International Journal of Development Mathematics Vol 2 Issue 4 Page No. 165 - 178

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals

Two Step Block Method for the Solution of Nonlinear Second Order Singular Boundary Value Problem of Ordinary Differential Equations

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ARTICLE INFO

Article history:
Received 01 August 2025
Received in revised form 20 October 2025
Accepted 10 November 2025

Keywords:
Block method, Interpolation, Collocation, Singular boundary value problems

MSC 2020 Subject classification:
65L04, 65L05

ABSTRACT

This paper presents a two-step second derivative block method for the numerical solution of nonlinear second-order singular boundary value problems (SBVPs), which are prevalent in the fields like fluid dynamics and quantum mechanics but challenging to solve with standard techniques due to accuracy loss near singular points. The proposed method formulated using interpolation and collocation at selected grid points with an approximate solution incorporating power series and Legendre polynomial, resulting in a continuous linear multistep scheme that is then discretized for block implementation. A rigorous theoretical analysis proves that the method is of order seven, consistent, zero-stable, convergent and A-stable. Numerical experiments on several nonlinear SBVPs, including applications from physiology and stellar structure, demonstrate the method's high effectiveness and liability. The results confirm that the new approach achieves superior accuracy compared to exiting methods in the literature, establishing it as an efficient and robust computational tool.

1. Introduction

In the fields of science and engineering, mathematical models are systematically constructed to facilitate a deeper understanding of physical phenomena. Such models frequently give rise to equations involving derivatives of an unknown function of one or more independent variables; these are referred to as Differential Equations (DEs) (Adesanya, *et al.*, 2012; Ogunniran, *et al.*, 2022; Kida, et al., 2022; Adamu *et al.*, 2025). Since differential equations modeling physical processes rarely have closed-form analytical solutions, the development of numerical methods for obtaining approximate solutions is essential (Jiang, et al., 2018). A variety of numerical techniques such as finite difference methods, exponentially fitted methods (Kumar, et al., 2023). and Nystrom methods (Yahaya, et al., 2021), have been devised, with the choice of method often dictated by the structure and properties of the underlying differential equation. Both linear and nonlinear differential equations play a fundamental role across diverse disciplines, including engineering, the natural sciences, and the social sciences.

This study proposes a Second Derivative Block Method (SDBM) for the numerical solution of singular boundary value problems of ordinary differential equations (ODEs). The method is constructed using a collocation and interpolation framework, employing a combination of two basis functions to approximate the solution. The proposed approach is designed to solve singular boundary value problems with high accuracy and efficiency, and a rigorous analysis of its convergence and stability is also presented.

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https://doi.org/10.62054/ijdm/0204.11

The general form of second order singular differential equations is denoted as:

$$v''(\zeta)+\frac{R(\zeta)}{S(\zeta)}v'(\zeta)+\tau(\zeta)v(\zeta)=g(\zeta,v); \quad a\le\zeta\le b \tag{1}$$

with Dirichlet boundary conditions,

$$v(a)=\alpha,\ v(b)=\beta \tag{2}$$

Equation (1) and (2) combined are called Boundary Value Problems (BVPs), while equation (1) is singular at $S(\zeta)=0$, $\tau(\zeta), R(\zeta)$, $g(\zeta,v)$ are also continuously differentiable functions in [a, b] and $\tau(\zeta);\ \ g(\zeta,v)$ are non-linear continuous functions, the parameters α and b are real constants. It is well known that some of these problems have proved to be either difficult to solve or cannot be solved analytically due to its singularity as the approximate solutions lost their accuracy in the neighborhood of the singular points; hence the need to derive numerical methods for the solution of such problems becomes necessary.

Singular boundary value problems (SBVPs) of ordinary differential equations (ODEs) frequently arise in diverse areas of science and engineering, including fluid dynamics, chemical kinetics, and quantum mechanics (Omar, et al., 2012; Jiang, et al., 2018). A defining feature of these problems is the occurrence of a singularity at one or both endpoints of the domain, which complicates the process of obtaining accurate numerical solutions. Standard numerical techniques, such as finite difference methods and shooting strategies, often exhibit reduced accuracy or fail altogether when applied to such problems due to the inherent singular behavior (Jiang, et al., 2018; Ramos, 2020).

The Second Derivative Block Method (SDBM) is derived from the concept of block methods formulated within the framework of Linear Multistep Methods (LMM). Second derivative methods are well known for their stability, convergence, and suitability for numerical integration (Jiang, et al., 2018; Kumar, et al., 2023). Moreover, they retain the traditional advantages of one-step methods, including being self-starting and allowing for straightforward step-size adjustments (Adesanya, et al., 2012; Adamu, et al., 2019; Adamu *et al.*, 2025). Block methods offer additional benefits over predictor--corrector approaches, as they are computationally less expensive in terms of function evaluations, provide simultaneous approximations at multiple grid points without overlap (as occurs in predictor--corrector schemes), and efficiently generate solutions at all required points (Adesanya, et al., 2012; Kida, et al., 2022; . Jiang, et al., 2018).

Many applied mathematicians and physicists have devoted significant attention to the study of second order singular problems in ordinary differential equations (ODEs). Various researchers have addressed different aspects of such equations. For example, (Hasan & Zhu, 2007) investigated singular initial value problems both linear and nonlinear, homogeneous and non-homogeneous using the Taylor series method. In (Jiang, et al., 2018), nonlinear singular boundary value problems of the first order with singularities of the first kind were examined. A seventh-order numerical method for solving singularly perturbed two-point boundary value problems was presented in (Chakravarthy, et al., 2007), while (Yahaya, et al., 2021) proposed a two-step hybrid block method with four off-step points for singular initial and boundary value problems.

Furthermore, hybrid Simpson's block methods for stiff systems were developed in (Fotta, et al., 2015; Biala, et al., 2015; Omar, et al., 2012) for solving first-order initial value problems (IVPs) of ODEs. In (Ramos, 2020), a continuous linear multistep method was introduced for general second-order IVPs, whereas (Anake, et al., 2012) proposed a one-step implicit hybrid block method for solving IVPs of general second-order ODEs. Two-step second derivative methods for stiff systems were discussed in (Joshua, 2022), and a new class of second-derivative implicit linear multistep methods was formulated in (Khalsaraei, et al., 2012). Additionally, (Ngwane & Jator, 2012) developed a Hybrid Second Derivative Method (HSDM) with two off-step points, while (James, et al., 2013) introduced a hybrid block method for second-order IVPs with constant step sizes. Finally, (Abba, 2019) formulated several second-derivative methods for approximating stiff systems.

2. Preliminaries

Definition 1.1 (Lambert, 1973)

A block method is said to be A-stable if the whole of the left-half plane {z: Re(z) $\leq$ 0} is contained in the region {z: $|\mathrm{Re}(z)|$ $\leq$ 1}, where R(z) is called the stability polynomial of the method.

Definition 1.2 (Adesanya, et al., 2012)

A block method is said to be $A(\alpha)-$ stable for some $\alpha \subset \left[0, \frac{\pi}{2}\right]$ if the wedge

$$S_\alpha = \{z \ : \ | \, Arg(-z) \, | < \alpha, z \neq 0\} \tag{3}$$

is contained in its region of absolute stability. The largest $\alpha(i.e. \, \alpha_{\max})$ is called the angle of absolute stability.

Definition 1.3 (Kida, et al., 2022)

A solution $v(\zeta)$ is said to be stable if given any $\varepsilon > 0$ there exists $\delta > 0$ such that any other solution $v(\zeta)$ satisfies

$$| \, \mathbf{v}(\alpha) - \bar{\mathbf{v}}(\zeta) | \leq \delta \tag{4}$$

also satisfies

$$| \, \mathbf{v}(\alpha) - \bar{\mathbf{v}}(\zeta) | \leq \varepsilon \tag{5}$$

The solution $v(\zeta)$ is asymptotically stable if in addition to equation (4) $| \, v(\alpha) - \bar{\mathbf{v}}(\zeta) | \to \mathbf{0}$ as $\zeta \to \infty$

Theorem 1.1. Consistency (Adesanya, et al., 2014): A block method is said to be consistent if it has order p $\geq$ 1.

Theorem 1.2. Zero-stable (Adesanya, et al., 2014): A block method is said to be zero stable if as $h \to 0$, the roots $r_j; \quad j = 1(1)k$ of the first characteristic polynomials $\rho(r) = 0$ that is

$$\rho(r) = \det | \sum A^{(0)} R^{k-1} | = \mathbf{0} \tag{6}$$

Satisfying $| \, \mathbf{R} \, | \leq \mathbf{1}$ must be simple.

Theorem 1.3. Convergent (Adesanya, et al., 2014): Consistent and zero stability are sufficient conditions for a block method to be convergent.

3. Methods and Materials

We consider an approximate solution in the form

$$v(\zeta) = \sum_{n=0}^{k} \alpha_n \zeta^n + \sum_{n=0}^{k} \alpha_n P_n(\zeta) \qquad a \leq \zeta \leq b \tag{7}$$

where, $n = 0,1,2,\ldots,k$ are unknown coefficients to be determined and $P_n(\zeta)$ indicates the Legendre polynomials by Rodrigue formula which are originally studied in 1970 by Bicknell.

$$P_n(\zeta) = \frac{1}{2^n n!} \frac{d^n}{d\zeta^n} \left(\zeta^2 - 1\right)^n \quad n = 0 \tag{8}$$

By using (8), the first and second Legendre polynomials respectively are given by

$P_0 = 1,$

$P_1 = \zeta,$

$P_2 = \frac{1}{2}\left(3\zeta^2 - 1\right),$

The first and second derivative of (7) give,

$$v'(\zeta) = \sum_{n=1}^{k} n a_n \zeta^{n-1} + \sum_{n=1}^{k} a_n P_n'(\zeta) \tag{9}$$

$$v''(\zeta) = \sum_{n=2}^{k} n(n-1) a_n \zeta^{n-2} + \sum_{n=2}^{k} a_n P_n''(\zeta) \tag{10}$$

$k = s + r - 1$ where s is the number of interpolation points and r is the number of collocation points respectively. Interpolating (7) at ζ_{n+k}, $k = 0,1,2,3,\ldots,r$ and collocating (9) and (10) at ζ_{n+k}, $k = 0,1,2,3,\ldots,s$ gives the system of nonlinear equation of the form,

$$\zeta A = U \tag{11}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_k \end{bmatrix}^{\tau}$$

$$U = \begin{bmatrix} v_n & v_{n+1} \ldots & v_{n+r,} & g_n & g_{n+1} \ldots & g_{n+s} \end{bmatrix}^{\tau}$$

and imposing the following conditions on (7)

$$v(\zeta_{n+j}) = v_{n+j},\ j = 0,1,\ldots,k$$
$$v''(\zeta_{n+j}) = g_{n+j},\ j = 0,1,\ldots,k$$

$$\zeta \begin{bmatrix} 1 & \zeta_n & \zeta_n^2 & \zeta_n^3 & \cdots & \zeta_n^m \\ 1 & \zeta_{n+1} & \zeta_{n+1}^2 & \zeta_{n+1}^3 & \cdots & \zeta_{n+1}^m \\ & \vdots & \vdots & & \ddots & \vdots \\ 1 & \zeta_{n+k} & \zeta_{n+k}^2 & \zeta_{n+k}^3 & \cdots & \zeta_{n+k}^m \\ 0 & 0 & 2 & 6\zeta_n & \cdots & m(m-1)a_m\zeta_n^{m-2} \\ 0 & 0 & 2 & 6\zeta_{n+1} & \cdots & m(m-1)(m-2)a_m\zeta_{n+1}^{m-3} \\ & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 2 & 6\zeta_{n+k} & \cdots & m(m-1)(m-2)a_m\zeta_{n+k}^{m-3} \end{bmatrix}$$

Solving the system (12) for the unknown parameter using Crammer's rules gives a Continuous LMM in the form

$$v_{n+\tau} = v_n + hv_n' + h\sum_{j=0}^{k}\alpha_j(\tau)\varpi_{n+j} + h^2\sum_{j=0}^{k}\beta_j(\tau)g_{n+j} \tag{12}$$

where $\alpha_j(\tau)$ and $\beta_j(\tau)$ are polynomial of degree $r+s-1$ and $\tau = \dfrac{\zeta-\zeta_n}{h}$. Evaluate (12) at $\zeta_{n+\tau}$ to give the discrete methods

$$v_{n+\tau} = \alpha_j v_{n+i} + h^2\sum_{j=0}^{k}\beta_j g_{n+i} \qquad i=0,..,2;\ j=0,1,...,2. \tag{13}$$

Evaluating (13) at $\tau = 1,2,...,k$ can be implemented in a block in the form

$$A^{(0)}v_m = A^{(1)}v_{m-1} + h^2\left[\gamma^{(0)}G_m + \gamma^{(1)}G_{m-1}\right] \tag{14}$$

where

$$v_m = \begin{bmatrix} v_{n+1} & v_{n+2} & \dots & v_{n+k}\end{bmatrix}^{\tau}, v_{m-1} = \begin{bmatrix} v_{n-1} & v_{n-2} & \dots & v_n\end{bmatrix}^{\tau}$$

$$G_m = \begin{bmatrix} g_{m+1} & g_{m+2} & \dots & g_{m+k}\end{bmatrix}^{\tau}, G_{m-1} = \begin{bmatrix} g_{m-1} & g_{n-2} & \dots & g_n\end{bmatrix}^{\tau}$$

$A^{(0)}$, $\beta^{(0)}$, and $\zeta^{(0)}$ are $k\times k$ identity matrix, $A^{(1)}$, $\beta^{(1)}$, and $\zeta^{(1)}$ are $k\times k$ matrix where b_n and $a_n \in \mathrm{R}$, are constants to be determined.

We seek approximation within the interval of integration $\pi_N < \zeta_1 < \zeta_2 < \cdots < \zeta_{N+1} < \zeta_N = b$ with a constant step length given by $h = \zeta_{j+1} - \zeta_j, j = 0,1,\cdots,N-1.$ r and s are chosen such that $r+s=k-1,$ where k is the sum of the collocation and interpolation points.

4. Specification of the method

We will consider a two-step method with grids $c,d,e,l,w\ and\ q.$ $c<d<e<l<w<q$ In (7), let $\rho = 8$ hence the approximate solution reduces to

$$v(\zeta) = \sum_{i=0}^{8} a_i\zeta^i + \sum_{i=0}^{8} a_i P_i(\zeta) \tag{15}$$

yield this

$$\begin{aligned} y(\zeta) = {} & 235\alpha_0 + 24\alpha_1\zeta - 580\alpha_2\zeta^2 + 1848\alpha_3\zeta^3 + 5098\alpha_4\zeta^4 \\ & -4408\alpha_5\zeta^5 - 10036\alpha_6\zeta^6 + 3560\alpha_7\zeta^7 + 6563\alpha_8\zeta^8 \end{aligned} \tag{16}$$

$$\begin{aligned} y''(\zeta) = {} & -1160\alpha_2 + 1088\alpha_3\zeta + 61176\alpha_4\zeta^2 - 88160\alpha_5\zeta^3 \\ & -301080\alpha_6\zeta^4 + 149520\alpha_7\zeta^5 + 367\,528\alpha_8\zeta^6 \end{aligned} \tag{17}$$

Evaluating (16) at points ζ_n, ζ_{n+c} and (17) at points $\zeta_n, \zeta_{n+c}, \zeta_{n+d}, \zeta_{n+e}, \zeta_{n+l}, \zeta_{n+w}, \zeta_{n+q}$ for $c, d, e, l, w \text{ and } q \in \mathrm{R},$ where $c < d < e < l < w < q$ (15) reduces to (11)
where

$$A = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \end{bmatrix}^{\tau}$$

$$U = \begin{bmatrix} y_n & y_{n+c} & f_n & f_{n+c} & f_{n+d} & f_{n+e} & f_{n+l} & f_{n+w} & f_{n+q} \end{bmatrix}^{\tau}$$

$$\zeta = \begin{bmatrix} 235 & 24\zeta_n & -580\zeta_n^2 & 1848\zeta_n^3 & 5098\zeta_n^4 & -4408\zeta_n^5 & -10\,036\zeta_n^6 & 3560\zeta_n^7 & 6563\zeta_n^8 \\ 235 & 24\zeta_{n+c} & -580\zeta_{n+c}^2 & 1848\zeta_{n+c}^3 & 5098\zeta_{n+c}^4 & -4408\zeta_{n+c}^5 & -10\,036\zeta_{n+c}^6 & 3560\zeta_{n+c}^7 & 6563\zeta_{n+c}^8 \\ 0 & 0 & -1160 & 11088\zeta_n & 61176\zeta_n^2 & -88160\zeta_n^3 & -301080\zeta_n^4 & 149\,520\zeta_n^5 & 367\,528\zeta_n^6 \\ 0 & 0 & -1160 & 11088\zeta_{n+c} & 61176\zeta_{n+c}^2 & -88160\zeta_{n+c}^3 & -301080\zeta_{n+c}^4 & 149\,520\zeta_{n+c}^5 & 367\,528\zeta_{n+c}^6 \\ 0 & 0 & -1160 & 11088\zeta_{n+d} & 61176\zeta_{n+d}^2 & -88160\zeta_{n+d}^3 & -301080\zeta_{n+d}^4 & 149\,520\zeta_{n+d}^5 & 367\,528\zeta_{n+d}^6 \\ 0 & 0 & -1160 & 11088\zeta_{n+e} & 61176\zeta_{n+e}^2 & -88160\zeta_{n+e}^3 & -301080\zeta_{n+e}^4 & 149\,520\zeta_{n+e}^5 & 367\,528\zeta_{n+e}^6 \\ 0 & 0 & -1160 & 11088\zeta_{n+l} & 61176\zeta_{n+l}^2 & -88160\zeta_{n+l}^3 & -301080\zeta_{n+l}^4 & 149\,520\zeta_{n+l}^5 & 367\,528\zeta_{n+l}^6 \\ 0 & 0 & -1160 & 11088\zeta_{n+w} & 61176\zeta_{n+w}^2 & -88160\zeta_{n+w}^3 & -301080\zeta_{n+w}^4 & 149\,520\zeta_{n+w}^5 & 367\,528\zeta_{n+w}^6 \\ 0 & 0 & -1160 & 11088\zeta_{n+q} & 61176\zeta_{n+q}^2 & -88160\zeta_{n+q}^3 & -301080\zeta_{n+q}^4 & 149\,520\zeta_{n+q}^5 & 367\,528\zeta_{n+q}^6 \end{bmatrix}$$

We considered grid point: $c = \frac{1}{3}, d = \frac{2}{3}, e = 1, l = \frac{4}{3}, w = \frac{5}{3}$ and $q = 2,$ using the continuous LMM in (12) we obtained

$$v_{n+\tau} = \alpha_0(\tau)v_n + \alpha_{\frac{1}{3}}(\tau)v_{n+\frac{1}{3}} + h^2 \begin{bmatrix} b_0(\tau)f_n + b_{\frac{1}{3}}(\tau)f_{n+\frac{1}{3}} + b_{\frac{2}{3}}(\tau)f_{n+\frac{2}{3}} + b_1(\tau)f_{n+1} \\ + b_{\frac{4}{3}}(\tau)f_{n+\frac{4}{3}} + b_{\frac{5}{3}}(\tau)f_{n+\frac{5}{3}} + b_2(\tau)f_{n+2} \end{bmatrix} \quad (18)$$

where

$$\alpha_0(\tau) = -(3\tau - 1)$$

$$\alpha_{\frac{1}{3}}(\tau) = 3\tau$$

$$b_0(\tau) = \frac{1}{362\,880}h^2\tau(3\tau - 1)\begin{pmatrix} -95\,793\tau + 157\,149\tau^2 - 142\,425\tau^3 \\ +72\,819\tau^4 - 19\,683\tau^5 + 2187\tau^6 + 28\,549 \end{pmatrix}$$

$$b_{\frac{1}{3}}(\tau) = -\frac{1}{60\,480}h^2\tau(3\tau - 1)\begin{pmatrix} -28875\tau + 94815\tau^2 - 110187\tau^3 \\ +64\,071\tau^4 - 18\,711\tau^5 + 2187\tau^6 - 9625 \end{pmatrix}$$

$$b_{\frac{2}{3}}(\tau) = \frac{1}{40\,320}h^2\tau(3\tau - 1)\begin{pmatrix} -17151\tau + 99\,747\tau^2 - 143019\tau^3 \\ +93\,717\tau^4 - 29\,565\tau^5 + 3645\tau^6 - 5717 \end{pmatrix}$$

$$b_1(\tau)=-\frac{1}{90720}h^2\tau(3\tau-1)\begin{pmatrix}-31863\tau+206811\tau^2-339687\tau^3\\+246483\tau^4-83835\tau^5+10935\tau^6-10621\end{pmatrix}$$

$$b_{\frac{4}{3}}(\tau)=\frac{1}{120960}h^2\tau(3\tau-1)\begin{pmatrix}-23109\tau+157473\tau^2-276021\tau^3\\+216351\tau^4-78975\tau^5+10935\tau^6-7703\end{pmatrix}$$

$$b_{\frac{5}{3}}(\tau)=-\frac{1}{20160}h^2\tau(3\tau-1)\begin{pmatrix}-1209\tau+8469\tau^2-15417\tau^3\\+12717\tau^4-4941\tau^5+729\tau^6-403\end{pmatrix}$$

$$b_2(\tau)=\frac{1}{362880}h^2\tau(3\tau-1)\begin{pmatrix}-2985\tau+21285\tau^2-39717\tau^3\\+33939\tau^4-13851\tau^5+2187\tau^6-995\end{pmatrix}.$$

Evaluating (18) at $\tau=1\left(\frac{3}{2}\right)2$ yields

$$v_{n+1}=-2v_n+3v_{n+\frac{1}{3}}+h^2\begin{bmatrix}\frac{2803}{181440}f_n+\frac{1265}{6048}f_{n+\frac{1}{3}}+\frac{1657}{20160}f_{n+\frac{2}{3}}+\frac{1777}{45360}f_{n+1}\\-\frac{1049}{60480}f_{n+\frac{4}{3}}+\frac{11}{2016}f_{n+\frac{5}{3}}-\frac{137}{181440}f_{n+2}\end{bmatrix} \tag{19}$$

$$v_{n+\frac{3}{2}}=\frac{1}{2}\left[-7v_n+9v_{n+\frac{1}{3}}\right]+h^2\begin{bmatrix}\frac{23681}{884736}f_n+\frac{275251}{737280}f_{n+\frac{1}{3}}+\frac{112003}{491520}f_{n+\frac{2}{3}}+\frac{241903}{1105920}f_{n+1}\\+\frac{31609}{1474560}f_{n+\frac{4}{3}}+\frac{1969}{245760}f_{n+\frac{5}{3}}-\frac{5099}{4423680}f_{n+2}\end{bmatrix} \tag{20}$$

$$v_{n+2}=-5v_n+6v_{n+\frac{1}{3}}+h^2\begin{bmatrix}\frac{1375}{36288}f_n+\frac{3259}{6048}f_{n+\frac{1}{3}}+\frac{1489}{4032}f_{n+\frac{2}{3}}+\frac{3751}{9072}f_{n+1}\\+\frac{2059}{12096}f_{n+\frac{4}{3}}+\frac{265}{2016}f_{n+\frac{5}{3}}+\frac{199}{36288}f_{n+2}\end{bmatrix}. \tag{21}$$

Considering equations (19), (20) and (21) to obtained a discrete block of the form

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_{n+1} \\ v_{n+\frac{3}{2}} \\ v_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 3 \\ 0 & -\frac{7}{2} & \frac{9}{2} \\ 0 & -5 & 6 \end{bmatrix}\begin{bmatrix} v_{n-\frac{4}{3}} \\ v_n \\ v_{n+\frac{1}{3}} \end{bmatrix} + h^2\begin{bmatrix} 0 & 0 & \frac{2803}{181\,440} \\ 0 & 0 & \frac{23\,681}{884\,736} \\ 0 & 0 & \frac{1375}{36\,288} \end{bmatrix}\begin{bmatrix} f_{n-\frac{4}{3}} \\ f_{n-\frac{1}{3}-} \\ f_n \end{bmatrix}
$$

$$
+h^2\begin{bmatrix} \frac{1265}{6048} & \frac{275\,251}{737\,280} & \frac{3259}{6048} \\ \frac{1657}{20\,160} & \frac{112\,003}{491\,520} & \frac{1489}{4032} \\ \frac{1777}{45\,360} & \frac{241\,903}{1105\,920} & \frac{3751}{9072} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{bmatrix} \tag{22}
$$

$$
+h^2\begin{bmatrix} -\frac{1049}{60\,480} & \frac{31\,609}{1474\,560} & \frac{2059}{12\,096} \\ \frac{11}{2016} & \frac{1969}{245\,760} & \frac{265}{2016} \\ -\frac{137}{181\,440} & -\frac{5099}{4423\,680} & \frac{199}{36\,288} \end{bmatrix}\begin{bmatrix} f_{n+\frac{4}{3}} \\ f_{n+\frac{5}{3}} \\ f_{n+2} \end{bmatrix}
$$

5. Analysis of the method

In this subsection, the analysis of basic properties of the newly derived methods shall be carried out. These properties include local truncation error, order, consistence, zero-stability, convergence, linear stability and stability regions.

5.1 Order of the Method (James, et al., 2013; Omar, et al., 2012)

The operator L is associated with the linear method defined by

$$
L[v(\zeta):\ h] = v_{n+\tau} - \sum_{j=0}^{k}\alpha_j(\tau)v_{n+j} + h\sum_{j=0}^{k}\beta_j(\tau)\varpi_{n+j} + h^2\sum_{j=0}^{k}\gamma_j(\tau)g_{n+j} \tag{23}
$$

where $v(\zeta)$ is an arbitrary function, continuously differentiable on interval of integration. Equation (23) can be written in Taylor expansion about the point ζ to obtain

$$
L[v(\zeta):\ h] = c_0 v(\zeta) + c_1 h v'(\zeta) + c_2 h^2 v''(\zeta) + \ldots + c_p h^p v^p(\zeta) \tag{24}
$$

where

$$
c_p = \frac{1}{p!}\left[\sum_{j=1}^{r} j^p\theta_j - \frac{1}{(p-1)!}\sum_{j=1}^{r} j^{p-1}\gamma_j - \frac{1}{(q-2)!}\sum_{j=1}^{r} j^{q-2}g_j\right]
$$

Equation (24) is of order p if

$$
L[v(\zeta):\ h] = 0\left(h^{p+2}\right), c_0 = c_1 = \ldots = c_p = c_{p+1} = 0, c_{p+2} \neq 0
$$

Hence c_{p+2} is called the error constant and $c_{p+2}h^{p+2}v^{(p+2)}(\zeta)$ is called the Local Truncation Error $(L\tau E)$

Thus, two-step second derivative methods with equidistant hybrid point spacing are of order seven on the application of the procedure above. The LTE for two-step second derivative method with equidistant hybrid point spacing is given by

$$L\tau E = \left[\frac{349}{1190\,427\,840}h^9, \frac{28\,061}{58\,047\,528\,960}h^9, \frac{349}{595\,213\,920}h^9\right]^\tau \tag{25}$$

5.2 Stability of the method (James, et al., 2013; Kida, et al., 2022)

The linear stability is derived by applying the test equation $v^{(k)} = \lambda^{(k)} v_n$ to yield $v_{n+1} = \mu(z) v_m$, $\mu(z)$ is the amplification equation given by

$$\mu(z) = -\left(A^{(1)} - z\beta^{(1)} - z^2\zeta^{(1)}\right)^{-1}\left(A^{(0)} + z\beta^{(0)}\right) \tag{26}$$

the matrix $\mu(z)$ has eigenvalues $(0,0,\ldots,\xi_k)$ where ξ_k is called the stability function which is the rational function with coefficients. Where $A^{(0)}, A^{(1),} B^{(0)}, B^{(1)}$ and $\gamma^{(1)}$ are from (22) is given as

$$\mu(z) = -\frac{\begin{array}{c} 5381\,327\,831\,400z^5 + 117\,670\,706\,371\,080z^4 \\ +1058\,969\,620\,775\,034z^3 + 2757\,054\,990\,646\,656z^2 \\ -6367\,709\,887\,297\,920z - 15\,002\,210\,669\,467\,392 \end{array}}{\begin{array}{c} 942\,508\,819\,875z^6 + 75161\,520\,406175z^5 \\ -1011\,416152\,302\,710z^4 + 4846\,898\,757\,098\,664z^3 \\ -14\,649\,798\,474\,692\,640z^2 + 23\,636\,711\,451\,636\,864z \\ -15\,002\,210\,669\,467\,392 \end{array}}, 0$$

$\mu(z)$ is called the stability function.

5.3. Convergence

The necessary and sufficient conditions for the second derivative block methods to be convergent are that they must be consistent and zero-stable. Hence, the two-step second derivative method with non-equidistant hybrid point spacing was convergent since it is consistent and zero-stable. (Joshua, 2022)

5.4. Region of absolute stability

Region of absolute stability is a region in the complex z plane, where $z = \lambda h$ for which the method is absolutely stable, (Joshua, 2022).

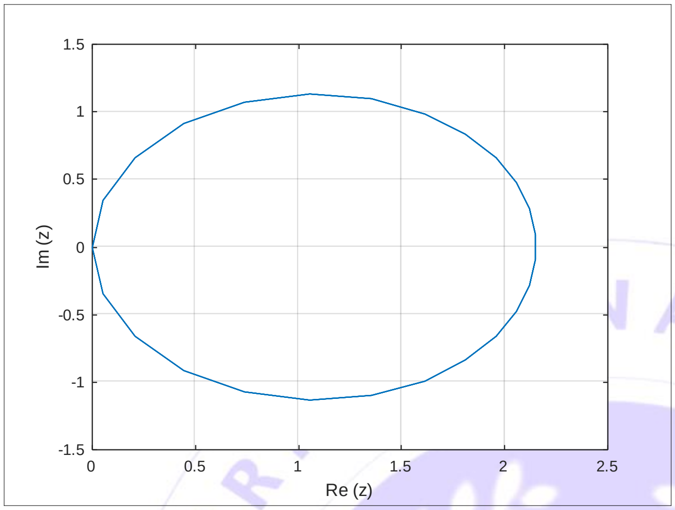

Figure 5.1: Region of absolute stability

Definition: L-stability

A two-step second derivative numerical method (11) is said to be L-stable if

(i) it is A-stable and

(ii) if $\lim_{z\to\infty}|R(z)| \to 0$

For our method $\lim_{z\to\infty}|R(z)| \to 0;$

Hence our method is A-stable.

6. Numerical Examples

We consider the following non-linear problems to test the efficiency of the developed methods. The following notation will be used in the presentation of the result.

Abbreviation and its Meaning

NSBVP = Absolute Error in New Method

Problem 1

Consider the nonlinear singular boundary value problem in physiology

$$u''(\tau)+\left(1+\frac{r}{\tau}\right)u'(\tau)=\frac{5\tau^3\left(5\tau^5 e^{u(\tau)}-\tau-r-4\right)}{4+\tau^5} \tag{29}$$

Subjected to $u'(0)=0, \quad u(1)+5u'(1)=\ln\left(\frac{1}{5}\right)-5$ (30)

Exact solution: $u(\tau)=\lim\left(\frac{1}{4+\tau^5}\right)$ (31)

Source: (Olabode, et al., 2023).

Table 1: comparison of result and errors for problem 1

t	Exact	Approximate	Error in Olabode	NSBVP
0.0	-1.3862943611198906	-1.3862943616542413	$1.1806x10^{-8}$	$5.3435x10^{-10}$
0.1	-1.3862968611167656	-1.3862968619898970	$1.1807x10^{-8}$	$8.7313x10^{-10}$
0.2	-1.3863743579200613	-1.3863743581401785	$1.1817x10^{-8}$	$2.2012x10^{-10}$
0.3	-1.3869016766664655	-1.3869016769252147	$1.1854x10^{-8}$	$2.5875x10^{-10}$
0.8	-1.4650316016572747	-1.4650316017587914	$1.1626x10^{-8}$	$1.0152x10^{-10}$
0.9	-1.5239867721873072	-1.5239867723458791	$1.0958x10^{-8}$	$1.5857x10^{-10}$
1.0	-1.6094379124341004	-1.6094379129548324	$1.0685x10^{-8}$	$5.2073x10^{-10}$

Problem 2

We consider the nonlinear SBVP,

$$u''(\tau)+\frac{2}{\tau}u'+u^5(\tau)=0 \qquad [0,1] \tag{32}$$

Subject to $u'(0)=0, \quad u(1)=\sqrt{\frac{3}{4}},$ (33)

The equation arise in the study of stellar structure.

Exact solution: $u(\tau)=\sqrt{\frac{3}{3+\tau^2}}$ (34)

Source: (Olabode, et al., 2023).

Table 2: comparison of result and errors for problem 2

t	Exact	Approximate	Error in Olabode	NSBVP
0.0	1.00000000000000000	1.00000000000000000	$1.3839x10^{-11}$	0
0.1	0.99833748845958268	0.99833748845955215	$1.3784x10^{-11}$	$3.0530x10^{-14}$
0.2	0.99339926779878258	0.99339926779873257	$1.3873x10^{-11}$	$5.0280x10^{-14}$
0.3	0.98532927816429315	0.98532927816421052	$1.3873x10^{-11}$	$8.2630x10^{-14}$
0.7	0.92714554082311955	0.92714554082310757	$1.3873x10^{-11}$	$1.1980x10^{-14}$
0.8	0.90784129900320363	0.90784129900310788	$1.3873x10^{-11}$	$9.5750x10^{-14}$
0.9	0.88735650941611377	0.88735650941613077	$1.3873x10^{-11}$	$1.7000x10^{-14}$
1.0	0.86602540378443865	0.86602540378443865	$1.3873x10^{-11}$	0

Problem 3

We consider the nonlinear SBVP,

$$v''(\tau) = -\frac{2}{\tau}v' - n^2\sin(n\tau) + \frac{2}{\tau}n\sin(n\tau) \qquad [0,1] \tag{35}$$

Subject to $v'(0) = 0, \quad v(0) = 0,$ (36)

The equation arise in the study of stellar structure.

Exact solution for $n = 3$ is $v(\tau) = 1 + \cos(n\tau)$ (37)

Source: (Yahaya, et al., 2021)

Table 3: Comparison of result and error for problem 3

h	Exact	Approximate	Error in Yahaya	NSBVP
$\frac{1}{5}2^{-2}$	1.731688869	1.73168823	$3.6865x10^{-3}$	$6.39x10^{-7}$
$\frac{1}{5}\ 2^{-4}$	1.982473313	1.982473156	$2.1320x10^{-4}$	$1.57x10^{-7}$
$\frac{1}{5}2^{-2}$	1.892133699	1.892133123	$1.3090x10^{-5}$	$5.76x10^{-7}$

Discussion

Table 1, for a physiological problem, and Table 2, for a stellar structure problem, both show the new method's approximate solutions aligning closely with exact solutions, demonstrating superior accuracy over the method by Olabode et al. Table 3 reinforces this conclusion, showing the new scheme's efficiency and higher precision compared to the approach by Yahaya et al. on another challenging problem. Collectively, the tables validate the new method as a highly effective and reliable computational tool for solving a range of nonlinear singular boundary value problems.

7. Conclusion

The study concludes that the developed two-step block method is a highly effective and accurate tool for solving nonlinear singular boundary value problems. This is confirmed by its theoretical properties—being seventh-order, convergent, and A-stable—and by numerical results showing superior performance over existing techniques. It is recommended that future work extend this robust method to systems of singular ODEs and develop adaptive step-size versions to enhance its computational efficiency for more complex real-world applications.

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