frac{5103}{512h^2} & \frac{19683}{2048h^2} & \frac{295245}{32768h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{5}{h^2} & \frac{25}{3h^2} & \frac{625}{54h^2} & \frac{3125}{216h^2} & \frac{21875}{1296h^2} & \frac{109375}{5832h^2} & \frac{78125}{3888h^2} & \frac{1953125}{93312h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2} & \frac{90}{h^2}
\end{bmatrix}$$

$$X = \begin{bmatrix} a_0 & a_1 & a_2 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_{n+\frac{1}{6}} & y_{n+\frac{1}{4}} & f_n & f_{n+\frac{1}{6}} & f_{n+\frac{1}{4}} & f_{n+\frac{1}{3}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{3}{4}} & f_{n+\frac{5}{6}} & f_{n+1} \end{bmatrix}^T$$

The unknown values of $a'_j s, \, j = 0(1)9$ are obtained by applying Gaussian elimination method and substituted into (3) to produce a continuous scheme with its derivatives of the form:

$$\gamma(\tau) = \sum_{j=\eta} \alpha_j^i(\tau) y_{n+j} + \sum_{j=0}^{1} \beta_j(\tau) f_{n+j} + \sum_{\varsigma \in \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}} \beta_\varsigma^i(\tau) f_{n+j}, \, \eta = \frac{1}{6}, \frac{1}{4}, \varsigma = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \tag{7}$$

Where the values of $\alpha_j, \, j = \eta$ and $\beta_\varsigma, \varsigma \in \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$ in equation (7) are

$$\alpha_{\frac{1}{6}} = 3 - 12\tau$$

$$\alpha_{\frac{1}{4}} = -2 + 12\tau$$

$$\beta_0 = \frac{109198219}{75246796800} - \frac{1562744609}{37623398400}\tau - \frac{1}{2}\tau^2 + \frac{601}{180}\tau^3 + \frac{1237}{90}\tau^4 - \frac{21959}{600}\tau^5 + \frac{28861}{450}\tau^6 - \frac{367}{5}\tau^7 + \frac{3693}{70}\tau^8 + \frac{108}{5}\tau^9 + \frac{95}{25}\tau^{10}$$

$$\beta_{\frac{1}{6}} = \frac{5336531}{154828800} - \frac{31769593}{7741400}\tau + \frac{162}{7}\tau^3 - \frac{11367}{70}\tau^4 + \frac{14013}{25}\tau^5 - \frac{28647}{25}\tau^6 + \frac{10152}{7}\tau^7 - \frac{5589}{5}\tau^8 + \frac{16848}{35}\tau^9 - \frac{15552}{175}\tau^{10}$$

$$\beta_{\frac{1}{4}} = -\frac{733019}{20995200} + \frac{41105527}{73483200}\tau - \frac{1024}{21}\tau^3 + \frac{123136}{315}\tau^4 - \frac{36864}{25}\tau^5 + \frac{721408}{225}\tau^6 - \frac{29696}{25}\tau^7 + \frac{16896}{5}\tau^8 - \frac{52224}{35}\tau^9 + \frac{49152}{175}\tau^{10}$$

$$\beta_{\frac{1}{3}} = \frac{25067281}{928972800} - \frac{40128343}{92897280}\tau + \frac{81}{2}\tau^3 - \frac{13797}{40}\tau^4 + \frac{55323}{40}\tau^5 - \frac{158139}{50}\tau^6 + \frac{30483}{7}\tau^7 - \frac{250533}{70}\tau^8 + 1620\tau^9 - \frac{7776}{25}\tau^{10}$$

$$\beta_{\frac{1}{2}} = -\frac{3043807}{250822656} + \frac{124090512}{627056640}\tau - 20\tau^3 + \frac{541}{3}\tau^4 - \frac{3866}{5}\tau^5 + \frac{28454}{15}\tau^6 - \frac{19512}{7}\tau^7 + \frac{16974}{7}\tau^8 - 1152\tau^9 + \frac{1152}{5}\tau^{10}$$

$$\beta_{\frac{2}{3}} = \frac{1213349}{103219200} - \frac{9986671}{51609600}\tau + \frac{81}{4}\tau^3 - \frac{1353}{20}\tau^4 + \frac{166617}{200}\tau^5 - \frac{106569}{50}\tau^6 + \frac{22977}{7}\tau^7 - \frac{209709}{70}\tau^8 + \frac{7452}{5}\tau^9 - \frac{7776}{25}\tau^{10}$$

$$\beta_{\frac{3}{4}} = -\frac{1367003}{146966400} + \frac{2257181}{14696640}\tau - \frac{1048}{3}\tau^3 + \frac{47872}{315}\tau^4 - \frac{2048}{3}\tau^5 + \frac{398848}{225}\tau^6 - \frac{19456}{7}\tau^7 + \frac{90624}{35}\tau^8 - \frac{92116}{7}\tau^9 + \frac{49152}{175}\tau^{10}$$

$$\beta_{\frac{5}{6}} = \frac{1216513}{464486400} - \frac{10067843}{232243200}\tau + \frac{162}{35}\tau^3 - \frac{3051}{70}\tau^4 + \frac{4941}{25}\tau^5 - \frac{12987}{25}\tau^6 + \frac{28944}{35}\tau^7 - \frac{27459}{35}\tau^8 + \frac{14256}{35}\tau^9 - \frac{15552}{175}\tau^{10}$$

$$\beta_1 = -\frac{6979241}{75246796800} + \frac{8281069}{5374771200}\tau - \frac{1}{6}\tau^3 + \frac{571}{360}\tau^4 - \frac{1459}{200}\tau^5 + \frac{8791}{450}\tau^6 - \frac{223}{7}\tau^7 + \frac{2181}{70}\tau^8 - \frac{84}{5}\tau^9 + \frac{96}{25}\tau^{10}$$

For $j = 0$, equation (7) is evaluated at the non-interpolating point $x_{n+\kappa},\ \kappa = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, 1$

For $j = \frac{1}{6}, \frac{1}{4}$ equation (7) are evaluated to produce the discreet schemes with its derivatives. The discreet scheme and its derivatives are combined in a block form as

$$\mathrm{A}\mathrm{Y}_m = Z\mathrm{N}_1 + h^2\left[\Omega\mathrm{N}_2 + B\mathrm{N}_3\right] \tag{8}$$

$$A = \begin{pmatrix}
1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 7 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{12}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\mathrm{Y}_m = \begin{pmatrix} y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{2}{3}} \\ y_{n+\frac{3}{4}} \\ y_{n+\frac{5}{6}} \\ y_{n+1} \end{pmatrix},
Z = \begin{pmatrix}
3 & 0 \\
-1 & 0 \\
-3 & 0 \\
-5 & 0 \\
-6 & 0 \\
-7 & 0 \\
-9 & 0 \\
-\frac{12}{h} & -1 \\
-\frac{12}{h} & 0 \\
-\frac{12}{h} & 0 \\
-\frac{12}{h} & 0 \\
-\frac{12}{h} & 0 \\
-\frac{12}{h} & 0 \\
-\frac{12}{h} & 0 \\
-\frac{12}{h} & 0 \\
-\frac{12}{h} & 0
\end{pmatrix},
N_1 = \begin{pmatrix} y_n \\ y'_n \end{pmatrix},
\Omega = \begin{pmatrix}
\frac{109198219}{75246796800h^2} \\
-\frac{695202}{225740390400h^2} \\
\frac{852491}{37623398400h^2} \\
-\frac{77509}{1672151040h^2} \\
\frac{571181}{10749542400h^2} \\
-\frac{478871}{8062156800h^2} \\
-\frac{826073}{5016453120h^2} \\
-\frac{1562744609}{37623398400h} \\
-\frac{126905078873}{4180377600h} \\
-\frac{1694300961851}{37623398400h} \\
-\frac{441985204589}{7524679680h} \\
-\frac{30120945190783}{37623398400h} \\
-\frac{358877107289}{4180377600h} \\
\frac{597039207199}{7524679680h} \\
\frac{2417108191009}{37623398400h} \\
\frac{8684951}{5374771200h}
\end{pmatrix},
N_2 = \left[f_n\right],
N_3 = \begin{pmatrix} f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{5}{6}} \\ f_{n+1} \end{pmatrix}$$

and

$$B=\begin{pmatrix}
\frac{109198219h^2}{75246796800} & \frac{5336531h^2}{154828800} & -\frac{733019h^2}{20995200} & \frac{25067281h^2}{928972800} & -\frac{1213349h^2}{103219200} & -\frac{1367003h^2}{146966400} & \frac{1216513h^2}{464486400} & \frac{6979241h^2}{75246796800} \\
-\frac{820373h^2}{1393459200} & \frac{2559607h^2}{440899200} & \frac{1467341h^2}{2786918400} & \frac{39581h^2}{752467968} & -\frac{174439h^2}{2786918400} & \frac{22417h^2}{440899200} & -\frac{20227h^2}{1393459200} & \frac{117499h^2}{22740390400} \\
\frac{173741}{77414400h^2} & \frac{1191037}{73483200h^2} & \frac{8930671}{464486400h^2} & \frac{632159}{125411328h^2} & -\frac{109861}{51609600h^2} & \frac{107227}{73483200h^2} & -\frac{86657h^2}{232243200} & \frac{439849h^2}{37623398400} \\
\frac{383057}{92897280h^2} & \frac{247577}{9797760h^2} & \frac{7878697}{185794560h^2} & \frac{2599903}{83607552h^2} & \frac{52613}{185794560h^2} & \frac{4981}{3265920h^2} & -\frac{47207}{92897280h^2} & \frac{94037}{5016453120h^2} \\
\frac{108341}{22118400h^2} & \frac{79919}{2624400h^2} & \frac{7036861}{132710400h^2} & \frac{1668539}{35831808h^2} & \frac{152069}{14745600h^2} & \frac{313}{328050h^2} & -\frac{22337}{66355200h^2} & \frac{167299}{10749542400h^2} \\
\frac{281603}{49766400h^2} & \frac{561937}{15746400h^2} & \frac{6328331}{99532800h^2} & \frac{1668539}{26873856h^2} & \frac{2077151}{99532800h^2} & \frac{97447}{15746400h^2} & \frac{21083}{22249766400h^2} & \frac{74989}{8062156800h^2} \\
\frac{20227}{206434h^2} & \frac{361087}{9797760h^2} & \frac{5973157}{61931520h^2} & \frac{6771913}{83607552h^2} & \frac{474569}{6881280h^2} & \frac{25553}{1399680h^2} & \frac{1127509}{30965760h^2} & \frac{1452743}{1003290624h^2} \\
-\frac{31769593h}{77414400} & \frac{41105527h}{73483200} & \frac{40128343h}{92897280} & \frac{124090513h}{627056640} & -\frac{9986671h}{51609600} & \frac{2257181h}{14696640} & -\frac{10067843h}{232243200} & \frac{8281069h}{5374771200} \\
\frac{21594670289h}{46448640} & -\frac{4280007077h}{3499200} & \frac{532876014413h}{464486400} & -\frac{134055047461h}{209018880} & \frac{320972388889h}{464486400} & -\frac{462338871h}{8164800} & \frac{5439073883h}{33177600} & -\frac{15177389611h}{2508226560} \\
\frac{7627652837h}{11059200} & -\frac{16665615493h}{91185400} & \frac{790472698507h}{464486400} & -\frac{596578867391h}{627056640} & \frac{52904100707h}{51609600} & -\frac{137780687h}{164025} & \frac{56479066153h}{232243200} & -\frac{337717095179h}{37623398400} \\
\frac{208927290389h}{2322243200} & -\frac{173897533321h}{73483200} & \frac{343686903407h}{154828800} & -\frac{778139715439h}{627056640} & \frac{88720288991h}{66355200} & -\frac{80511027823h}{73483200} & \frac{327411485h}{1032192} & -\frac{440495877253h}{37623398400} \\
\frac{94921775111h}{77414400} & -\frac{47404837813h}{14696640} & \frac{1405364732749h}{464486400} & -\frac{30300843589h}{17915904} & \frac{18810099997h}{10321920} & -\frac{109733517167h}{73483200} & \frac{100406334589h}{232243200} & -\frac{600381611141h}{37623398400} \\
\frac{305355437077h}{232243200} & -\frac{84720763523h}{24494400} & \frac{215281194251h}{66355200} & -\frac{379053845797h}{209018880} & \frac{907717913113h}{464486400} & -\frac{13074605239h}{8164800} & \frac{107668758653h}{232243200} & -\frac{214601277143}{12541132800} \\
\frac{94073939363h}{77414400} & -\frac{4194761041h}{1312200} & \frac{1392805460971h}{464486400} & -\frac{1051003431839h}{627056640} & \frac{93219247043h}{51609600} & -\frac{6797094653h}{4592700} & \frac{568634503h}{1327104} & -\frac{595026871403h}{37623398400} \\
\frac{32644966403h}{33177600} & -\frac{190202395273h}{73483200} & \frac{75184098019h}{30965760} & -\frac{850970104687h}{627056640} & \frac{679318145561h}{464486400} & -\frac{2515778117h}{2099520} & \frac{89534648373h}{25804800} & -\frac{481798339717h}{37623398400} \\
\frac{812443h}{15482880} & -\frac{6670217h}{73483200} & \frac{148718413h}{464486400} & -\frac{7292783h}{627056640} & \frac{28831121h}{51609600} & -\frac{36489839h}{73483200} & \frac{97427581h}{232243200} & -\frac{311983487h}{7524679680}
\end{pmatrix}$$

equation (8) is multiplied by the inverse of A to have a hybrid block method of the form

$$A^{(0)}\Upsilon_m = A^{-1}ZN_1 + h^2\left[A^{-1}\Omega N_2 + A^{-1}BN_3\right] \tag{9}$$

Equation (9) can be written as follows

$$y_{n+\frac{1}{6}} = y_n + \frac{hy'_n}{6} + h^2\left[\begin{aligned}&\frac{9649609}{1763596800} f_n + \frac{4925}{145152} f_{n+\frac{1}{6}} - \frac{200876}{3444525} f_{n+\frac{1}{4}} + \frac{979999}{21772800} f_{n+\frac{1}{3}} - \frac{612761}{29393280} f_{n+\frac{3}{4}} \\ &+ \frac{49583}{2419200} f_{n+\frac{2}{3}} - \frac{56132}{3444525} f_{n+\frac{3}{4}} + \frac{50143}{10886400} f_{n+\frac{5}{6}} - \frac{57859}{352719360} f_{n+1}\end{aligned}\right]$$

$$y_{n+\frac{1}{4}} = y_n + \frac{hy'_n}{4} + h^2\left[\begin{aligned}&\frac{1844099}{206438400} f_n + \frac{781353}{11468800} f_{n+\frac{1}{6}} - \frac{4701}{44800} f_{n+\frac{1}{4}} + \frac{1858113}{22937600} f_{n+\frac{1}{3}} - \frac{128467}{3440640} f_{n+\frac{3}{4}} \\ &+ \frac{839997}{22937600} f_{n+\frac{2}{3}} - \frac{11731}{403200} f_{n+\frac{3}{4}} + \frac{94257}{11468800} f_{n+\frac{5}{6}} - \frac{20123}{68812800} f_{n+1}\end{aligned}\right]$$

$$y_{n+\frac{1}{3}} = y_n + \frac{hy'_n}{3} + h^2\left[\begin{aligned}&\frac{68291}{5511240} f_n + \frac{8753}{85050} f_{n+\frac{1}{6}} - \frac{502016}{3444525} f_{n+\frac{1}{4}} + \frac{238}{2025} f_{n+\frac{1}{3}} - \frac{12349}{229635} f_{n+\frac{3}{4}} \\ &+ \frac{17923}{340200} f_{n+\frac{2}{3}} - \frac{144128}{3444525} f_{n+\frac{3}{4}} + \frac{67}{5670} f_{n+\frac{5}{6}} - \frac{5791}{13778100} f_{n+1}\end{aligned}\right]$$

$$y_{n+\frac{2}{3}} = y_n + \frac{2hy'_n}{3} + h^2\left[\begin{aligned}&\frac{90224}{3444525} f_n + \frac{3448}{14175} f_{n+\frac{1}{6}} - \frac{1077248}{3444525} f_{n+\frac{1}{4}} + \frac{12902}{42525} f_{n+\frac{1}{3}} - \frac{20368}{229635} f_{n+\frac{3}{4}} \\ &+ \frac{238}{2025} f_{n+\frac{2}{3}} - \frac{45056}{492075} f_{n+\frac{3}{4}} + \frac{1096}{42525} f_{n+\frac{5}{6}} - \frac{3154}{3444525} f_{n+1}\end{aligned}\right]$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3hy'_n}{4} + h^2\left[\begin{aligned}&\frac{136011}{4587520} f_n + \frac{3190833}{11468800} f_{n+\frac{1}{6}} - \frac{15867}{15867} f_{n+\frac{1}{4}} + \frac{8028477}{22937600} f_{n+\frac{1}{3}} - \frac{102897}{1146880} f_{n+\frac{3}{4}} \\ &+ \frac{3295809}{22937600} f_{n+\frac{2}{3}} - \frac{4701}{44800} f_{n+\frac{3}{4}} + \frac{67797}{2293760} f_{n+\frac{5}{6}} - \frac{24021}{22937600} f_{n+1}\end{aligned}\right]$$

$$y_{n+\frac{5}{6}} = y_n + \frac{5hy'_n}{6} + h^2\left[\begin{aligned}&\frac{2335225}{70543872} f_n + \frac{136375}{435456} f_{n+\frac{1}{6}} - \frac{54500}{137781} f_{n+\frac{1}{4}} + \frac{38375}{96768} f_{n+\frac{1}{3}} - \frac{533125}{5878656} f_{n+\frac{3}{4}} \\ &+ \frac{148375}{870912} f_{n+\frac{2}{3}} - \frac{15500}{137781} f_{n+\frac{3}{4}} + \frac{4925}{145152} f_{n+\frac{5}{6}} - \frac{83375}{70543872} f_{n+1}\end{aligned}\right]$$

$$y_{n+1} = y_n + hy'_n + h^2\left[\frac{503}{12600} f_n + \frac{27}{70} f_{n+\frac{1}{6}} - \frac{256}{525} f_{n+\frac{1}{4}} + \frac{351}{700} f_{n+\frac{1}{3}} - \frac{11}{105} f_{n+\frac{3}{4}} + \frac{351}{1400} f_{n+\frac{2}{3}} - \frac{256}{1575} f_{n+\frac{3}{4}} + \frac{27}{350} f_{n+\frac{5}{6}}\right]$$

$$y'_{n+\frac{1}{6}} = y_n + h\left[\begin{array}{l}\frac{6117617}{146966400}f_n + \frac{1571}{4050}f_{n+\frac{1}{6}} - \frac{673996}{1148175}f_{n+\frac{1}{4}} + \frac{802813}{1814400}f_{n+\frac{1}{3}} - \frac{15413}{76545}f_{n+\frac{3}{4}} + \frac{356563}{1814400}f_{n+\frac{2}{3}} \\ -\frac{178996}{1148175}f_{n+\frac{3}{4}} + \frac{1247}{28350}f_{n+\frac{5}{6}} - \frac{229633}{146966400}f_{n+1}\end{array}\right]$$

$$y'_{n+\frac{1}{4}} = y_n + h\left[\begin{array}{l}\frac{1070131}{25804800}f_n + \frac{300429}{716800}f_{n+\frac{1}{6}} - \frac{52279}{100800}f_{n+\frac{1}{4}} + \frac{1211031}{1814400}f_{n+\frac{1}{3}} - \frac{10481}{53760}f_{n+\frac{3}{4}} + \frac{547641}{2867200}f_{n+\frac{2}{3}} \\ -\frac{15289}{100800}f_{n+\frac{3}{4}} + \frac{30699}{716800}f_{n+\frac{5}{6}} - \frac{39299}{25804800}f_{n+1}\end{array}\right]$$

$$y'_{n+\frac{1}{3}} = y_n + h\left[\begin{array}{l}\frac{381439}{9185400}f_n + \frac{5897}{14175}f_{n+\frac{1}{6}} - \frac{545792}{1148175}f_{n+\frac{1}{4}} + \frac{53141}{113400}f_{n+\frac{1}{3}} - \frac{15286}{76545}f_{n+\frac{3}{4}} + \frac{22061}{113400}f_{n+\frac{2}{3}} \\ -\frac{177152}{1148175}f_{n+\frac{3}{4}} + \frac{617}{14175}f_{n+\frac{5}{6}} - \frac{14201}{9185400}f_{n+1}\end{array}\right]$$

$$y'_{n+\frac{1}{2}} = y_n + h\left[\begin{array}{l}\frac{8329}{201600}f_n + \frac{297}{700}f_{n+\frac{1}{6}} - \frac{116}{225}f_{n+\frac{1}{4}} + \frac{13149}{22400}f_{n+\frac{1}{3}} - \frac{11}{105}f_{n+\frac{3}{4}} + \frac{3699}{22400}f_{n+\frac{2}{3}} - \frac{212}{1575}f_{n+\frac{3}{4}} \\ +\frac{21}{700}f_{n+\frac{5}{6}} - \frac{281}{201600}f_{n+1}\end{array}\right]$$

$$y'_{n+\frac{2}{3}} = y_n + h\left[\begin{array}{l}\frac{47611}{1148175}f_n + \frac{5944}{14175}f_{n+\frac{1}{6}} - \frac{569344}{1148175}f_{n+\frac{1}{4}} + \frac{7904}{14175}f_{n+\frac{1}{3}} - \frac{752}{76545}f_{n+\frac{3}{4}} + \frac{4019}{14175}f_{n+\frac{2}{3}} \\ -\frac{28672}{164025}f_{n+\frac{3}{4}} + \frac{664}{14175}f_{n+\frac{5}{6}} - \frac{1844}{1148175}f_{n+1}\end{array}\right]$$

$$y'_{n+\frac{3}{4}} = y_n + h\left[\begin{array}{l}\frac{118827}{2867200}f_n + \frac{43011}{102400}f_{n+\frac{1}{6}} - \frac{5583}{11200}f_{n+\frac{1}{4}} + \frac{1608903}{2867200}f_{n+\frac{1}{3}} - \frac{261}{17920}f_{n+\frac{3}{4}} + \frac{945513}{2867200}f_{n+\frac{2}{3}} \\ -\frac{1473}{11200}f_{n+\frac{3}{4}} + \frac{31347}{716800}f_{n+\frac{5}{6}} - \frac{4443}{2867200}f_{n+1}\end{array}\right]$$

$$y'_{n+\frac{5}{6}} = y_n + h\left[\begin{array}{l}\frac{243865}{5878656}f_n + \frac{475}{1134}f_{n+\frac{1}{6}} - \frac{22700}{45927}f_{n+\frac{1}{4}} + \frac{40325}{72576}f_{n+\frac{1}{3}} - \frac{125}{15309}f_{n+\frac{3}{4}} + \frac{22475}{72576}f_{n+\frac{2}{3}} \\ -\frac{2900}{45927}f_{n+\frac{3}{4}} + \frac{85}{1134}f_{n+\frac{5}{6}} - \frac{10025}{5878656}f_{n+1}\end{array}\right]$$

$$y'_{n+1} = y_n + h\left[\begin{array}{l}\frac{503}{12600}f_n + \frac{81}{175}f_{n+\frac{1}{6}} - \frac{1024}{1575}f_{n+\frac{1}{4}} + \frac{1053}{1400}f_{n+\frac{1}{3}} - \frac{22}{105}f_{n+\frac{3}{4}} + \frac{1053}{1400}f_{n+\frac{2}{3}} - \frac{1024}{1575}f_{n+\frac{3}{4}} \\ +\frac{81}{175}f_{n+\frac{5}{6}} - \frac{503}{12600}f_{n+1}\end{array}\right]$$

3. Basic Properties of the New Method

We will scrutinize the assessment of the novel approach, encompassing various properties such as order, error constant, consistency, convergence, zero-stability, and stability region (Olanegan, Ogunware & Alakofa, 2018; Kwari, Sunday, Ndam, Shokari & Wang, 2023), among others.

3.1 Order and Error Constant of the Method

In determining the order and error constant of the new method (9), we define the linear difference operator L associated with equation (9) as

$$L[y(x); h] = Y_m - A^{-1}ZN_1 - h^2\left[A^{-1}\Omega N_2 + A^{-1}BN_3\right] \tag{10}$$

Corollary 1 (Kwari, Sunday, Ndam, Shokari & Wang, 2023)

Compare the linear operator (10) with the truncation error $C_{09}h^{09}y^{09}\left(x_n\right)+0\left(h^{10}\right)$.

Proof

The linear difference operators (10) is compared with the new method (9) as

$$\left.\begin{aligned}
l_{\frac{1}{6}}\left[y(x_n);h\right] &= y\left(x_n+\frac{1}{6}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_{\varsigma}(x)f_{n+\varsigma}+\beta_{\varsigma}(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{1}{4}}\left[y(x_n);h\right] &= y\left(x_n+\frac{1}{4}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_{\varsigma}(x)f_{n+\varsigma}+\beta_{\varsigma}(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{1}{3}}\left[y(x_n);h\right] &= y\left(x_n+\frac{1}{3}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_{\varsigma}(x)f_{n+\varsigma}+\beta_{\varsigma}(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{1}{2}}\left[y(x_n);h\right] &= y\left(x_n+\frac{1}{2}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_{\varsigma}(x)f_{n+\varsigma}+\beta_{\varsigma}(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{2}{3}}\left[y(x_n);h\right] &= y\left(x_n+\frac{2}{3}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_{\varsigma}(x)f_{n+\varsigma}+\beta_{\varsigma}(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{3}{4}}\left[y(x_n);h\right] &= y\left(x_n+\frac{3}{4}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_{\varsigma}(x)f_{n+\varsigma}+\beta_{\varsigma}(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{5}{6}}\left[y(x_n);h\right] &= y\left(x_n+\frac{5}{6}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_{\varsigma}(x)f_{n+\varsigma}+\beta_{\varsigma}(x)f_{n+\varsigma}\right)\right)\\
l_{1}\left[y(x_n);h\right] &= y\left(x_n+h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_{\varsigma}(x)f_{n+\varsigma}+\beta_{\varsigma}(x)f_{n+\varsigma}\right)\right)
\end{aligned}\right\} \qquad (11)$$

Corollary 2 (Kwari, Sunday, Ndam, Shokari & Wang, 2023)

The local truncation error of (9) is assume $y(x)$ to be sufficiently differentiable and expanding $y(x_n+qh)$ and $y(x_n+jh)$ about x_n using Taylor series to have

$l_{\frac{1}{6}}\left[y(x_n);h\right]=\left(1.2415\times10^{-12}\right)$, $l_{\frac{1}{4}}\left[y(x_n);h\right]=\left(2.2041\times10^{-12}\right)$, $l_{\frac{1}{3}}\left[y(x_n);h\right]=\left(3.1629\times10^{-12}\right)$,
$l_{\frac{1}{2}}\left[y(x_n);h\right]=\left(5.0339\times10^{-12}\right)$, $l_{\frac{2}{3}}\left[y(x_n);h\right]=\left(6.9050\times10^{-12}\right)$, $l_{\frac{3}{4}}\left[y(x_n);h\right]=\left(7.8638\times10^{-12}\right)$,
$l_{\frac{5}{6}}\left[y(x_n);h\right]=\left(8.8264\times10^{-12}\right)$, $l_{1}\left[y(x_n);h\right]=\left(1.0068\times10^{-11}\right)$

Proof

Expanding the term Y_m and N_3 using a Taylor series about x_n respectively and then collecting their like elements to the power of h gives

$$l_{\frac{1}{6}}\left[y(x_n);h\right]=\left(1.2415\times10^{-12}\right)h^9y^{(9)}(x_n)+0\left(h^{10}\right)$$
$$l_{\frac{1}{4}}\left[y(x_n);h\right]=\left(2.2041\times10^{-12}\right)h^9y^{(9)}(x_n)+0\left(h^{10}\right)$$
$$l_{\frac{1}{3}}\left[y(x_n);h\right]=\left(3.1629\times10^{-12}\right)h^9y^{(9)}(x_n)+0\left(h^{10}\right)$$
$$l_{\frac{1}{2}}\left[y(x_n);h\right]=\left(5.0339\times10^{-12}\right)h^9y^{(9)}(x_n)+0\left(h^{10}\right)$$
$$l_{\frac{2}{3}}\left[y(x_n);h\right]=\left(6.9050\times10^{-12}\right)h^9y^{(9)}(x_n)+0\left(h^{10}\right)$$
$$l_{\frac{3}{4}}\left[y(x_n);h\right]=\left(7.8638\times10^{-12}\right)h^9y^{(9)}(x_n)+0\left(h^{10}\right)$$
$$l_{\frac{5}{6}}\left[y(x_n);h\right]=\left(8.8264\times10^{-12}\right)h^9y^{(9)}(x_n)+0\left(h^{10}\right)$$
$$l_{1}\left[y(x_n);h\right]=\left(1.0068\times10^{-11}\right)h^9y^{(9)}(x_n)+0\left(h^{10}\right)$$

Hence, from the above results, the order of the new method (9) is 9, and the error constants is

$$C=\begin{pmatrix}1.2415\times10^{-12},2.2041\times10^{-12},3.1629\times10^{-12},5.0339\times10^{-12},\\ 6.9050\times10^{-12},7.8638\times10^{-12},8.8264\times10^{-12},1.0068\times10^{-11}\end{pmatrix}^T.$$

3.2 Consistency

Definition 1 (Kwari, Sunday, Ndam, Shokari & Wang, 2023)

The new method (9) is consistent because it is of order 9.

3.3 Zero-Stability of the Method

For zero stability, we consider the characteristic function of the equation below:

$$\left|\left[\lambda B^{(0)}-B^{i}\right]\right|=\left|\lambda\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\right|=0$$

$$\lambda^8-\lambda^7=0,0,0,0,0,0,0,0,1$$

Since the roots of the equations lies between 0 and 1, hence the new method is zero stable see (Olanegan, Ogunware & Alakofa, 2018).

3.4 Convergent

Theorem 1 (Kwari, Sunday, Ndam, Shokari & Wang, 2023)

According to Dalquist theorem, the new method is convergent since it is consistence and zero-stable see (Olanegan, Ogunware & Alakofa, 2018).

Linear Stability

Definition 3 (Arevalo, Soderlind, Hadjimichael & Fekete, 2021)
The stability region of a new method is the set of complex values λh for which all solutions of the test problem $y''=-\lambda^2 y$ remain bounded as $n \to \infty$.

The concept of A-stability according to Arevalo, Soderlind, Hadjimichael and Fekete (2021) is discussed by applying the test equation

$$y^{(k)} = \lambda^{(k)} y \tag{12}$$

To yield

$$Y_m = \mu(z) Y_{m-1},\ z = \lambda h \tag{13}$$

Where $\mu(z)$ is the amplification matrix of the form

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^2\eta^{(0)}\right)^{-1}\left(\xi^1 - z\eta^{(1)} - z^2\eta^{(1)}\right) \tag{14}$$

The matrix $\mu(z)$ has Eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function of new method (9) is given as

$$\zeta = -\frac{\begin{pmatrix} 24799949\ 719675695z^8 - 1167073163\ 739266043z^7 + 27\,128030061\,143833235z^6 - \\ 515\,556735008\ 654413944z^5 + 6539\,326196102\ 856181344z^4 - 65866\ 416469167\ 064393152z^3 \\ + 430104\ 648937877\ 518309632z^2 - 1874456\ 030584895\ 333990400z + 3669028\,117771997\ 675520000 \end{pmatrix}}{\begin{matrix} 29255954\ 595840000z^8 - 1172188580\ 806656000z^7 + 28\,951692668\ 043264000z^6 - \\ 513\,945205576\ 040448000z^5 + 6754\,848844724\ 305920000z^4 - 64936\ 984916199\ 997440000z^3 \\ + 435626\ 312980066\ 467840000z^2 - 1834514\ 058885998\ 837760000z + 3669028\,117771997\ 675520000 \end{matrix}}$$

4. Mathematical Illustration

The novel approach was utilized to simulate a range of oscillatory differential equations. Initially, we performed a numerical simulation of oscillatory differential equation (1) to investigate the traits of mass in a spring, dynamic mass, and the equilibrium in harmonic form. Subsequently, we simulated oscillatory differential equation (1) with an external force "F" to assess its influence on the system's behavior. Finally, we conducted oscillatory simulations of (1) in both linear and nonlinear formats.

The notations below are used in the results
ES: Exact Solution
CS: Computed Solution
NM: New Method
ENM: Error in New Method
E[19]: Error in [19]
E[12]: Error in [12]
E[13]: Error in [13]
E[20]: Error in [20]
E[21]: Error in [21]
E[22]: Error in [22]
E[23]: Error in [23]

Example 1
Consider the mechanical oscillatory differential equation in harmonic motion, of an object which stretches a spring 6 inches in equilibrium.

i. Set up the equation of motion and find its general solution.
ii. Find the displacement of the object for $t > 0$, if it's initially displaced 18 inches above equilibrium and given a downward velocity of $3\frac{ft}{s}$.

From Newton's second law of motion, we have

$$mu''+cu'+ku=F \tag{15}$$

By setting $c=0$ and $F=0$, we get

$$mu''+ku=0 \Rightarrow u''+\frac{k}{m}u=0 \tag{16}$$

The equation of the weight of the object is given as follow:

$$mg=k\Delta l \Rightarrow \frac{k}{m}=\frac{g}{\Delta l} \tag{17}$$

Substituting $g=32\frac{ft}{s^2}$, $\Delta l=\frac{6}{12}ft$ into (17) we obtain

$$\frac{k}{m}=\frac{32}{\frac{6}{12}}=64$$

(17)

Substituting equation (18) into the equation (16) we get

$$u''+64u=0 \tag{18}$$

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus, $u(0)=\frac{3}{2}, u'(0)=-3$ and $h=0.1$. We make use of (18) as

$$dsolver\left(\left\{u''(v)+64u(v)=0, u(0)=\frac{3}{2}, u'(0)=-3\right\}\right) \tag{20}$$

We obtain the exact solution (20) as

$$u(v)=-\frac{3}{8}\sin(8v)+\frac{3}{2}\cos(8v) \tag{21}$$

Source: (Sabo, 2021).

Example 2

The second order mechanical oscillatory differential equation in a spring of motion is consider.

A $128lb$ weight is attached to a spring having a spring constant of $64lb/ft$. The weight is started in motion with no initial velocity by displacing it $6inches$ above the equilibrium position and by simultaneously applying to the weight an external force $F_4(v)=8\sin 4v$. Assuming no air resistance, compute the subsequent motion of the weight at $t: 0.01 \le v \le 0.10$.

Now, we model this problem into a mathematical model and then apply our method to compute the motion on the weight attached to the spring. Here,

$$m=4, k=64, b=0, and\ F_4(v)=8\sin 4v$$

Thus, problem 3 boils down to

$$\frac{d^2u}{dv^2}+16u=2\sin 4v, u(0)=-\frac{1}{2}, u'(0)=0 \tag{22}$$

with the exact solution of (22) is given by,

$$u(v)=-\frac{1}{2}\cos 4v+\frac{1}{16}\sin 4v-\frac{1}{4}v\cos 4v \tag{23}$$

Source: (Fatunla, 1980; Skwame, Bakari & Sunday, 2017).

Example 3

Consider the mass in a dynamic motion that is coined into linear oscillatory form of differential equation (1).

A mass of $10\,kg$ is attached to a spring having a constant spring of $140\ N/M$. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(v)=5\sin v$. Find the subsequent motion of the mass $(v:0.10\le v\le 1.00)$ if the force due to air resistance is $90\left(\frac{du}{dv}\right)N$.

We apply the same procedure, where $m=10, k=140, a=90\ and\ F(v)=5\sin v$ example 3 reduces to

$$dsolver\left(\left\{\frac{d^2u}{dv^2}+9\frac{du}{dv}+14y(u)=\frac{1}{2}\sin(v), u(0)=0, u'(0)=-1\right\}\right) \tag{24}$$

with the exact solution of (24) is given by,

$$u(v)=\frac{1}{500}\left(-90\exp(-2v)+99\exp(-7v)+13\sin v-9\cos v\right) \tag{25}$$

Source: (Fatunla, 1980; Skwame, Bakari & Sunday, 2017; Areo & Rufai, 2016).

Example 4

Consider the linear oscillatory differential equation in Betiss and Stiefel form

$$\frac{d^2u_1}{dv^2}+\frac{du_1}{dv}=0.001\cos(v), u_1(0)=1, \frac{du_1}{dv}=0 \tag{26}$$

$$\frac{d^2u_2}{dv^2}+\frac{du_2}{dv}=0.001\sin(v), u_1(0)=0, \frac{du_1}{dv}=0.9995 \tag{27}$$

With exact solution of (26) and (27) as

$$u_1(v)=\cos(v)+0.0005v\sin(v) \tag{28}$$

$$u_2(v)=\sin(v)-0.0005v\cos(v) \tag{29}$$

Source: (Skwame, Donald, Kyagya, Sabo & Bambur, 2020; Lydia, Joshua, Ndam & James, 2021).

Example 5:

Consider the nonlinear oscillatory differential equation

$$\frac{d^2u}{dv^2}-4yu'+8u=v^3, u(0)=2, u'(0)=4, \tag{30}$$

Whose exact solution is

$$y(v)=\exp(2v)\left(2\cos(2v)-\frac{3}{64}\sin(2v)\right)+\frac{3v}{32}+\frac{3v^2}{16}+\frac{v^2}{8} \tag{31}$$

Source: [22, 23].

5. Results and Discussion

Table I Computation of NM with (Sabo, 2021) when solving example 1

v	**ES**	**CS**	**ENM**	**E[19]**
0.1	0.77605152993342709579	0.77605152993274408426	6.8301(-13)	3.3496(-07)
0.2	- 0.41863938459249752594	-0.41863938459387367324	1.3762(-12)	1.6371(-06)
0.3	- 1.3593892660185498469	-1.35938926601955541960	1.0056(-12)	3.2716(-06)
0.4	- 1.4755518599067871611	-1.47555185990606872960	7.1843(-13)	3.5979(-06)
0.5	- 0.69666449555494477770	-0.69666449555213113975	2.8136(-12)	1.3589(-06)
0.6	0.50481020347261010590	0.50481020347619324768	3.5831(-12)	2.9143(-06)
0.7	1.4000738069674951883	1.40007380696939826270	1.9031(-12)	6.7226(-06)
0.8	1.4460714263183540043	1.44607142631665691830	1.6971(-12)	7.0589(-06)

0.9	0.61490152285494961183	0.61490152284989092499	5.0587(-12)	2.6543(-06)
1.0	- 0.58925939319668845548	0.58925939320237650700	5.6881(-12)	4.6056(-06)

See (Lydia, Joshua, Ndam & James, 2021; Olabode & Momoh, 2016).

Table II Computation of NM with (Fatunla, 1980; Skwame, Bakari & Sunday, 2017) when solving example 2

v	**ES**	**CS**	**ENM**	**E[12]**	**E[13]**
0.1	- 0.49959872021047678004	- 0.49959872021047678004	0.0000(00)	1.6621(-09)	1.0000(-19)
0.2	- 0.49839019330974949646	- 0.49839019330974949646	0.0000(00)	1.1586(-08)	4.1000(-19)
0.3	- 0.49636836974027966301	- 0.49636836974027966301	0.0000(00)	2.9743(-08)	9.1000(-19)
0.4	- 0.49352852660817937130	- 0.49352852660817937130	0.0000(00)	5.6076(-08)	1.6600(-18)
0.5	- 0.48986728796894500998	- 0.48986728796894500998	0.0000(00)	9.0504(-08)	2.6200(-18)
0.6	- 0.48538264289709933476	- 0.48538264289709933476	0.0000(00)	1.3291(-07)	3.8000(-18)
0.7	- 0.48007396129056685722	- 0.48007396129056685722	0.0000(00)	1.8317(-07)	5.2000(-18)
0.8	- 0.47394200736436189072	- 0.47394200736436189072	0.0000(00)	2.4110(-07)	6.8500(-18)
0.9	- 0.46698895079202783994	- 0.46698895079202783994	0.0000(00)	3.0653(-07)	8.7500(-18)
1.0	- 0.45921837545722401274	- 0.45921837545722401274	0.0000(00)	3.7922(-07)	1.0850(-17)

See (Fatunla, 1980; Skwame, Bakari & Sunday, 2017).

Table III Computation of NM with (Fatunla, 1980; Skwame, Bakari & Sunday, 2017; Areo & Rufai, 2016) when solving example 3

v	**ES**	**CS**	**ENM**	**E[12]**	**E[13]**	**E[20]**
0.1	-0.06436205154552458248	- 0.06436205154550692713	1.7655(-14)	1.2744(-08)	2.0453(-10)	4.4268(-09)
0.2	-0.08430720522644774945	- 0.08430720522643379455	1.3955(-14)	3.0442(-08)	4.8485(-10)	2.2383(-08)
0.3	-0.08405225313390041905	- 0.08405225313389384414	6.5749(-15)	4.1501(-08)	6.6174(-10)	3.5865(-08)
0.4	-0.07529304213333374810	- 0.07529304213333305897	6.8913(-16)	4.5385(-08)	7.2649(-10)	4.2157(-08)
0.5	-0.06357063960355798563	-0.06357063960356088722	2.9016(-15)	4.4298(-08)	7.1295(-10)	4.2895(-08)
0.6	-0.05142117069384508163	- 0.05142117069384974188	4.6603(-15)	4.0466(-08)	6.5550(-10)	4.0288(-08)
0.7	-0.03993052956438697070	- 0.03993052956439220056	5.2299(-15)	3.5475(-08)	5.7884(-10)	3.6051(-08)
0.8	-0.02949865862803573900	- 0.02949865862804086216	5.1232(-15)	3.0285(-08)	4.9808(-10)	3.1287(-08)
0.9	-0.02021269131259124546	- 0.02021269131259391333	2.6679(-15)	2.5408(-08)	4.2140(-10)	2.6618(-08)
1.0	-0.01202699425403169607	- 0.01202699425403402038	2.3243(-15)	2.1071(-08)	3.5257(-10)	2.2352(-08)

See (Fatunla, 1980; Skwame, Bakari & Sunday, 2017; Areo & Rufai, 2016).

Table IV Computation of NM with (Skwame, Donald, Kyagya, Sabo & Bambur, 2020; Lydia, Joshua, Ndam & James, 2021) when solving (26)

v	**ES**	**CS**	**ENM**	**E[21]**	**E[22]**
0.1	0.09978366643856425102	0.09978366643856425102	0.0000(00)	1.2567(-12)	1.0170(-12)
0.2	0.19857132413727709130	0.19857132413727709130	0.0000(00)	2.1140(-12)	1.4285(-11)
0.3	0.29537690618797073421	0.29537690618797073421	0.0000(00)	2.3764(-12)	4.9557(-11)
0.4	0.38923413010984991465	0.38923413010984991465	0.0000(00)	3.4242(-12)	1.0161(-10)
0.5	0.47920614296373040709	0.47920614296373040709	0.0000(00)	3.3944(-12)	1.7416(-10)
0.6	0.56439487271056245371	0.56439487271056245371	0.0000(00)	3.3436(-12)	2.6425(-10)
0.7	0.64394999247214148272	0.64394999247214148272	0.0000(00)	4.2949(-12)	3.7579(-10)
0.8	0.71707740821578389546	0.71707740821578389546	0.0000(00)	4.2574(-12)	5.0602(-10)
0.9	0.78304718514176158945	0.78304718514176158945	0.0000(00)	5.2344(-12)	6.5904(-10)
1.0	0.84120083365496243679	0.84120083365496243679	0.0000(00)	6.2265(-12)	8.3225(-10)

See (Skwame, Donald, Kyagya, Sabo & Bambur, 2020; Lydia, Joshua, Ndam & James, 2021).

Table V Computation of NM with (Skwame, Donald, Kyagya, Sabo & Bambur, 2020; Lydia, Joshua, Ndam & James, 2021) when solving (27)

v	**ES**	**CS**	**ENM**	**E[21]**	**E[22]**

0.1	0.99500915694885810751	0.99500915694885810750	0.0000(00)	2.8269(-12)	1.0169(-11)
0.2	0.98008644477432113724	0.98008644477432113723	0.0000(00)	5.8994(-12)	2.0390(-11)
0.3	0.95538081715660522058	0.95538081715660522057	0.0000(00)	6.8309(-12)	1.5451(-13)
0.4	0.92113887767134681290	0.92113887767134681288	0.0000(00)	1.4991(-12)	8.1063(-11)
0.5	0.87770241827502376687	0.87770241827502376685	0.0000(00)	1.8395(-12)	2.5377(-10)
0.6	0.82550500765169680785	0.82550500765169680783	0.0000(00)	1.6559(-11)	5.4848(-10)
0.7	0.76506766347502161813	0.76506766347502161811	0.0000(00)	1.2970(-11)	9.9571(-10)
0.8	0.69699365178352523002	0.69699365178352523001	0.0000(00)	8.4312(-11)	1.6260(-10)
0.9	0.62196246537999682400	0.62196246537999682400	0.0000(00)	5.3240(-11)	2.4697(-10)
1.0	0.54072304136054366565	0.54072304136054366565	0.0000(00)	3.2126(-11)	3.5575(-10)

See (Skwame, Donald, Kyagya, Sabo & Bambur, 2020; Lydia, Joshua, Ndam & James, 2021).

Table VI Computation of NM with (Lydia, Joshua, Ndam & James, 2021; Olabode & Momoh, 2016) when solving example 5

v	**ES**	**CS**	**ENM**	**E[22]**	**E[23]**
0.1	2.39411257699639561 81	2.39411257699639563790	1.9800(-17)	7.1426(-08)	5.1070(-06)
0.2	2.748141332426423525 6	2.74814133242642358080	5.5200(-17)	1.7491(-07)	1.4959(-05)
0.3	3.00786694051106788 59	3.00786694051106799770	1.1180(-16)	3.6449(-07)	2.7853(-05)
0.4	3.10176240577420781 85	3.10176240577420801430	1.9580(-16)	6.1898(-07)	4.2891(-05)
0.5	2.93954310074526207 74	2.93954310074526238920	3.1180(-16)	6.9889(-07)	6.7031(-05)
0.6	2.41183653441571472 55	2.41183653441571519130	4.6580(-16)	1.4794(-06)	1.0264(-04)
0.7	1.39155483048984331 04	1.39155483048984396930	6.5890(-16)	2.1022(-06)	1.4491(-04)
0.8	-0.262326758334357631	- 0.26232675833435674263	8.8837(-16)	2.8409(-06)	1.9091(-04)
0.9	-2.697771160773070925	- 2.69777116077306977980	1.1452(-15)	3.6689(-06)	2.3973(-04)
1.0	-6.058560720845666951	- 6.05856072084566553990	1.4111(-15)	4.5617(-06)	2.9467(-04)

See (Lydia, Joshua, Ndam & James, 2021; Olabode & Momoh, 2016).

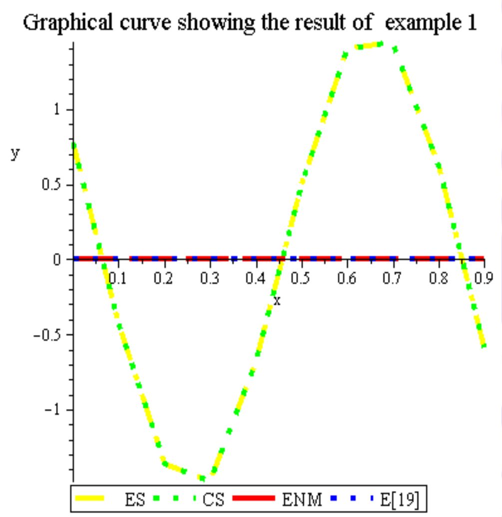

Figure 1: Textual graph of Table 1

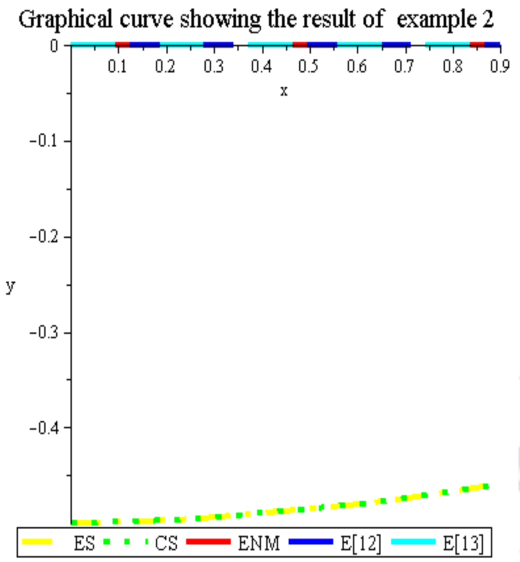

Figure 2: Textual graph of Table 2

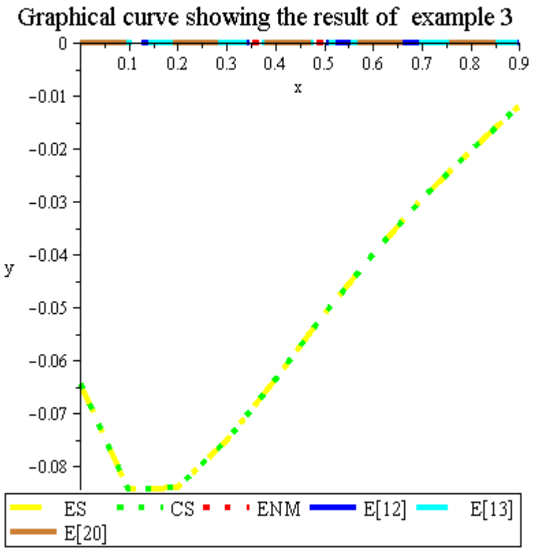

Figure 3: Textual graph of Table 3

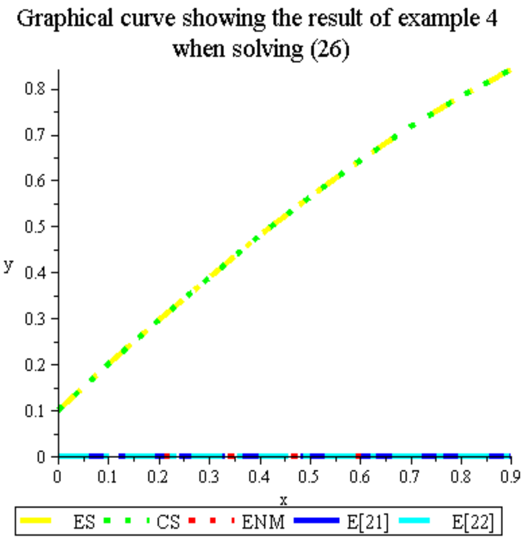

Figure 4: Textual graph of Table 4

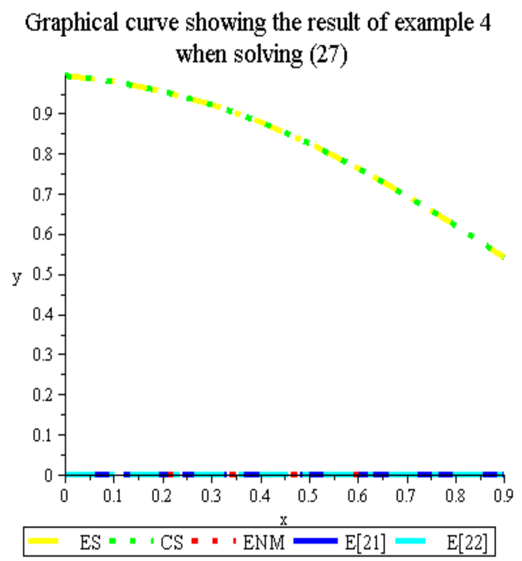

Figure 5: Textual graph of Table 5

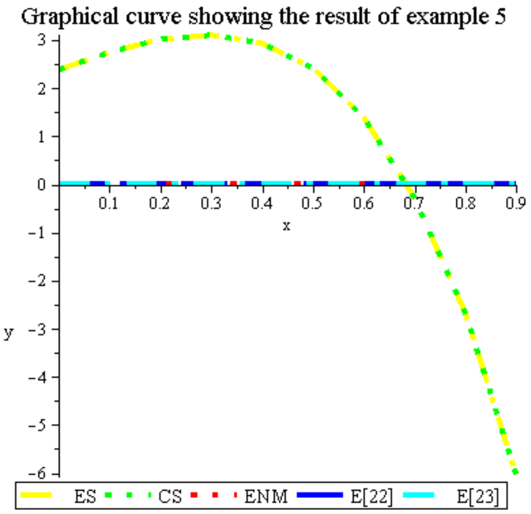

Figure 6: Textual graph of Table 6

6. Discussion

The innovative method was utilized to simulate five distinct oscillatory differential equations. The outcomes, presented in Tables 1 to 6, unequivocally showcase the superior performance of the new method in comparison to the methods it was benchmarked against.

Moreover, the simulation results illustrated in Figures 1 to 6 affirm the efficacy of the new method in simulating oscillatory differential equations. Specifically, when applied to simulate oscillatory differential equations in harmonic motion, the new method demonstrated superior convergence compared to the approach outlined in reference (Sabo, 2021), as evident in Table 1 and Figure 1.

Table 2 offers a comparative analysis of the new method against references (Fatunla, 1980; Skwame, Bakari & Sunday, 2017) when resolving oscillatory differential equations in spring motion. This comparison provides insights into the impact of the external force "F" on the system's behavior.

Likewise, the new method was employed to simulate second-order oscillatory differential equations in mass dynamic motion, Betiss and Stiefel equations, and nonlinear oscillatory differential equations (examples 3 to 5). The findings in Tables 3 to 6 and Figures 3 to 6 undeniably affirm that the new method surpasses the approaches outlined in references (Fatunla, 1980; Skwame, Bakari & Sunday, 2017; Areo &Rufai, 2016; Skwame, Donald, Kyagya, Sabo & Bambur, 2020; Lydia, Joshua, Ndam & James, 2021; Olabode & Momoh, 2016).

In conclusion, both the tabulated results in Tables 1 to 6 and the graphical representations in Figures 1 to 6 validate the efficiency and effectiveness of the new method in handling second-order oscillatory differential equations.

7. Conclusion

This study explored the numerical approximation and practical implementation of oscillations in a moving mass. The development of a novel method, grounded in power series polynomials, underwent rigorous analysis of its properties. The findings reveal that this new method demonstrates computational reliability surpassing the approaches considered for solving analogous oscillatory differential equations.

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