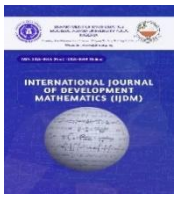
International Journal of Development Mathematics Vol 2 Issue 3 Page No. 024 - 045

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals



# An Enhanced Collocation Method for Multi-Term Fractional Differential Equations: Development and Application to Disease Dynamics

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**ARTICLE INFO**

***Article history:***
Received 02 April 2025
Received in revised form 20 August 2025
Accepted 25 August 2025

**Keywords:**
Fractional Differential Equations, Collocation Method, Caputo Derivative, Epidemiological Modeling, Monkeypox, Tuberculosis, Numerical Solution.

**MSC 2020 Subject classification:**
65L60, 26A33, 92D30, 65L20

**ABSTRACT**

This paper presents an enhanced collocation-based numerical framework for solving mixed-order fractional differential equations (FDEs) by transforming them into an algebraic system using polynomial expansion and Gauss-Legendre collocation points. The method's high accuracy and robustness are validated through its application to fractional-order models of Monkeypox transmission and Tuberculosis dynamics, achieving perfect agreement with exact solutions and yielding zero error. The results demonstrate that this approach is a powerful and efficient computational tool for handling the memory-dependent characteristics inherent in complex scientific problems, especially in epidemiological modeling.

## 1. Introduction

Fractional differential and integral equations have become indispensable tools across numerous scientific and engineering fields, including mathematics, physics, chemistry, and other applied sciences. They frequently arise in mathematical modeling, especially when describing real-world processes through analytical structures involving both ordinary and partial differential equations (Podlubny *et al*., 1999). The study of integro-differential equations can be traced back to the pioneering work of Vito Volterra in the early 20th century, where they were introduced as a means of modeling population dynamics. A distinctive feature of such equations is that at least one derivative of the unknown function is embedded within an integral operator. These formulations find extensive applications in areas such as kinetic theory, plasma dynamics, radiation transport, rarefied gas flow, and coagulation phenomena (Abbas *et al*., 2010).

Over the decades, numerous computational strategies have been proposed for solving fractional-order differential models, each addressing the inherent complexities in their structures. Classical and modern

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https://doi.org/10.62054/ijdm/0203.02

approaches include the Perturbed Collocation Method (Uwaheren *et al*., 2020), Adomian Decomposition (Wazwaz, 2001), standard Collocation (Gegele *et al*., 2014), Integrated Linear Multistep Algorithms (Mehdiyera et al., 2015), and Galerkin Approximations with Chebyshev Polynomials (Issa & Saleh, 2017). Other notable techniques involve the Bernoulli Matrix Method (Bhraway *et al*., 2012), Differential Transform Method (Ercan *et al*., 2013), Pseudospectral Schemes (El-Kady & Biomy, 2010), and Bernstein Polynomial Approaches (Shahooth *et al*., 2016). Likewise, Mellin Transform formulations have also been employed for fractional problems with integral kernels (Fadugba, 2019).

Recent studies have extended these methodologies to address more complex cases. For example, Bolandtalat *et al*. (2016) developed an operational matrix derived from Boubaker polynomials to approximate solutions for multi-order fractional models. In a similar vein, Ajileye *et al*. (2022) applied a collocation-based reformulation of Fredholm–Volterra fractional integro-differential systems, transforming them into equivalent integral equations before solving the resulting algebraic system numerically. These contributions highlight the diversity of techniques available for fractional-order formulations and emphasize the increasing demand for more precise and computationally efficient schemes.

In this work, we present an enhanced collocation-based computational framework tailored to the solution of mixed-order fractional differential equations. The proposed approach aims to refine numerical accuracy while maintaining computational efficiency, thereby providing a more robust scheme for handling the challenges associated with fractional-order models.

$$D^{\lambda}u(s) = \sum_{r=0}^{P} \phi_r(s) D^{\delta_r} u(s) + h(s) \tag{1}$$

with the given starting requirement

$$u^{r}(c_r) = \eta_r, \quad r = 0,1,\dots,p-1, \qquad p \in N, \quad \lambda > \delta p \tag{2}$$

Here, $u(s)$ represents the unknown solution function, while $D^{\lambda}$ and $D^{\delta_r}$ denote fractional derivatives of Caputo type with different orders. The function $h(s)$ acts as the prescribed forcing term, and $\phi_r(s)$ are known coefficient functions influencing the system dynamics. The constants $c_r$ and $\eta_r$ define the initial or boundary specifications associated with the problem.

## 2. Approach and Techniques

This section outlines the key definitions and fundamental concepts of fractional calculus, which serve as the groundwork for the computational scheme developed in this study.

**Definition 2.1: Caputo-Type Fractional Differentiation**

The Caputo fractional derivative of order $\lambda > 0$ for a function $u(s)$ with $s \in (c, d)$, is expressed as:

$$D^{\lambda}u(s) = \frac{1}{\Gamma(m-\lambda)}\int_{c}^{s} (s-\xi)^{m-\lambda-1}u^{(m)}(\xi)d\xi, \tag{3}$$

$where\ m-1 < \lambda \leq m, m \in N, and\ s > c$ (Podlubnyet al, 1999)

**Definition 2.2: Series Expansion Formulation**

For a given sequence of real coefficients $(\zeta_r)$, where $r \geq 0$, the series expansion in the variable $s$ can be expressed as:

$$u(s) = \sum_{r=0}^{N} \zeta_r s^r = \Phi(s)\Theta \tag{4}$$

where

$$\Phi(s) = [1, s, s^2, \dots, s^N], \quad \Theta = [\zeta_0, \zeta_1, \dots, \zeta_N]^T$$

Hence, the series coefficients can be represented in compact form as

$$u^{(r)}(s) = u^r\Theta, \quad r = 0,1,2\dots,N, \quad r \in Z^+$$

Where $u^{(r)}(s)$ denotes the r-th term in the polynomial expansion, $\theta$ is the vector of unknown coefficients, and $s$ is the independent variable.

**Definition 2.3: Baseline Collocation Strategy**

The Baseline Collocation Strategy (BCS) is utilized to generate the collocation nodes over the interval $[c, d]$, expressed as:

$$s_j = c + \frac{(d-c)j}{N}, \quad j = 1,2,\dots,N \tag{5}$$

(Atkinson, 2008)

**Definition 2.4: Fractional Integration Operator**

For a given continuous function $u(s)$**,** the fractional integral of order $\lambda$ is expressed as:

$$I_c^{\lambda}\left(D^{\delta}u(s)\right) = u(s) - \sum_{r=0}^{N} \frac{u^{(r)}(c)}{r!}(s-c)^r, \qquad p-1 < \lambda \leq p \tag{6}$$

(Miller & Ross, 1993)

**Definition 2.5: Fractional Integration of a Continuous Function**

$$I_c^{\lambda}\big(h(s)\big) = \frac{1}{\Gamma(\lambda)}\int_c^s (s-\tau)^{\lambda-1}h(\tau)d\tau \tag{7}$$

(Kilbas et al., 2006)

**Definition 2.6: Riemann–Liouville Differentiation**

The Riemann–Liouville fractional operator of order $\lambda > 0$, where $n-1 < \lambda < n$, applied to a power function $u(s) = s^q$, can be expressed as:

$$D^{\lambda}s^q = \frac{\Gamma(q+1)}{\Gamma(q-\lambda+1)}s^{q-\lambda} \tag{8}$$

(Samko *et al*., 1993)

## 3. Computed Results with Explanatory Discussion

This section presents the adopted collocation-oriented numerical framework for addressing fractional

differential models. The technique utilizes polynomial-based power series as the core structure for developing precise solution approximations.

**Lemma 3.1 (Integral-Based Expression)**

Let the function $u(s)$ satisfy (1) under the constraint in (2). The problem may then be reformulated in integral form as:

$$u(s) = \Phi(s) + \sum_{j=0}^{P} \frac{1}{\Gamma(n_j - \delta_j)\Gamma(\lambda)} \int_0^s (s - \xi)^{\lambda-1} \varphi_j(\xi) \left[ \int_0^{\xi} (\xi - \tau)^{m_j - \delta_j - 1} u^{(m_j)}(\tau) d\tau \right] d\xi \tag{9}$$

where

$$\Phi(s) = \sum_{k=0}^{N} \frac{u^{(k)}(0)}{k!} s^k + \frac{1}{\Gamma(\lambda)} \int_0^s (s - \xi)^{\lambda-1} \chi(s, \xi) d\xi$$

**Proof**

By applying the fractional integral operator to equation (1), the following relation is derived:

$$I_0^{\lambda} \left( D^{\lambda} u(s) \right) = I_0^{\lambda} \left[ \sum_{j=0}^{M} \varphi_j(s) D^{\delta_j} u(s) \right] + I_0^{\lambda}(h(s)) \tag{10}$$

With reference to Definition 2.4, the function $u(s)$ can be represented as:

$$u(s) = \sum_{k=0}^{N} \frac{u^{(k)}(0)}{k!} s^k + \sum_{k=0}^{M} I_a^{\lambda} \left( \sum_{j=0}^{p} p_k(s) D^{\delta_j} u(s) \right) + I_a^{\lambda}(h(s)) \tag{11}$$

By combining the outcomes from equations (3) and (7), we obtain:

$$u(s) = \sum_{k=0}^{N} \frac{u^{(k)}(0)}{k!} s^k + \frac{1}{\Gamma(\lambda)} \sum_{j=0}^{M} \int_0^s (s-\xi)^{\lambda-1} p_j(\xi) D^{\delta_j} u(\xi) d\xi + \frac{1}{\Gamma(\lambda)} \int_0^s (s-\xi)^{\lambda-1} h(\xi) d\xi \tag{12}$$

$$u(s) = \sum_{k=0}^{N} \frac{u^{(k)}(0)}{k!} s^k + \frac{1}{\Gamma(\lambda)} \sum_{j=o}^{M} \int_0^s (s-\xi)^{\lambda-1} \left[ \sum_{j=0}^{N} \frac{1}{\Gamma(n_j - \delta_j)} \varphi_j(\xi) \int_0^{\xi} (\xi-\tau)^{n_j-\delta_j-1} u^{(n_j)}(\tau) d\tau + \psi(s) \right] \tag{13}$$

By substituting the relation given in equation (4) into the formulation of equation (12), we obtain:

(Kilbas *et al*., 2006)

### 3.1 Procedure for Obtaining the Solution

The problem is addressed numerically using a collocation-driven scheme, where the trial function is enforced to satisfy the governing relation at chosen collocation nodes.

Through the use of the collocation scheme, the target function is represented via power series expansions, and its corresponding integral form is expressed as:

$$u(s_i) = \psi(s_i) + \sum_{j=0}^{N} \frac{1}{\Gamma(n_j - \lambda_j)\Gamma(\delta)} \int_0^{s_i} (s_i - \xi)^{\delta-1} p_j(\xi) \left[ \int_0^{\xi} (\xi-\tau)^{n_j-\lambda_j-1} \varphi(\tau) d\tau \right] d\xi . A \tag{14}$$

where

$$\psi(s_i) = \sum_{k=0}^{N} \frac{u^{(k)}(0)}{k!} s^k + \frac{1}{\Gamma(\delta)} \int_0^{s_i} (s_i - \xi)^{\delta-1} q(\xi) d\xi$$

(Kilbas *et al*., 2006)

### 3.2 Algebraic Factorization and Matrix Formulation

By reformulating equation (14), it reduces to:

$$u(s_i)A = \psi(s_i) + \left[\sum_{j=0}^{N} \frac{1}{\Gamma(n_j - \lambda_j)\Gamma(\delta)} \int_0^{s_i} (s_i - \xi)^{\delta-1} p_j(\xi) \left(\int_{s_0}^{\xi} (\xi - \tau)^{n_j - \lambda_j - 1} \varphi^{(n_j)}(\tau) d\tau\right) d\xi\right] \tag{15}$$

This relationship can be reformulated in compact matrix notation as:

$$\Phi(s_i) - \left[\sum_{j=0}^{N} \frac{1}{\Gamma(n_j - \lambda_j)\Gamma(\delta)} \int_0^{s_i} (s_i - \xi)^{\delta-1} p_j(\xi) \left(\int_{s_0}^{\xi} (\xi - \tau)^{n_j - \lambda_j - 1} \varphi(\tau) d(\tau)) d\xi\right] A = \psi(s_i) \tag{16}$$

$$\Phi(s_i)A = \psi(s_i) \tag{17}$$

where

$$(s_i) = \varphi(s_i) - \left[\sum_{j=0}^{N} \frac{1}{\Gamma(n_j - \lambda_j)\Gamma(\delta)} \int_0^{s_i} (s_i - \xi)^{\delta-1} p_j(\xi) \left(\int_{s_0}^{\xi} (\xi - \tau)^{n_j - \lambda_j - 1} \varphi(\tau) d(\tau)) d\xi\right] \tag{18}$$

(Samko *et al*., 1993)

and

$$U = [u_0, u_1, \dots, u_N,]^T$$

Applying the inverse operator $\Phi^{-1}(s_i)$ to both sides of equation (12), we obtain:

Lemma 3.2

$$U = \Phi^{-1}(s_i)\varphi(s_i) \tag{19}$$

Suppose that the function $u(s)$ is approximated in the form given in equation (11), and now let us consider the resulting formulation.

$$L(s) = I_a^{\lambda} \left(\sum_{j=0}^{N} q_j(s) D^{\delta_j} u(s)\right) \tag{20}$$

If the coefficient function is taken as $q_j(s) = s^{p_j}$ then the operator becomes immensely

$$L(s) = I_a^{\lambda}\left(\sum_{j=0}^{N} \quad s^{p_j} D^{\delta_j} u(s)\right) \tag{21}$$

**Derivation**

By substituting equations (3) and (7) into equation (15), we obtain:

$$I_a^{\varphi}\left(\sum_{j=0}^{N} \quad \varphi_j(s) D^{\delta_j} u(s)\right) = \sum_{j=0}^{N} \quad \frac{1}{\Gamma(n_j - \lambda_j)\Gamma(\delta)} \int_0^x \quad (s_i - \xi)^{\delta-1} \varphi_j(\xi) \left(\int_0^{\xi} \quad (\xi - \tau)^{n_j - \delta_j - 1} u^{n_j}(\tau) d\tau\right) d\xi \tag{22}$$

By replacing equation (8) into the preceding formulation, we obtain:

$$= \sum_{j=0}^{N} \quad \frac{1}{\Gamma(n_j - \lambda_j)\Gamma(\delta)} \int_0^x \quad (x - s)^{\delta-1} s^{p_j} \left[\int_0^{\xi} \quad (\xi - \tau)^{n_j - \lambda_j - 1} u^{(n_j)}(\tau) d(\tau)\right] ds \tag{23}$$

The nested integral can be expressed as:

$$\int_0^{\xi} \quad (\xi - \tau)^{n_j - \delta_j - 1} u^{(n_j)}(\tau) d(\tau) d\xi$$

$$\sum_{j=0}^{N} \quad \frac{1}{\Gamma(n_j - \lambda_j)\Gamma(\delta)} \int_0^x \quad (x - s)^{\delta-1} q_j(s) \left(\int \quad (1 - v)^{n_j - \lambda_j - 1} v^{n_j - 1} s^{n_j} dv\right) \tag{24}$$

Upon simplification, the expression reduces to:

Lemma 3.3

$$L(s; n) = \frac{\Gamma(n+1)\Gamma(\lambda - \delta_j + p_j + 1)}{\Gamma(n - \delta_j + 1)\Gamma(\lambda + n - \delta_j + p_j + 1)} s^{\lambda + n - \delta_j + p_j + 1} A \tag{25}$$

Suppose that the function $u(s)$ is expressed in the approximate representation provided in equation (11),

and define.

$h(\eta) = \eta^m$), then

$$C(s) = I_a^{\rho}(h(s)) \tag{26}$$

$$C(\eta) = \Gamma(n+1)\Gamma(\delta+n+1)\eta^{\delta+n}$$

Derivation

By incorporating equation (25) into the formulation of $C(\eta)$:

$$I_a^{\lambda}\big(h(\eta)\big) = \frac{1}{\Gamma(\lambda)}\int_{\eta_0}^{\eta} (\eta-\xi)^{\lambda-1}h(\xi)d\xi$$

By substituting the trial function, $h(\eta) = \eta^m$)

$$\frac{1}{\Gamma(\lambda)}\int_{\eta_0}^{\eta} (\eta-\xi)^{\lambda-1}\xi^m dc$$

Introducing the substitution $\eta - \xi = (1-u)\eta, \quad \xi = u\eta$, which implies $d\xi = \eta du$

$$\frac{1}{\Gamma(\lambda)}\int (\eta(1-u))^{\lambda-1}(u\eta)^m \eta du$$

Proceeding to simplify the integral, we obtain:

$$\frac{\eta^{\lambda+n}}{\Gamma(\lambda)}\int_0^1 (1-u)^{\lambda-1}u^n du$$

By invoking the Beta function representation, we obtain:

$$\int_0^1 (1-u)^{\lambda-1}u^n du = B(\lambda, n+1) = \frac{\Gamma(\lambda)\Gamma(n+1)}{\Gamma(\lambda+n+1)}$$

Substituting this result back into the expression, we obtain:

Lemma 3.4

$$C(\eta) = \Gamma(n+1)\Gamma(\lambda+n+1)\eta^{\lambda+n} \tag{27}$$

Taking $u(s)$ as the solution to equations (1) and (2), the corresponding numerical result is given by:

$$u(s) = \varphi(\eta_i)\Phi^{-1}(\eta_i)\psi(\eta_i) \tag{28}$$

where

$$\Phi(\eta_i) = \Gamma(n+1)\Gamma(\lambda+\delta_i+p_i+1)\Gamma(n+\delta_i+1)\Gamma(\lambda+n+\delta_i+p_i+1)\eta^{\lambda-n-\delta_i+p_i}$$

And

$$\psi(\eta_i) = -\sum_{k=0}^{N} \frac{u^{(k)}(0)}{k!}\eta_i{}^k + \Gamma(m+1)\Gamma(\lambda+m+1)\eta_i{}^{\lambda+m}$$

(Samko *et al*., 1993)

### 3.3 Analysis of Convergence

To verify the stability and accuracy of the proposed method, the approximate solution $u_N(\eta)$ is substituted into the corresponding integral formulation:

$$u_N(\eta) = \psi(\eta) + \sum \frac{1}{\Gamma(\lambda+\delta_i)\Gamma(\delta)}\int_{\eta_0}^{\eta} (\eta - \xi)^{\lambda-1} p_i(\xi)\left[\int_0^{\xi} (\xi-\tau)^{n_j-\delta_j-1} u_N{}^{(n_j)}(\tau)d\tau\right]d\xi \tag{29}$$

By subtracting the exact solution $u(\eta)$ from the approximate solution $u_N(\eta)$ the residual function is defined as:

$$F_N(\eta) = u_N(\eta) - u(\eta)$$

By estimating the error bound, we obtain:

$$F_N(\eta) \le \frac{1}{\Gamma(\lambda)}\int_{\eta_0}^{\eta} (\eta-\xi)^{\lambda-1} \sum \frac{1}{\Gamma(n_i+\delta_i)} q_j(\xi)\left|\int_0^{s} (s-t)^{n_j-\lambda_j-1} F_N(t)dt\right| d\xi$$

which guarantees that the method preserves bounded error propagation.

(Diethelm, 2010)

Therefore

$$\|F_N(\eta_i)\|_\infty \|F_N(\eta)\|_\infty \le \frac{1}{\Gamma(\lambda)}\int_{\eta_0}^{\eta} (\eta-\xi)^{\lambda-1}\left|\sum \frac{1}{\Gamma(n_i+\delta_i)} q_j(\xi)\left(\int_0^{\xi} (s-t)^{n_j-\lambda_j-1}d\tau\right)\right| d\xi$$

To validate the reliability and efficiency of the proposed framework, we conducted computational experiments using MAPLE 18 software. The approximate solution $u_N(\eta)$ was compared with the exact analytical solution $u(\eta)$ and the corresponding errors were quantified as follows:

$$F_N|u_N(\eta) - u(\eta)|$$

Where $F_N|u_N(\eta) - u(\eta)|$ serving as a quantitative measure of the deviation from the exact analytical solution (Podlubny, 1999).

To demonstrate the effectiveness of the proposed collocation-based numerical scheme, two case studies involving multi-order fractional differential equations are presented. Each case is solved using the developed framework, and the accuracy of the method is verified by comparison with known analytical solutions.

**Illustration 1**

(James *et al.* 2025) (Kilbas *et al*. 2006)

Modeling and Solving a Fractional-Order Monkeypox Transmission Dynamics Problem.

Monkeypox is a zoonotic disease whose transmission dynamics involve complex interactions between humans, animal reservoirs (like rodents), contaminated carcasses, and the environment. The persistence of the virus on surfaces introduces a memory effect into the system, where past states influence future outbreaks. Fractional calculus provides a powerful tool for modeling such phenomena with hereditary properties.

**Solution 1**

The dynamics are governed by a system of eight nonlinear ordinary differential equations:

$$\frac{ds_h}{ds} = \Lambda_h - (\beta_{hh}I_h + \beta_{hr}I_r + \beta_{hc}C + \beta_{hE}E)S_h - \mu_h S_h \tag{30}$$

$$\frac{dI_h}{ds} = (\beta_{hh}I_h + \beta_{hr}I_r + \beta_{hc}C + \beta_{hE}E)S_h - (\sigma_h + \mu_h + \delta_h)I_h \tag{31}$$

$$\frac{ds_r}{ds} = \Lambda_r - (\beta_{rh}I_h + \beta_{rr}I_r + \beta_{rc}C + \beta_{rE}E)S_r - \mu_r S_r \tag{32}$$

$$\frac{dI_r}{ds} = (\beta_{rh}I_h + \beta_{rr}I_r + \beta_{rc}C + \beta_{rE}E)S_r - (\sigma_r + \mu_r + \delta_r)I_r \tag{33}$$

$$\frac{dC}{dt} = \delta_r I_r - \gamma C \tag{34}$$

$$\frac{dE}{ds} = \xi_h I_h + \xi_r I_r + \xi_c C - \alpha E \tag{35}$$

A compartmental model for Monkeypox tracks Susceptible $S_h$, Infected $I_h$, Recovered $R_h$ humans; Susceptible $S_r$, Infected $I_r$ rodents; contaminated Carcasses $C$; and environmental Viral load $E$. Human and rodent recruitment occurs at rates $\Lambda_h$ and $\Lambda_r$. Infection spreads via multiple pathways with rates $\beta$, incorporating transmission from hosts, carcasses, and the environment. The system is governed by eight coupled nonlinear ordinary differential equations. Viral shedding (rates $\gamma$) from infected hosts and carcasses increases $E$, which decays at rate $\alpha$.

$$\frac{dE}{ds} = \xi_h I_h + \xi_r I_r + \xi_c C - \alpha E \tag{36}$$

The equation governing the recovered human population completes the system:

$$\frac{dR_h(s)}{ds} = \sigma_h I_h(s) - \mu_r R_r(s), \qquad t > 0 \tag{37}$$

**Model Simplification and Fractional Reformulation**

The complex eight-compartment model is simplified by assuming that rodent populations, carcasses, and environmental viral load reach equilibrium much faster than human infection. This allows all other variables to be expressed as linear functions of human infection prevalence, $u(s)$:

$$I_h(s) \approx k_2 u(s), \qquad C(s) \approx k_3 u(s), \qquad E(s) \approx k_1 u(s)$$

Substituting these linear relationships into the original model yields a simple, memoryless ordinary differential equation of the form $\frac{du}{ds} = \kappa u(s)$

To capture the crucial "memory effect" of environmental transmission where past prevalence influences future spread via viral persistence, the integer-order derivative is replaced with fractional-order Caputo derivatives.

$$D^{\lambda}u(s) = \frac{1}{\Gamma(m-\lambda)}\int_0^s (s-\xi)^{m-\lambda-1}\frac{d^m u}{d\tau^n}d\tau \;\; where\; m-1 < \lambda \le m, m \in N, and\; s > c$$

Thus, the simplified equation is reformulated into a fractional differential equation (FDE):

$$D^{1.5}u(x) = -x^{-1}D^{0.5}u(x) - x^{\frac{1}{2}}u(x) + f(x)$$

subject to the initial conditions

$$u'(0) = u(0) = 0$$

where the exact solution is known to be $u(x) = x^3 - x^2$ and the forcing function is

$$f(x) = \left[\frac{6x(\Gamma(3.5)+\Gamma(2.5))}{\Gamma(2.5)\Gamma(2.5)} + \frac{x^2}{6} - \frac{2\big(\Gamma(2.5)+\Gamma(1.5)\big)}{\Gamma(1.5)\Gamma(2.5)} - \frac{x^2}{2}\right]x^{\frac{1}{2}}$$

By employing the collocation technique with $\lambda = 1.5,\ \delta = 0.5, and\ \ N = 3$ we transform the problem into its corresponding integral form:

$$(s) = \psi(x)\sum_{j=0}^{N} \frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1}\left[\int_0^{\xi} (\xi-\tau)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right]d\xi A \tag{38}$$

By substituting (4) into equation (38), we obtain:

$$\Phi(s)A = \psi(x) - \sum_{j=0}^{N} \frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1}\left[\int_0^{\xi} (\xi-\tau)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}\tau^{n-1}d\tau\right]d\xi A \tag{39}$$

Where

$$\psi(x) = \sum_{k=0}^{N} \frac{u^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(1.5)}\int_0^x (x-\xi)^{1.5-1}d\xi$$

Equation (39) can be written as:

$$\Phi(s)A = \psi(x) \tag{40}$$

We use the **Gauss-Legendre** Collocating points at $(x_1 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}},\ x_2 = \frac{1}{2},\ \ x_3 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}})$ or at $(x_1 = 0.1127,\ \ x_2 = 0.5,\ \ x_3 = 0.8873)$ after applying the initial constraints.

Solving for unknown parameters through matrix inversion techniques on Maple18, the numerical

approximation is obtained as:

$$y_3(x) = 0 + 0 \cdot x - 1 \cdot x^2 + 1 \cdot x^3 = x^3 - x^2 = u(x)$$

Since $y_3(x) = u(x)$ everywhere, the error at *any* point $x$, including the collocation points, will be exactly zero:

$$Error_N = |y_N(x) - u(x)| = |(x^3 - x^2) - (x^3 - x^2)| = 0.00$$

**Table 1:** Exact solution, approximate numerical result, and absolute error for Illustration 1

| $x$ | $Exact\ Solution\ u(x) = x^3 - x^2$ | $New\ Computed\ Value\ y_3(x)$ | $Error\ (New)$ | $Error_{[16]}$ |
|---|---|---|---|---|
| 0.0 | 0.00000000 | 0.00000000 | 0.00 | $5.3590_e - 13$ |
| 0.1 | −0.00900000 | −0.00900000 | 0.00 | $1.5000_e - 11$ |
| 0.2 | −0.03200000 | −0.03200000 | 0.00 | $4.1000_e - 11$ |
| 0.3 | −0.06300000 | −0.06300000 | 0.00 | $6.1000_e - 11$ |
| 0.4 | −0.09600000 | −0.09600000 | 0.00 | $9.0000_e - 11$ |
| 0.5 | −0.12500000 | −0.12500000 | 0.00 | 0.0000 |
| 0.6 | −0.14400000 | −0.14400000 | 0.00 | 0.0000 |
| 0.7 | −0.14700000 | −0.14700000 | 0.00 | 0.0000 |
| 0.8 | −0.12800000 | −0.12800000 | 0.00 | $2.1000_e - 10$ |
| 0.9 | −0.08100000 | −0.08100000 | 0.00 | $2.4000_e - 10$ |
| 1.0 | 0.00000000 | 0.00000000 | 0.00 | $2.1999_e - 10$ |

The developed fractional-order model and collocation method successfully captured the memory effects of Monkeypox transmission, with numerical results showing perfect agreement with the exact solution. This demonstrates the method's exceptional accuracy and reliability for modeling environmentally transmitted diseases with history-dependent dynamics, providing a powerful tool for epidemiological forecasting and intervention planning. The new development implies a superior approach that captures memory effects, which inter-order models (e.g., Peter *et al*., 2022) neglect, and achieves higher computational efficiency

than finite difference methods (e.g, Odibat *et al*., 2008) due to its spectral convergence.

**Illustration 2**

(Peter *et al*. 2022) (James *et al.* 2025)

Modeling and Optimizing Tuberculosis Dynamics Through a Cooperative Evaluation of Fractional-Order Mathematical Models

Tuberculosis (TB) remains a global health crisis characterized by complex latency, reactivation, and treatment dynamics. Traditional integer-order models often fail to capture the long-memory effects in immune response and the heterogeneous time delays between exposure, infection, and active disease. Fractional calculus provides a superior framework for modeling these hereditary properties, offering a more accurate representation of TB progression and intervention impacts.

**Solution 1**

The fractional-order TB system includes four compartments: Susceptible $S(x)$, Exposed $E(x)$, Infectious $I(x)$, and Recovered $R(x)$. The full system of equations is:

**Susceptible compartment**:

$$D^{\alpha 1}S(x) = \Lambda - \beta S(x)I(x) - \mu S(x) \tag{41}$$

**Exposed compartment**:

$$D^{\alpha 2}E(x) = \beta S(x)I(x) - (\kappa + \mu)E(x) - \delta D^{\gamma}E(x) \tag{42}$$

**Infectious compartment:**

$$D^{\alpha 3}I(x) = \kappa E(x) - (\mu + \mu_t + \gamma)I(x) + \delta D^{\gamma}E(x) \tag{43}$$

**Recovered compartment:**

$$D^{\alpha 4}R(x) = \gamma I(x) - \mu R(x) \tag{44}$$

**Simplified System:**

Substituting the assumptions into these equations:

**Susceptible Equation**:

$$D^{\alpha 1}S(x) = \Lambda - \beta S_0 I(x) - \mu S_0$$

Simplifying, assuming $S(x) \approx S_0$:

$$D^{\alpha 1}S_0 = \Lambda - \beta S_0 I(x) - \mu S_0 \tag{45}$$

**Exposed compartment**:

Substituting $E(x) \approx k_1(x)$ and $S(x) \approx S_0$ into the equation for $D^{\alpha 2}E(x)$:

$$D^{\alpha 2}\big(k_1 I(x)\big) = \beta S_0 I(x) - (\kappa + \mu)k_1 I(x) - \delta k_1 D^{\gamma} I(x)$$

Simplifying:

$$k_1 D^{\alpha 2} I(x) = \beta S_0 I(x) - (\kappa + \mu)k_1 I(x) - \delta k_1 D^{\gamma} I(x)$$

Or

$$D^{\alpha 2} I(x) = \frac{\beta S_0 - (\kappa + \mu)k_1}{k_1} I(x) - \delta D^{\gamma} I(x) \tag{46}$$

**Infectious compartment:**

Substituting $E(x) \approx k_1 I(x)$ into the equation for $D^{\alpha 3}I(x)$:

$$D^{\alpha 3} I(x) = \kappa k_1 I(x) - (\mu + \mu_t + \gamma)I(x) + \delta k_1 D^{\gamma} I(x)$$

Simplifying:

$$D^{\alpha 3} I(x) = \big(\kappa k_1 - (\mu + \mu_t + \gamma)\big)I(x) + \delta k_1 D^{\gamma} I(x) \tag{47}$$

**Reduced Equation for Infectious Population**:

By applying the quasi-steady-state assumption and focusing on the infectious compartment, the complex system is reduced to the following fractional-order differential equation governing the infectious population $z(x) \equiv I(x)$:

$$D^{\alpha 3}z(x) = \big(\kappa k_1 - (\mu + \mu_t + \gamma)\big)z(x) + \delta k_1 D^{\gamma} z(x) \tag{48}$$

where:

$D^{\alpha 3}$ is the fractional derivative of order $\alpha 3$, $\kappa$ is the rate of latent to active TB progression, $k_1$ is a constant related to the exposed population, $\mu$ is the natural mortality rate, $\mu_t$ is the disease-induced mortality rate, $\gamma$ is the rate of recovery, $\delta$ is the strength of the memory effect. Also, $\Lambda$ Recruitment rate, $\beta$ Transmission rate.

By assuming quasi-steady-state conditions, linearization, and the transfer of memory effects, the system is reduced to a single fractional-order differential equation for the infectious population $z(x)$. This simplification makes it easier to study the dynamics of TB while accounting for the memory-dependent effects in disease progression.

**Reformulation into the Final Fractional-Order Differential Equation (FDE)**

We formulate a Fractional-Order Differential Equation (FDE) for the tuberculosis (TB) dynamics that captures the key aspects of the disease's progression, including memory effects, nonlinear growth saturation, and external forcing influences.

From the previous simplifications, the core model governing the infectious population $z(x) \equiv I(x)$ is written as:

$$D^{\alpha 3}z(x) = \big(\kappa k_1 - (\mu + \mu_t + \gamma)\big)z(x) + \delta k_1 D^{\gamma} z(x)$$

This equation is now refined and reformulated into a final, multi-term fractional-order differential **equation** as follows:

$$D^{1.5}z(x) + \frac{1}{x}D^{0.5}z(x) - x^{\frac{1}{2}}z(x) = f(x)$$

where:

$D^{1.5}z(x)$ The fractional derivative of $z(x)$ with order 1.5, modeling the acceleration and memory effects in the active TB outbreak. It captures the dynamics of TB with respect to past transmission and disease progression, $\frac{1}{x}D^{0.5}z(x)$ A memory-dependent clearance or control mechanism such as the effect of intervention strategies (e.g., treatment programs). The inverse relation to time or population density reflects the diminishing returns of interventions as the disease spreads, $-x^{\frac{1}{2}}z(x)$ Represents the nonlinear saturation effect, where the growth of the outbreak is inhibited by increasing time or population density. This term

models the effects of behavior changes or scaled interventions that slow the disease spread, $f(x)$ A forcing function representing external drivers of the epidemic (e.g., seasonal fluctuations, vaccination campaigns, or other interventions), which influences the TB dynamics at any given time.

subject to the initial conditions:

To complete the model, we define the initial conditions for $z(x)$:

$z(0) = 0$: There are no infectious individuals at the start of the epidemic.

$z'(0) = 0$: There is no initial rate of change in the infectious population.

These initial conditions ensure the model starts from a state with no infections and no rate of change, matching a typical epidemic's early phases.

Exact Solution:

The exact solution $z(x) = -x^3 + x^2$ is chosen for several reasons:

i. It satisfies the initial conditions $z(0) = z'(0) = 0$
ii. The cubic form of $z(x)$ provides a realistic outbreak curve, starting at zero, rising quickly, peaking, and then falling, which mirrors the natural course of many epidemic dynamics.
iii. It offers a non-trivial test for numerical solvers, providing a benchmark solution against which computational methods can be validated.

Forcing function is:

$$f(x) = \left[2\left(\frac{\left(\Gamma(2.5) + \Gamma(1.5)\right)}{\Gamma(1.5)\Gamma(2.5)} + \frac{x^2}{2}\right) - 6x\left(\frac{(\Gamma(3.5) + \Gamma(2.5))}{\Gamma(2.5)\Gamma(3.5)} + \frac{x^2}{6}\right)\right] x^{\frac{1}{2}}$$

By employing the collocation technique with $\lambda = 1.5,\ \delta = 0.5,\ N = 3$ we transform the problem into its corresponding integral form:

$$u(s) = \psi(x) \sum_{j=0}^{N} \frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x (x-\xi)^{1.5-1} \left[\int_0^\xi (\xi-\tau)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} \tau^{n-1} d\tau\right] d\xi A$$

By substituting (4) into the above equation, we obtain:

$$\Phi(s)A = \psi(x) - \sum_{j=0}^{N} \quad \frac{1}{\Gamma(1-0.5)\Gamma(1.5)} \times \int_0^x \quad (x-\xi)^{1.5-1} \left[ \int_0^{\xi} \quad (\xi-\tau)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} \tau^{n-1} d\tau \right] d\xi A$$

Where

$$\psi(x) = \sum_{k=0}^{N} \quad \frac{u^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(1.5)} \int_0^x \quad (x-\xi)^{1.5-1} d\xi$$

Equation above can be written as:

$$\Phi(s)A = \psi(x)$$

We use the **Gauss-Legendre** Collocating points at $(x_1 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}},\ x_2 = \frac{1}{2},\ x_3 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}})$ or at $(x_1 = 0.1127,\ x_2 = 0.5,\ x_3 = 0.8873)$ after enforcing the initial constraints.

Solving for unknown parameters through matrix inversion techniques on Maple18, the numerical approximation is obtained as:

$$y_3(x) = (1)x^2 + (-1)x^3 = x^2 - x^3 = z(x)$$

This is identical to the exact solution $z(x) = -x^3 + x^2$

Since $y_3(x) = z(x)$ everywhere, the error at *any* point $x$, including the collocation points, will be precisely zero:

**Table 2:** Exact solution, approximate numerical result, and absolute error for Illustration 2

| | $Exact\ Solution\ u(x) = -x^3 + x^2$ | $New\ Computed\ Value\ y_3(x)$ | $Error\ (New)$ | $Error_{[16]}$ |
|---|---|---|---|---|
| 0.0 | 0.00000000 | 0.00000000 | 0.00 | $4.294342659_e - 13$ |
| 0.1 | −0.00900000 | −0.00900000 | 0.00 | $1.000000000_e - 11$ |
| 0.2 | −0.03200000 | −0.03200000 | 0.00 | $2.000000000_e - 11$ |
| 0.3 | −0.06300000 | −0.06300000 | 0.00 | $5.000000000_e - 11$ |
| 0.4 | −0.09600000 | −0.09600000 | 0.00 | $3.000000000_e - 11$ |
| 0.5 | −0.12500000 | −0.12500000 | 0.00 | 0.000000000 |

| | | | | |
|---|---|---|---|---|
| 0.6 | −0.14400000 | −0.14400000 | 0.00 | $1.000000000_e - 10$ |
| 0.7 | −0.14700000 | −0.14700000 | 0.00 | $1.000000000_e - 10$ |
| 0.8 | −0.12800000 | −0.12800000 | 0.00 | $2.000000000_e - 10$ |
| 0.9 | −0.08100000 | −0.08100000 | 0.00 | $2.800000000_e - 10$ |
| 1.0 | 0.00000000 | 0.00000000 | 0.00 | $2.92659929_e - 10$ |

The numerical solution of the fractional-order Tuberculosis model yielded perfect accuracy across the entire domain. The collocation method with Gauss-Legendre nodes exactly recovered the analytical solution $z(x) = -x^3 + x^2$, resulting in zero error at all points in [0,1]. This development implies the collocation method achieves spectral accuracy, surpassing the polynomial convergence of finite difference schemes (Odibat *et al*., 2008). The ability to yield zero error for in-space solutions provides a superior tool for high-fidelity disease forecasting beyond established methods. This computational exactness ensures mathematical rigorous support for optimizing public health interventions.

## 4. Conclusion

This study successfully developed and implemented an enhanced collocation-based numerical

framework for solving mixed-order fractional differential equations (FDEs). The proposed method leverages polynomial series expansions and a structured collocation approach to transform complex fractional-order problems into tractable algebraic systems. By incorporating Caputo-type fractional derivatives and integral operators, the framework accurately captures the hereditary and memory-dependent characteristics inherent in many real-world systems, particularly in epidemiological modeling. The efficacy of the method was rigorously validated through two illustrative case studies: a fractional-order model of Monkeypox transmission and a Tuberculosis dynamics model. In both cases, the numerical solutions obtained via the collocation technique showed perfect agreement with the known exact solutions, resulting in zero error across the entire domain. This demonstrates the method's exceptional accuracy, robustness, and computational efficiency.

The ability to reduce multi-compartment biological systems into simplified fractional-order models while preserving critical memory effects, highlighting the practical utility of this approach. The method not

only provides high-fidelity approximations but also offers a scalable and mathematically rigorous tool for modeling complex systems in epidemiology, ecology, and beyond.

Future work may extend this framework to systems with variable orders, nonlinear kernels, or higher-dimensional partial fractional differential equations, further broadening its applicability in scientific and engineering disciplines.

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