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Introduction to Fully Fuzzy Parameterized Soft Group

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In this paper the concept of full fuzzy parameterized soft group (FFPS-group) is introduced which extends the investigation of the full fuzzy parameterized soft set (FFPS-set) as introduced in 2022 by Edeghagba and Muhammad. In this study, the definition of the new concept is given, along with the joint-intersection and meet-union of FFPS-groups. Furthermore, the basic properties of FFPS-group and FFPS-subgroup are studied. Finally, we investigated the fundamental of FFPS-group properties via external product, homomorphism, isomorphism and factoring of FFPS-group are defined and investigated.

1. Introduction

In modern day Mathematics, despite the fact that fuzzy set (Zadeh, 1965) and soft set (Molodstov, 1999) theories were proved to have high degree of effectiveness in solving the inherent problems of uncertainty, vagueness and parameter dependence, it still remains a critical challenge for these problems to be completely addressed.

However, in order to enhance the applicability and efficiency of the theories of fuzzy set and soft set, researchers have developed hybrid structures that combine the two concepts. Maji et al. (2001) were the first to contribute in this direction, defining the concept of fuzzy soft set theory by incorporating fuzzy logic in a parameterized environment. This formation led to several developments in different aspects of Mathematics like algebra, topology and information science (Atkas and Cagman, 2007; Roy and Maji, 2007; Ali et al., 2009). Further developments include the work of Cagman et al. in 2010 and 2011 on fuzzy parameterized fuzzy soft set and fuzzy parameterized soft set respectively.

So many efforts by different researchers have been explored in the direction of a fundamental algebraic structure (Group Theory). These efforts include the definition of soft groups by Cagman et al. (2011) and fuzzy groups by (Rosenfeld, 1971). These led to the introduction of fuzzy soft groups by Jun and Song (2009). In essence, Jun and Song's work provides a powerful framework for dealing with uncertainties in algebraic structures using the concepts of fuzzy and soft sets, opening up new possibilities for applications in various fields including decision-making and medical diagnosis. The authors also studied algebraic structures and properties including fuzzy normal soft groups and fuzzy soft homomorphisms. This contributes immensely in algebraic structures with imprecise or parameter-dependent information.

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Over the years, significant studies have been made in order to explore the algebraic aspect of fuzzy and soft structures. In 2014, Hong and Qin investigated the algebraic structure of fuzzy soft sets, analyzing closures and homomorphisms. In the same direction, Enginoglu and Cagman in 2020 introduced and study fuzzy parameterized fuzzy soft matrices and its basic properties. The authors create a new algorithm (Prevalence Effect Method) and used it in a performance-based value assignment. Reeta et al. (2021) studied lattice-ordered fuzzy soft groups ($\mathrm{l}-$FSGs), including congruence relations and quotient structures and went ahead to prove that the set of all $\mathrm{l}-$FSG quotient set is a complete lattice. Also, in 2021, Akin defined multi-fuzzy soft groups and examined their structural properties.
In another direction, one notable advancement in the direction of merging the ideas of fuzzy set and soft set was full fuzzy parameterized soft set, introduced by Elijah and Muhammad (2021) in order to enhance fuzzy parameterized soft set by allowing complete fuzziness in the parameter domain. In the framework of full fuzzy parameterization, the concept of full fuzzy parameterized soft expert set (Elijah et al., 2024) was introduced. This fully fuzzified experts' opinions soft expert set.

However, Acharjee and Medhi in 2024, took their time to scrutinize the existing model in the aspect of fuzzy soft theory and focus on refining concepts like t-norms, t-conorms, negations and implications- ensuring they align with Molodtsov's foundational principles of soft set theory. The idea of promoting complete fuzzification to both underlying sets and parameter domains, and the fact that the existing models often don't integrate full group structures; therefore the need for a holistic algebraic framework that supports simultaneous full fuzzification and parameterization within group theory. Hence, the introduction of full fuzzy parameterized soft group.

The purpose of this study is to introduce a basic form of full fuzzy parameterized soft group theory, which extends the notion of a full fuzzy parameterized soft set theory to include the algebraic structures of full fuzzy parameterized soft sets. This is in line with works like (Rosenfeld, 1971), who proposed the concept of fuzzy groups in order to establish the algebraic structures of fuzzy sets, (Biswas and Nanda, 1994), who gave the first definitions of rough group, rough subgroup and some properties of them, (Aktaş and Çagman, 2007) who introduced the soft group structure depending on the definition of soft sets given by Molodsov and showed a way to construct algebraic structures based on the concept of a soft set.

2. Preliminaries

In this section, we provide some basic relevant definitions.

Definition 2.1. *(Zadeh, 1965)* ***Fuzzy Set****: A pair (μ, U) is called a fuzzy set, where U is a non-empty universe set and μ is a function from U into the unit interval $[0,1]$, (i.e, $\mu: U \to [0,1]$) and for all $x \in U$, $\mu(x)$ is referred to as the degree of membership of $x \in U$.*
Definition 2.2. *(Molodstov, 1999)* ***Soft Set****: Let U be a universe set, and E be the set of parameters. A pair (F, E) is called a soft set over U if and only if F is a mapping from E into the set of all subsets of the universe set U, i.e., $F: E \to P(U)$, where $P(U)$ is the power set of U.*
In other words, Soft set over U is a parameterized family of subsets of U.
Every set $F(e)$, for every $e \in E$, from this family may be considered as the set of e-elements of the soft set (F, E) or considered as the set of e-approximate elements of the soft set. Accordingly, we can view a soft set (F, E) as a collection of approximations: $(F, E) = \{F(e): e \in E\}$.

Definition 2.3. *(Maji et al., 2001)* ***Fuzzy Soft Set****: Let I^U be the set of all fuzzy sets of U. Then a pair (f, A) is called a fuzzy soft set over U, where A is a subset of the set of parameters E, and f is a mapping from A into I^U. That is, $f: A \to I^U$, and for each $a \in A$, $f(a) = f_a: U \to I$, is a fuzzy set on U.*

Definition 2.4. *(Cagman et al., 2011a)* ***Fuzzy Parameterized Soft Set****: Let U be an initial universe, E be the set of parameters and A be the fuzzy set over E. A fuzzy parameterized soft set (FP-soft set) $\digamma_A$ on the universe U is defined by the set of ordered pairs:*
$\digamma_A = \{(\mu_A(x)/x, \gamma_A(x)): x \in E, \gamma_A(x) \in P(U), \mu_A(x) \in [0,1]\}$,
Where $P(U)$ is the power set of U and the function $\gamma_A: E \to P(U)$ is called approximate function such that $\gamma_A(x) = \varnothing$ if $\mu_A(x) = 0$

Characterized by the membership function $\mu_A: E \rightarrow [0,1]$
The value $\mu_A(x)$ *of an element* x *of the parameters represents its degree of importance. And it is solely based on the desirability of the decision maker.*
Hence, this means that the approximate function is defined from fuzzy subset of E *to the crisp subset of the Universe set* U.

Definition 2.5. *(Edeghagba and Muhammad, 2021)* ***Full Fuzzy Parameterized Soft Set***: *Let* $\tilde{A} \subset \tilde{E}$. *A Full Fuzzy Parameterized Soft Set (FFP-soft set)* $F_{\tilde{A}}$ *on the universe* U *is given as:*
$F_{\tilde{A}} = \{(\hat{y}, f_{\tilde{A}}(\hat{y})) : \hat{y} \in \tilde{A}, f_{\tilde{A}}(\hat{y}) \in P(U), \mu_{\hat{y}}(x) \in [0,1], x \in E\}$
Where E *is the set of all parameters,* $\tilde{E}$ *is the set of all possible fuzzy set over* E *and* $\tilde{A}$ *is a finite subset of* $\tilde{E}$, $f_{\tilde{A}}: \tilde{A} \rightarrow P(U)$ *represents the approximation function of* $F_{\tilde{A}}$ *such that:* $f_{\tilde{A}}(\hat{y}) = \emptyset$ *whenever* $\mu_{\hat{y}}(x) = 0$ $\forall x \in E$ *and* $\hat{y} \in \tilde{A}$. *Therefore, a fuzzy set* $\hat{y}$ *over* E *is as:*
$\hat{y} = (\mu_{\hat{y}}(x_1)/x_1, \mu_{\hat{y}}(x_2)/x_2, ..., \mu_{\hat{y}}(x_n)/x_n)$,
characterized by the membership function
$\mu_{\hat{y}}: E \rightarrow [0,1]$, $E = \{x_1, x_2, \ldots, x_n\}$.

Definition 2.6. *(Herstein, 1975)* ***Group***: *A non-empty set* G *under binary operation* $\star$ *i.e* $(G,\star)$ *is said to be a group if it satisfies the following properties:*
1 Closure:$\forall a, b \in G, a * b \in G$.
2.Associativity:$\forall a, b, c \in G, a * (b * c) = (a * b) * c$.
3.Identity element:There exist an identity element $e \in G$*,such that* $a * e = e * a = a, \forall a \in G$.
4.Inverse: $\forall a \in G, \exists a^1 \in G$ *,such that* $a * a^1 = a^1 * a = e$.

Definition 2.7. *(Rosenfeld, 1971)* ***Fuzzy Subgroup***: *A fuzzy subset* μ *of a group* G *is said to be a fuzzy subgroup of* G *if*

1. $\mu(x, y) \geq min\{\mu(x), \mu(y)\}$ $\forall x, y \in G$

2. $\mu(x^{-1}) = \mu(x)$ $\forall x \in G$

holds for all $x, y \in G$.

Equivalently, $\mu(xy^1) \leq \mu(x) \wedge \mu(y)$ *for all* $x, y \in G$.

Definition 2.8. *(Atkas and Cagman, 2007)* ***Soft Group***: *Let* G *be a group and* (F, A) *be a soft set over* G. *Then* (F, A) *is called a soft group over* G *if and only if* $F(x)$ *is a soft group of* G *for all* $x \in G$.
Definition 2.9. *(Jun and Song, 2009)* ***Fuzzy Soft Group***: *Let* G *be a group and* (F, A) *be a fuzzy set over* G *.Then* (F, A) *is said to be fuzzy soft group over* G *if and only if for each* $a \in A$ *and* $x, y \in G$*,then*
1. $fa(xy) \geq T(fa(x).fa(y))$
2. $fa(x^-1) \geq fa(x)$
That is for each $a \in A$, fa *is a fuzzy subgroup in Rosenfield's sense.*

Definition 2.10. *(Cagman et al., 2011b)* ***Fuzzy Normal Subgroup***: *A fuzzy subset* μ *of a group* G *is said to be a fuzzy normal subgroup of* G *if the following axiom hold*
$\mu(xyx^1) \geq \mu(y)$ *for all* $x, y \in G$,
Equivalently, $\mu(xyx^1) = \mu(y)$ *for all* $x, y \in G$ *or* $\mu(xy) = \mu(yx)$ *for all* $x, y \in G$.

3. Results

In this section, we introduce the concept of FFPS-group which is an extension of the notion of FFPS-set theory to include algebraic structure. The FFPS-group is defined by using the concepts of the FFPS-set and classical group together, and this exhibits an algebraic structure containing parameters and uncertainty. First, we give the definition of **FFPS-group**, then define some of its basic operations and properties.

3.1 The Concept of FFPS-group

***Definition 3.1.1**. (**FFPS-group**): Let U be the initial universe,E the set of parameters and $\tilde{E}$ be the set of all fuzzy subsets of E .Therefore,for $\tilde{A} \subset \tilde{E}$, $F_{\tilde{A}}$ is said to be **FFPS-group** over U if*

1. *$F_{\tilde{A}}$ is a full fuzzy parameterized soft set over U .*
2. *$f_{\tilde{A}}(\hat{x}) \leq U, \forall \hat{x} \in \tilde{A}$.*
3. *If E is a group then in addition to (1) and (2) $\hat{y}$ is a fuzzy subgroup of E for all $\hat{y} \in \tilde{A}$.*

***Definition 3.1.2**. Let $F_{\tilde{A}}$ be an FFPS-group over the group G, and e be the identity element of G*

1. *If for all $\hat{a} \in \tilde{A}, f_{\tilde{A}}(\hat{A}) = \{e\}$, then $F_{\tilde{A}}$ is called the trivial FFPS-group over G. This is denoted by $e_{\tilde{A}}$*
2. *If for all $\hat{a} \in \tilde{A}, f_{\tilde{A}}(\hat{a}) = G$ is called the improper FFPS-group over G. This is denoted by $U_{\tilde{A}}$*

We noted that it could be the case that $|e_{\tilde{A}}| = 1$

***Example 3.1.3**. Let $G = S_3$ and the subgroupness property set $E = \{e_1, e_2, e_3, e_4\}$, e_1 = normality, e_2=cyclicness, e_3=commutativity, e_4=centrality.*
Clearly the trivial subgroup $\{e\}$ of S_3 is normal, cyclic, commutative, and it is center of S_3. Therefore $\hat{a} = \left(\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}, \frac{1}{e_4}\right) \in \tilde{A}$ is the only possible fuzzy set over E for $I_{\tilde{A}}$, thus $f_{\tilde{A}}(\hat{a}) \in I_{\tilde{A}}$ is the only element in $e_{\tilde{A}}$
***Definition 3.1.4.** Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPS-groups over the group G. Then $F_{\tilde{A}}$ is called FFPS-subgroup of $G_{\tilde{B}}$ if for each $\hat{a} \in \tilde{A}$ there exists $\hat{b} \in \tilde{B}$ such that $\hat{b} \leq \hat{a}$ $f_{\tilde{A}}(\hat{a}) \leq g_{\tilde{B}}(\hat{b})$ written as $F_{\tilde{A}} \leq G_{\tilde{B}}$*

***Proposition 3.1.5**. If $F_{\tilde{A}}$ is an FFPS-group over the group G and has a trivial element, then $e_{\tilde{A}}$ is an FFPS-subgroup of $F_{\tilde{A}}$*
Proof: Let $F_{\tilde{A}}$ be an FFPS-group over G with approximate function $f_{\tilde{A}}$, and suppose $F_{\tilde{A}}$ has a trivial element in the sense that for every fuzzy parameter $\hat{a} \in \tilde{A}$, the subgroup $f_{\tilde{A}}(\hat{a})$ contains the identity element e, so {e} is a subgroup of each $f_{\tilde{A}}(\hat{a})$. The trivial FFPS-group $e_{\tilde{A}}$ assigns the singleton subgroup $\{e\}$ to each parameter $\hat{a}$. To show $e_{\tilde{A}}$ is an FFPS-subgroup of $F_{\tilde{A}}$, we check the condition from Definition 3.1.4. Therefore, for an arbitrary parameter $\hat{a} \in \tilde{A}$ (of course $\hat{a}$ is considered to be in the domain of $e_{\tilde{A}}$) let $\hat{b} = \hat{a}$, then by the reflexive property $\hat{b} \leq \hat{a}$ and since for every fuzzy parameter $\hat{x} \in \tilde{A}$, $e \in f_{\tilde{A}}(\hat{x})$ and $f_{e_{\tilde{A}}}(\hat{a}) = \{e\}$, so it is the case that $\{e\} = f_{e_{\tilde{A}}}(\hat{a}) \leq f_{\tilde{A}}(\hat{b})$ satisfying the inclusion condition. Since the choice of $\hat{a}$ was arbitrary, the condition holds for every $\hat{a} \in \tilde{A}$. Hence $e_{\tilde{A}}$ is an FFPS-subgroup of $F_{\tilde{A}}$.
We note that there exists an inverse relationship between the order of parameter and the size of the set. So, a higher fuzzy parameter value produces a smaller approximate set.

***Definition 3.1.6**. Let $F_{\tilde{A}}$ be an FFPS-group over the universe U with the approximate function $f_{\tilde{A}}$, then a function $O_{\tilde{A}}$ is said to be a $f_{\tilde{A}}$ - single valued generated function if for all $\hat{a} \in \tilde{A}$, $O_{\tilde{A}}(\hat{a}) \in f_{\tilde{A}}(\hat{a})$.*

***Example 3.1.7**. Let $E = \{x_1, x_2, x_3, x_4\}$ be set of all parameters and $\tilde{A} = \{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$ be the finite set of fuzzy set over E and $U = \{U_1, U_2, U_3, U_4\}$. Let $F_{\tilde{A}}$ be a FFPS-group over U defined by approximate functions $f_{\tilde{A}}(\hat{a}_1) = \{U_1, U_2\}$, $f_{\tilde{A}}(\hat{a}_2) = \{U_1, U_3\}$, $f_{\tilde{A}}(\hat{a}_3) = \{U_4, \}$. Then the functions $O_{\tilde{A}}(\hat{a}_1) = U_2$, $O_{\tilde{A}}(\hat{a}_2) = U_1$, $O_{\tilde{A}}(\hat{a}_3) = U_3$. Clearly, $O_{\tilde{A}}$ is a $f_{\tilde{A}}$-single valued function.*
***Remark 3.1.8**. $f_{\tilde{A}}(\hat{a})$ needs not to be empty for all $\hat{a} \in \tilde{A}$. We must note that $O_{\tilde{A}}$ is generally not unique since each $O_{\tilde{A}}$ is an arbitrary element in $f_{\tilde{A}}(\hat{a})$ for all $\hat{a} \in \tilde{A}$.*
The above remark indicates the need to have at least one $\hat{a} \in \tilde{A}$ for which $f_{\tilde{A}}(\hat{a}) \neq \emptyset$. Otherwise choosing a value for $O_{\tilde{A}}$ would be able impossible and hence the definition fails. Since each of $f_{\tilde{A}}(\hat{a})$ may be sets with more than one element then it turns out that $O_{\tilde{A}}$ has several choices hence $O_{\tilde{A}}$ is not unique.
Example 3.1.9 demonstrates the definition of FFPS-subgroup.

Example 3.1.9. *Let $U = S_4$ and let $E = \{e_1, e_2, e_3, e_4\}$ is the set of parameters where parameters: e_1 = Normality, e_2 = Cyclicness, e_3 = Commutativity and e_4 = Centrality. We exhibit three FFPS-families $\tilde{A}, \tilde{B}, \tilde{C} \subset \tilde{E}$ with their approximate functions. Each approximate image is a subgroup of S_4.*

For the family $\tilde{A} = \{\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4\}$

Let $\hat{y}_1 = \left\{\frac{1}{e_1}, \frac{0.4}{e_2}, \frac{1}{e_3}, \frac{0.2}{e_4}\right\}$, define $f_{\tilde{A}}(\hat{y}_1) = \{e, (12)(34), (13)(24), (14)(23)\}$. Clearly, this set is normal and abelian, so a parameter with maximal normal and commutative values naturally selects it

Let $\hat{y}_2 = \left\{\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}, \frac{1}{e_4}\right\}$, define $f_{\tilde{A}}(\hat{y}_2) = \{e\}$. Clearly, the fuzzy with ones encodes the strongest possible requirement on every sub-groupness property; only the identity element satisfies all properties simultaneously.

Let $\hat{y}_3 = \left\{\frac{0.35}{e_1}, \frac{0.9}{e_2}, \frac{1}{e_3}, \frac{0.1}{e_4}\right\}$, define $f_{\tilde{A}}(\hat{y}_3) = \{e, (13)(24), (1234), (1432)\}$. Clearly, this shows a subgroup that is cyclic and abelian. The moderate values in the fuzzy values reveal weaker properties than $\hat{y}_2$.

Let $\hat{y}_4 = \left\{\frac{0.35}{e_1}, \frac{0.4}{e_2}, \frac{0.2}{e_3}, \frac{0.1}{e_4}\right\}$, define $f_{\tilde{A}}(\hat{y}_4) = \{e, (12), (13), (23), (123), (132)\}$. Clearly, this subgroup is non-abelian (so commutativity is moderate) and contains cyclic subgroups; the fuzzy values reflect these mixed properties.

Suppose $\tilde{B} = \{\hat{z}_1, \hat{z}_2, \hat{z}_3\}$,

Let $\hat{z}_1 = \left\{\frac{1}{e_1}, \frac{0.4}{e_2}, \frac{0.55}{e_3}, \frac{0.1}{e_4}\right\}$,

Define

$G_{\tilde{B}}(\hat{z}_1) = \{e, (12)(34), (13)(24), (14)(32), (123), (132), (134), (143), (124), (142), (234), (243)\}$. *Clearly, this subgroup is Normal in S_4 (so normality is high), non-abelian (so commutativity is moderate) and contains cyclic subgroups; the fuzzy values reflect these mixed properties.*

Let $\hat{z}_2 = \left\{\frac{0.3}{e_1}, \frac{0.9}{e_2}, \frac{0.95}{e_3}, \frac{0.1}{e_4}\right\}$, define $G_{\tilde{B}}(\hat{z}_2) = \{e, (1234), (1432), (13)(24)\}$. Clearly, this shows a subgroup with high cyclic and abelian properties. The moderate values in the fuzzy values reveal a weaker properties.

Let $\hat{z}_3 = \left\{\frac{0.3}{e_1}, \frac{0.35}{e_2}, \frac{0.2}{e_3}, \frac{0.1}{e_4}\right\}$, define $G_{\tilde{B}}(\hat{z}_3) = \{e, (12), (13), (23), (123), (132)\}$. Clearly, this shows a subgroup with weak normality, cyclic and abelian properties, and very weak centrality. The moderate value in the fuzzy values reveal weaker properties.

Suppose $\tilde{C} = \{\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3\}$,

Let $\tilde{\tau}_1 = \left\{\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}, \frac{1}{e_4}\right\}$, define $H_{\tilde{C}}(\hat{\tau}_1) = \{e\}$.

Let $\tilde{\tau}_2 = \left\{\frac{1}{e_1}, \frac{0.35}{e_2}, \frac{0.3}{e_3}, \frac{0.05}{e_4}\right\}$, define $H_{\tilde{C}}(\hat{\tau}_2) = \{S_4\}$

$\tilde{\tau}_3 = \left\{\frac{1}{e_1}, \frac{0.5}{e_2}, \frac{1}{e_3}, \frac{0.2}{e_4}\right\}$, *define $H_{\tilde{C}}(\hat{\tau}_3) = \{e, (12)(34), (13)(24), (14)(23)\}$*

From above the approximate functions generate subgroups of S_4 determined by fuzzy membership levels of the parameters, where $f_{\tilde{A}}(\hat{y}_i)$ is a subgroup of S_4 whose elements match with fuzzy group $\hat{y}_i$ over E. Therefore $f_{\tilde{A}}(\hat{y}_2)$ represents a group $\{e\}$ which is considered to be normal, cyclic, commutative and center with to the degree 1. However, $\{e\}$ or $f_{\tilde{A}}(\hat{y}_2)$ is a subgroup of itself as well as $f_{\tilde{A}}(\hat{y}_1)$, $f_{\tilde{A}}(\hat{y}_3)$ and $f_{\tilde{A}}(\hat{y}_4)$. Here, as stated, we are to consider all the parameters i.e., $\hat{y}_i$. Unlike in Cagman et al. (2011a) whereby all the parameters in E are attached with choice numerical values and all together assigned against possible approximate value set. In (Edeghagba and Muhammad, 2021), parameters of approximate value set are intra-dependently related, i.e., (putting into consideration, the impact parameters have on one another).

In this work, we will consider the parameters group wise instead of element wise and we will also be dealing with multiple entries.

Summary: *From the above we have that:*

$\hat{y}_1 \geq \hat{z}_1$ *this implies that* $f_{\tilde{A}}(\hat{y}_1) \leq g_{\tilde{B}}(\hat{z}_1)$

$\hat{y}_3 \geq \hat{z}_2$ *this implies that* $f_{\tilde{A}}(\hat{y}_2) \leq g_{\tilde{B}}(\hat{z}_2)$

$\hat{y}_1 \geq \hat{z}_3$ *this implies that* $f_{\tilde{A}}(\hat{y}_1) \leq g_{\tilde{B}}(\hat{z}_3)$

$\hat{y}_4 \geq \hat{z}_3$ *this implies that* $f_{\tilde{A}}(\hat{y}_4) \leq g_{\tilde{B}}(\hat{z}_3)$

This shows that an FFPS-group $(F,\tilde{A})$ is a FFPS-subgroup of $(G,\tilde{B})$. i.e, $(F,\tilde{A}) \lessapprox (G,\tilde{B})$. But $(F,\tilde{A})$ is not a FFPS-subgroup of $(H,\tilde{C})$ since $\hat{y}_3$ is not greater than any $\hat{\tau}_i$. Also $(H,\tilde{C})$ is not a FFPS-subgroup of $(G,\tilde{B})$ since $\hat{\tau}_2$ is not greater than any $\hat{z}_i$ for which the reverse order holds.

3.2 Basic Operations of FFPS-group

In this section, we introduce the concepts of joint-intersection and meet- union of FFPS-group, we then define the two concepts. The symbol $\vee$ and $\wedge$ means joint and meet respectively. Therefore, for $\hat{x} \in \tilde{A}$ and $\hat{y} \in \tilde{B}$, $\hat{x} \vee \hat{y} = max\{\mu_{\hat{x}}(x), \mu_{\hat{y}}(x)\}, \forall x \in E$. Similarly, $\hat{x} \wedge \hat{y} = min\{\mu_{\hat{x}}(x), \mu_{\hat{y}}(x)\}, \forall x \in E$

Definition 3.2.1. *Let $(F,\tilde{A})$ and $(G,\tilde{B})$ be any two FFPS-groups over the universe U, then the joint-intersection of $(F,\tilde{A})$ and $(G,\tilde{B})$ denoted by:*

$$(F,\tilde{A}) \cap_V (F,\tilde{A}) = (H,\tilde{A} \cap_V \tilde{B})$$

where, $\tilde{A} \cap_V \tilde{B} = \{\hat{\tau} = \hat{x} \vee \hat{y},\ \hat{x} \in \tilde{A} \text{ and } \hat{y} \in \tilde{B}\}$ we have the approximation function $h_{\tilde{A}\cap_V\tilde{B}}: \tilde{A} \cap_V \tilde{B} \longrightarrow P(U)$ is defined by

$$h_{\tilde{A}\cap_V\tilde{B}}(\hat{\tau}) = \begin{cases} f_{\tilde{A}}(\hat{\tau}) \ \cap \ g_{\tilde{B}}(\hat{\tau}), & if \quad \hat{\tau} \in \tilde{A} \cap \tilde{B} \\ f_{\tilde{A}}(\hat{\tau}) \ \cap \ g_{\tilde{B}}(\hat{\tau}), & if \quad \hat{\tau} = \hat{x} \vee \hat{y}, \quad \hat{x} \neq \hat{y} \end{cases}$$

The definition ensures that the approximation function is unique and well-defined for every parameter $\hat{\tau}$.

Definition 3.2.2. *Let $(F,\tilde{A})$ and $(G,\tilde{B})$ be any two FFPS-groups over the universe U, then the meet-union of $(F,\tilde{A})$ and $(G,\tilde{B})$ denoted by:*

$$(F,\tilde{A}) \cup_\wedge (F,\tilde{A}) = (H,\tilde{A} \cup_\wedge \tilde{B})$$

where, $\tilde{A} \cup_\wedge \tilde{B} = \{\hat{\tau} = \hat{x} \wedge \hat{y},\ \hat{x} \in \tilde{A} \text{ and } \hat{y} \in \tilde{B}\}$ we have the approximation function $h_{\tilde{A}\cap_V\tilde{B}}: \tilde{A} \cup_\wedge \tilde{B} \longrightarrow P(U)$ is defined by

$$h_{\tilde{A}\cup_\wedge\tilde{B}}(\hat{\tau}) = \begin{cases} f_{\tilde{A}}(\hat{\tau}) \ \cup \ g_{\tilde{B}}(\hat{\tau}), & if \quad \hat{\tau} \in \tilde{A} \cap \tilde{B} \\ f_{\tilde{A}}(\hat{\tau}) \cup g_{\tilde{B}}(\hat{\tau}), & if \ \hat{\tau} = \hat{x} \wedge \hat{y}, \hat{x} \neq \hat{y} \end{cases}$$

Remark 3.2.3.

1. *Clearly joint- intersection of FFPS-groups is an FFPS-group but and meet-union of FFPS-groups is generally not an FFPS-group.*
2. *Generally, joint-intersection and meet-union are not unique. This follows from the definition.*

Proposition 3.2.4. *Let $(F,\tilde{A}) = F_{\tilde{A}}$ and $(G,\tilde{B}) = G_{\tilde{B}}$ be any two FFPS-groups over the group G, then the joint-intersection of $(F,\tilde{A})$ and $(G,\tilde{B})$ is also FFPS-group over G.*

Proof. Suppose $H_{\tilde{C}} = F_{\tilde{A}} \cap_V G_{\tilde{B}}$and by definition, it follows that $\tilde{C} = \tilde{A} \cap \tilde{B}$. Therefore, for all $\hat{c} \in \tilde{C}$, if there exists $\hat{a} \in \tilde{A}$ and $\hat{b} \in \tilde{B}$ such that $\hat{c} = \hat{a} = \hat{b}$, then, $h_{\tilde{C}}(\hat{c}) = f_{\tilde{A}}(\hat{a})$ or $h_{\tilde{C}}(\hat{c}) = g_{\tilde{B}}(\hat{b})$. Otherwise, there exist some $\hat{a} \in \tilde{A}$ and $\hat{b} \in \tilde{B}$ such that $\hat{c} = \hat{a} \vee \hat{b}$, thus; $h_{\tilde{C}}(\hat{c}) = f_{\tilde{C}}(\hat{c}) = f_{\tilde{A}}(\hat{a}) \cap g_{\tilde{B}}(\hat{b})$. Therefore $H: \tilde{C} \to P(G)$ is a mapping. Hence $(H,\tilde{C}) = H_{\tilde{C}}$ is a soft set over G. Now, for each $\hat{a} \in \tilde{A}, f_{\tilde{A}}(\hat{a}) < G$ and for each $\hat{b} \in \tilde{B}, g_{\tilde{B}}(\hat{b}) < G$, then it follows that for any fixed $\hat{c} \in \tilde{C}$ for which $\hat{c} = \hat{a} = \hat{b}$, then $h_{\tilde{C}}(\hat{c}) = f_{\tilde{A}}(\hat{a}) < G$ or $h_{\tilde{C}}(\hat{c}) = g_{\tilde{B}}(\hat{b}) < G$. Thus, whichever of the two sets is chosen as $h_{\tilde{A}\cup_\wedge\tilde{B}}(\hat{c})$, it is a subgroup of G and hence a FFPS-group over G. Next for for $\hat{a} \neq \hat{b}$ then $\hat{c} = \hat{a} \vee \hat{b}$, it follows that $h_{\tilde{C}}(\hat{c}) = f_{\tilde{A}}(\hat{a}) \cap g_{\tilde{B}}(\hat{b})$, since $f_{\tilde{A}}(\hat{a}) \leq G$ and $g_{\tilde{B}}(\hat{b}) \leq G$, their intersection is also a subgroup of G. Hence, $H_{\tilde{C}}$ is an FFPS-group over the group G. Therefore, the result follows.

□

Proposition 3.2.5. *Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPS-group over the group G. Then meet-union of $F_{\tilde{A}}$ and $G_{\tilde{B}}$ is an FFPS-group over G, if either $\forall\ \hat{a} \in \tilde{A}$, $\exists\ \hat{b} \in \tilde{B}$ such that $f_{\tilde{A}}(\hat{a}) \leq g_{\tilde{B}}(\hat{b})\ \forall\ \hat{b} \in \tilde{B}$, or $\exists\ \hat{a} \in \tilde{A}$ such that $g_{\tilde{B}}(\hat{b}) \leq f_{\tilde{A}}(\hat{a})$.*
Proof. *Given that $F_{\tilde{A}}$ and $G_{\tilde{B}}$ are two FFPS-groups over the group G and $\hat{c} \in \tilde{A} \cup_{\wedge} \tilde{B}$. Let $H_{\tilde{C}}$ be their meet-union denoted by $H_{\tilde{C}} = F_{\tilde{A}} \cup_{\wedge} G_{\tilde{B}}$, where, for each $\hat{c} \in \tilde{C}$, if $\hat{c} = \hat{a} = \hat{b}$ such that $g_{\tilde{B}}(\hat{b}) \subseteq f_{\tilde{A}}(\hat{a})$ or $f_{\tilde{A}}(\hat{a}) \subseteq g_{\tilde{B}}(\hat{b})$. In any case $h_{\tilde{C}}(\hat{c}) = f_{\tilde{A}}(\hat{a})$ or $h_{\tilde{C}}(\hat{c}) = g_{\tilde{B}}(\hat{b})$, coinciding with the larger subgroup $h_{\tilde{C}}(\hat{c}) < G$, since for each $\hat{a} \in \tilde{A}$, $f_{\tilde{A}}(\hat{a}) < G$ and for each $\hat{b} \in \tilde{B}$, $g_{\tilde{B}}(\hat{b}) < G$. Since $\hat{c}$ is arbitrary it is the case that for each $\hat{c} \in \tilde{A} \cup_{\wedge} \tilde{B}$, then $h_{\tilde{A}\cup_{\wedge}\tilde{B}}(\hat{c})$ is a subgroup of G. Hence, $H_{\tilde{C}} = F_{\tilde{A}} \cup_{\wedge} G_{\tilde{B}}$ is an FFPS-group over the group G.*
$\square$

Example 3.2.6. *With reference to example 3.1.9 above, S_4 is the universe, then*

$$f_{\tilde{A}}(\hat{y}_1) = \{e, (12)(34), (13)(24), (14)(23)\}$$
$$f_{\tilde{A}}(\hat{y}_2) = \{e\}$$
$$f_{\tilde{A}}(\hat{y}_3) = \{e, (13), (24), (13)(24), (12)(34), (14)(23), (1234), (1432)\}$$
$$f_{\tilde{A}}(\hat{y}_4) = \{e, (12), (13), (23), (123)(132)\}$$

and then;

$$\tilde{A} = \left(\hat{y}_1 = \left\{\frac{1}{e_1}, \frac{0.6}{e_2}, \frac{1}{e_3}, \frac{0.2}{e_4}\right\}, \hat{y}_2 = \left\{\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}, \frac{1}{e_4}\right\}, \hat{y}_3 = \left\{\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.3}{e_3}, \frac{0.5}{e_4}\right\}, \hat{y}_4 = \left\{\frac{0.5}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.3}{e_4}\right\}\right)$$

Therefore, for all $\hat{y}_3, \hat{y}_4 \in \tilde{A}$, $f_{\tilde{A}}(\hat{y}_3) \cap_{V} f_{\tilde{A}}(\hat{y}_4) = f_{\tilde{A}}(\hat{y}_3) \cap f_{\tilde{A}}(\hat{y}_4) = \{e, (13)\}$ and $(\hat{y}_3 \vee \hat{y}_4) = \left\{\frac{0.5}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.5}{e_4}\right\}$
Therefore, the joint-intersection of $f_{\tilde{A}}(\hat{y}_3)$ and $f_{\tilde{A}}(\hat{y}_4)$, i.e,

$$f_{\tilde{A}}(\hat{y}_3) \cap_{V} f_{\tilde{A}}(\hat{y}_4) = \left\{\left\{\frac{0.5}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.5}{e_4}\right\}, \{e, (13)\}\right\}$$

Now for all $\hat{y}_3, \hat{y}_4 \in \tilde{A}$,
$f_{\tilde{A}}(\hat{y}_3) \cup_{\wedge} f_{\tilde{A}}(\hat{y}_4) = f_{\tilde{A}}(\hat{y}_3) \cup f_{\tilde{A}}(\hat{y}_4)$
$= \{e, (12), (13), (23), (24), (13)(24), (12)(34), (14)(23), (123), (132), (1234), (1432)\}$, *and* $(\hat{y}_3 \wedge \hat{y}_4) = \left\{\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.3}{e_3}, \frac{0.3}{e_4}\right\}$

Therefore the meet-union of $f_{\tilde{A}}(\hat{y}_3)$ and $f_{\tilde{A}}(\hat{y}_4)$ i.e,

$$f_{\tilde{A}}(\hat{y}_3) \cup_{\wedge} f_{\tilde{A}}(\hat{y}_4) = \{\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.3}{e_3}, \frac{0.3}{e_4}, \{e, (12), (13), (23), (24), (13)(24), (12)(34), (14)(23), (123), (132), (1234), (1432)\}\}$$

Definition 3.2.7. *Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPS-group over the group G. Then $F_{\tilde{A}}$ is said to be a normal FFPS-subgroup of $G_{\tilde{B}}$, if $F_{\tilde{A}}$ is a FFPS-subgroup of $G_{\tilde{B}}$ and for every $\hat{a} \in \tilde{A}$, $\exists\ \hat{b} \in \tilde{B}$ with $\hat{b} \leq \hat{a}$, such that*

$$x f_{\tilde{A}}(\hat{a}) x^{-1} = f_{\tilde{A}}(\hat{a})$$

For all $x \in g_{\tilde{B}}(\hat{b})$. This is denoted by $F_{\tilde{A}} \trianglelefteq G_{\tilde{B}}$.
Remark 3.2.8. *When the fuzzy parameters collapse to crisp values, this definition reduces exactly to the classical notion of a normal subgroup. The reverse-order condition $\hat{b} \leq \hat{a}$ reflects the semantic interpretation that higher degrees of sub-groupness correspond to smaller, more structured subgroups, such as the trivial subgroup, which is normal in every group.*

Example 3.2.9. *Let $G = S_3$ and $E = \{e_1, e_2, e_3\}$ such that e_1 represent normality, e_2 represent cyclicness and e_3 represent commutativity. For the FFPS-families $\tilde{A}, \tilde{B} \subset \tilde{E}$, let $\tilde{A} = \{\hat{a}\}$ and $\tilde{B} = \{\hat{b}\}$, with $\hat{a} = \{\frac{1}{e_1}, \frac{0.9}{e_2}, \frac{1}{e_3}\}$ and $\hat{b} = \{\frac{0.4}{e_1}, \frac{0.9}{e_2}, \frac{1}{e_3}\}$, clearly $\hat{a} \geq \hat{b}$. Define $F_{\tilde{A}}$ and $G_{\tilde{B}}$ by the approximate functions*

$$f_{\tilde{A}}(\hat{a}) = \{e\} \text{ and } g_{\tilde{B}}(\hat{b}) = \{e, (12)\}$$

Thus $F_{\tilde{A}}$ and $G_{\tilde{B}}$ are both FFPS-group over S_3. Now for $\hat{a} \in \tilde{A}$ choose $\hat{b} \in \tilde{B}$, then for $\hat{a} \geq \hat{b}$

$$f_{\tilde{A}}(\hat{a}) \leq g_{\tilde{B}}(\hat{b})$$

It then follows that $F_{\tilde{A}}$ is an FFPS-subgroup of $G_{\tilde{B}}$.
Finally, for any $b \in g_{\tilde{B}}(\hat{b})$,

$$gf_{\tilde{A}}(\hat{a})g^{-1} = f_{\tilde{A}}(\hat{a}) = \{e\},$$

Since conjugation fixes the identity element. Hence $F_{\tilde{A}} \trianglerighteq G_{\tilde{B}}$.

Proposition 3.2.10. *Let* $F_{\tilde{A}}$ *be an FFPS-group over the group* G. *For the family* $\{G_{i,\tilde{B}_i}: i \in I\}$ *of FFPS-subgroups of* $F_{\tilde{A}}$. *The joint-intersection of the family is an FFPS-subgroup of* $F_{\tilde{A}}$. *I is the indexing set.*

Proof: Let $K_{\tilde{B}} = \cap_V\{G_{i,\tilde{B}_i}: i \in I\}$ *and* $\tilde{B} = \cap_V\{\tilde{B}_i: i \in I\}$. *Suppose* $\hat{c} \in \tilde{B}$, *then by Definition 3.2.1 the approximation function of* $K_{\tilde{B}}$ *is given as*

$$k_{\tilde{B}}(\hat{c}) = \cap\{g_{i,\tilde{B}_i}(\hat{c}): i \in I\},$$

For the approximation function $g_{i,\tilde{B}_i}$ *of* $G_{i,\tilde{B}_i}$. *By definition* $g_{i,\tilde{B}_i}(\hat{c}) \leq G$ *for each* $\hat{c} \in \tilde{B}_i$, *since each* $G_{i,\tilde{B}_i}$ *is an FFPS-subgroups of* $F_{\tilde{A}}$. *Then it follows that* $k_{\tilde{B}}(\hat{c}) \leq G$, *since the intersection of an arbitrary family of subgroups of a group is again a subgroup. Therefore,* $K_{\tilde{B}}$ *satisfies the FFPS-group condition. Now since each* $G_{i,\tilde{B}_i}$ *is an FFPS-subgroup of* $F_{\tilde{A}}$. *Then it is the case that for each* $\hat{c} \in \tilde{B}_i$ *there exists for each* $\hat{a}_i \in \tilde{A}$ *such that*

$$\hat{a}_i \leq \hat{c} \text{ and } g_{i,\tilde{B}_i}(\hat{c}) \leq f_{\tilde{A}}(\hat{a}_i).$$

Suppose

$$\hat{a} = \vee\{\hat{a}_i: i \in I\},$$

Then

$$\hat{a} \leq \hat{c} \text{ and } k_{\tilde{B}}(\hat{c}) = \cap\{g_{i,\tilde{B}_i}(\hat{c}): i \in I\} \leq f_{\tilde{A}}(\hat{a}).$$

Hence, the joint-intersection of FFPS-subgroups of $F_{\tilde{A}}$ *is an FFPS-subgroup of* $F_{\tilde{A}}$. ◻

Theorem 3.2.11. *If* $F_{\tilde{A}}$ *and* $G_{\tilde{B}}$ *are normal FFPS-groups of an FFPS-group* $H_{\tilde{C}}$ *over the group* G. *Then* $F_{\tilde{A}} \cap_V G_{\tilde{B}}$ *is a normal FFPS-group of* $H_{\tilde{C}}$ *over the group* G.

Proof: Let $K_{\tilde{A}\cap_V\tilde{B}} = F_{\tilde{A}} \cap_V G_{\tilde{B}}$ *and* $\hat{c} \in \tilde{A} \cap_V \tilde{B}$. *By definition* $\hat{c} \in \tilde{A} \cap \tilde{B}$ *or* $\hat{c} = \hat{a} \vee \hat{b}$ *for suitable parameters* $\hat{a} \in \tilde{A}, \hat{b} \in \tilde{B}$. *For either case*

$$k_{\tilde{A}\cap_V\tilde{B}}(\hat{u}) = f_{\tilde{A}}(\hat{a}) \cap g_{\tilde{B}}(\hat{b}).$$

Since $F_{\tilde{A}}$ *and* $G_{\tilde{B}}$ *are FFPS-groups,* $f_{\tilde{A}}(\hat{a})$ *and* $g_{\tilde{B}}(\hat{b})$ *are subgroups of* G, *their intersection,* $k_{\tilde{A}\cap_V\tilde{B}}(\hat{u})$ *is a subgroup of* G. *Hence* $K_{\tilde{A}\cap_V\tilde{B}}$ *is an FFPS-group. Next, since* $F_{\tilde{A}} \trianglelefteq H_{\tilde{C}}$ *and* $G_{\tilde{B}} \trianglelefteq H_{\tilde{C}}$, *for each* $\hat{a} \in \tilde{A}$, *and* $\hat{b} \in \tilde{B}$ *there exists* $\hat{c}, \hat{d} \in \tilde{C}$, *such that* $\hat{c} \leq \hat{a}$ *and* $\hat{d} \leq \hat{b}$, *and*

$$f_{\tilde{A}}(\hat{a}) \subseteq h_{\tilde{C}}(\hat{c}) \text{ and } g_{\tilde{B}}(\hat{b}) \subseteq h_{\tilde{C}}(\hat{d}).$$

Let $\hat{z} = \hat{c} \vee \hat{d}$, *then* $\hat{z} \leq \hat{u}$ *and* $k_{\tilde{A}\cap_V\tilde{B}}(\hat{u}) \subseteq h_{\tilde{C}}(\hat{z})$. *Then*

$$k_{\tilde{A}\cap_V\tilde{B}} \leq H_{\tilde{C}}$$

Finally, let $F_{\tilde{A}}$ *and* $G_{\tilde{B}}$ *be normal FFPS-groups of the FFPS-group* $H_{\tilde{C}}$, $g \in h_{\tilde{C}}(\hat{z})$ *and* $h \in k_{\tilde{A}\cap_V\tilde{B}}(\hat{u})$, *then*

$$ghg^{-1} \in f_{\tilde{A}}(\hat{a}) \text{ and } ghg^{-1} \in g_{\tilde{B}}(\hat{b})$$

Thus,

$$ghg^{-1} \in k_{\tilde{A}\cap_V\tilde{B}}(\hat{u})$$

Therefore,

$$gk_{\tilde{A}\cap_V\tilde{B}}(\hat{u})g^{-1} = k_{\tilde{A}\cap_V\tilde{B}}(\hat{u}).$$

Therefore, the joint-intersection of two normal FFPS-subgroups of an FFPS-group is again a normal FFPS-subgroup. ◻

Proposition 3.2.12. *Let* $F_{\tilde{A}}$ *be an FFPS-group over the group* G. *Let* $G_{\tilde{B}}$ *and* $H_{\tilde{C}}$ *be two FFPS-subgroups of* $F_{\tilde{A}}$ *such that* $H_{\tilde{C}}$ *is a normal FFPS-subgroup of* $F_{\tilde{A}}$. *Then* $H_{\tilde{C}} \cap_V G_{\tilde{B}}$ *is a normal FFPS-subgroup of* $G_{\tilde{B}}$.

Proof. Let $F_{\tilde{A}}$ *be an FFPS-subgroup over the group* G, *where* $G_{\tilde{B}}$ *is an FFPS-subgroup of* $F_{\tilde{A}}$ *and* $H_{\tilde{C}}$ *is normal FFPS-subgroup of* $F_{\tilde{A}}$. *Let* $J_{\tilde{D}} = G_{\tilde{B}} \cap_V H_{\tilde{C}}$ *and* $\tilde{D} = \tilde{B} \cap_V \tilde{C}$. *Since* $G_{\tilde{B}} \precsim F_{\tilde{A}}$ *and* $H_{\tilde{C}} \precsim F_{\tilde{A}}$. *It follows that* $J_{\tilde{D}} \precsim G_{\tilde{B}}$. *Next, we show that* $G_{\tilde{B}} \cap_V H_{\tilde{C}}$ *is a normal FFPS-subgroup of* $G_{\tilde{B}}$. *Therefore, for any* $\hat{d} \in \tilde{D}$, *there exist* $\hat{b} \in \tilde{B}$ *and* $\hat{c} \in \tilde{C}$ *such that*

$$j_{\tilde{D}}(\hat{d}) = g_{\tilde{B}}(\hat{b}) \cap h_{\tilde{C}}(\hat{c})$$

where $\hat{d} = \hat{b} \vee \hat{c}$ *for* $\hat{d} = \hat{b} = \hat{c}$ *or* $\hat{b} \neq \hat{c}$. *Since* $G_{\tilde{B}}$ *and* $H_{\tilde{C}}$ *are FFPS-subgroups of* $F_{\tilde{A}}$,

$$g_{\tilde{B}}(\hat{b}) \leq f_{\tilde{A}}(\hat{a}_1) \text{ and } h_{\tilde{C}}(\hat{c}) \leq f_{\tilde{A}}(\hat{a}_2)$$

for some $\hat{a}_1, \hat{a}_2 \in \tilde{A}$.

Thus, $j_{\tilde{D}}(\hat{d})$ *is the intersection of two subgroups of* G, *hence itself a subgroup. Therefore,* $J_{\tilde{D}}$ *is an FFPS-group over*

G. For $\hat{d} \in \tilde{D}$ fix $\hat{b} \in \tilde{B}$ such that $\hat{b} \leq \hat{d}$ and

$$j_{\tilde{D}}(\hat{d}) \leq g_{\tilde{B}}(\hat{b})$$

Let $y \in g_{\tilde{B}}(\hat{b})$ and $x \in j_{\tilde{D}}(\hat{d}) = g_{\tilde{B}}(\hat{b}) \cap h_{\tilde{C}}(\hat{c})$. Since $H_{\tilde{C}}$ is a normal FFPS-subgroup of $F_{\tilde{A}}$, then exists $\hat{a} \in \tilde{A}$ such that

$$h_{\tilde{C}}(\hat{c}) \leq f_{\tilde{A}}(\hat{a})$$

and

$$y h_{\tilde{C}}(\hat{c}) y^{-1} = h_{\tilde{C}}(\hat{c})$$

for all $y \in f_{\tilde{A}}(\hat{a})$. Since $y \in g_{\tilde{B}}(\hat{b}) \leq f_{\tilde{A}}(\hat{a})$, it follows that

$$yxy^{-1} \in h_{\tilde{C}}(\hat{c}).$$

Since $g_{\tilde{B}}(\hat{b})$ is a subgroup,

$$yxy^{-1} \in g_{\tilde{B}}(\hat{b}).$$

Hence

$$yxy^{-1} \in j_{\tilde{D}}(\hat{d})$$

Thus, we have that $y j_{\tilde{D}}(\hat{d}) y^{-1} = j_{\tilde{D}}(\hat{d})$ for all $y \in g_{\tilde{B}}(\hat{b})$. Hence since our choice of $\hat{b}$, $\hat{c}$ and $\hat{d}$ are arbitrary, then we conclude that $J_{\tilde{D}} = G_{\tilde{B}} \cap_V H_{\tilde{C}}$ is a normal FFPS-subgroup of $G_{\tilde{B}}$.

□

3.3 Fundamental Properties of FFPS-group

***Definition 3.3.1**. Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPS-groups over the groups G_1 and G_2 respectively. Then their external product i.e $H_{\tilde{A}\times\tilde{B}} = F_{\tilde{A}} \tilde{\times} G_{\tilde{B}}$ is defined by; $h_{\tilde{A}\times\tilde{B}}(\hat{a},\hat{b}) = f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}(\hat{b})$, where $(\hat{a},\hat{b}) = \hat{a} \wedge \hat{b}$ for all $(\hat{a},\hat{b}) \in \tilde{A} \times \tilde{B}$*

***Proposition 3.3.2**. Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPS-groups over the groups G_1 and G_2 respectively. Then, the external product $F_{\tilde{A}} \times G_{\tilde{B}}$ is an FFPS-group over $G_1 \times G_2$.*

Proof. The approximation function for the external product i.e., $H_{\tilde{A}\times\tilde{B}} = F_{\tilde{A}} \tilde{\times} G_{\tilde{B}}$ is given by

$$h_{\tilde{A}\times\tilde{B}}: \tilde{A} \times \tilde{B} \longrightarrow G_1 \times G_2$$

With the approximation function $h_{\tilde{A}\times\tilde{B}}(\hat{a},\hat{b}) = f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}(\hat{b})$, For each $(\hat{a},\hat{b}) \in \tilde{A} \times \tilde{B}$ with $h_{\tilde{A}\times\tilde{B}}(\hat{a},\hat{b}) = \{(x,y) \in f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}(\hat{b}) : x \in f_{\tilde{A}}(\hat{a}), y \in g_{\tilde{B}}(\hat{b})\}$. Since $f_{\tilde{A}}(\hat{a})$ and $g_{\tilde{B}}(\hat{b})$ are subgroups of G_1 and G_2 respectively. Then $(e_{G_1}, e_{G_2}) \in f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}(\hat{b}) = h_{\tilde{A}\times\tilde{B}}(\hat{a},\hat{b})$

Thus, (e_{G_1}, e_{G_2}) is an identity element in $h_{\tilde{A}\times\tilde{B}}(\hat{a} \wedge \hat{b})$ for e_{G_1} and e_{G_2} are the identity elements in $f_{\tilde{A}}(\hat{a})$ and $g_{\tilde{B}}(\hat{b})$ respectively. Given that $(x,y) \in h_{\tilde{A}\times\tilde{B}}(\hat{a},\hat{b})$, we show that $(x,y)^{-1} = (x^{-1},y^{-1}) \in h_{\tilde{A}\times\tilde{B}}(\hat{a} \wedge \hat{b})$. Again, since $f_{\tilde{A}}(\hat{a})$ and $g_{\tilde{B}}(\hat{b})$ are subgroups of G_1 and G_2 respectively, then for $x \in f_{\tilde{A}}(\hat{a}), x^{-1} \in f_{\tilde{A}}(\hat{a})$ and for $y \in g_{\tilde{B}}(\hat{b}), y^{-1} \in g_{\tilde{B}}(\hat{b})$. So, it follows that $(x^{-1},y^{-1}) \in f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}(\hat{b})$. So obviously we have that $(x,y)^{-1} \in h_{\tilde{A}\times\tilde{B}}(\hat{a},\hat{b})$. Next, let $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in h_{\tilde{A}\times\tilde{B}}(\hat{a} \wedge \hat{b})$ since $f_{\tilde{A}}(\hat{a}) \leq G_1$ and $g_{\tilde{B}}(\hat{b}) \leq G_2$, then for $x_1,x_2,x_3 \in f_{\tilde{A}}(\hat{a})$ and $y_1,y_2,y_3 \in g_{\tilde{B}}(\hat{b})$

$x_1(x_2x_3) = (x_1x_2)x_3$ *in* $f_{\tilde{A}}(\hat{a})$.(1)

$y_1(y_2y_3) = (y_1y_2)y_3$ *in* $g_{\tilde{B}}(\hat{b})$ (2)

Respectively combining (1) and (2), we have

$(x_1,y_1)[(x_2,y_2)(x_3,y_3)] = (x_1,y_1)(x_2x_3,y_2y_3)$

$= x_1(x_2x_3), y_1(y_2y_3) = ((x_1x_2)x_3, (y_1y_2)y_3)$

$= (x_1x_2, y_1y_2)(x_3,y_3) = [(x_1,y_1)(x_2,y_2)](x_3,y_3)$

Thus, associativity holds.

Lastly, let $(x_1,y_1),(x_2,y_2) \in h_{\tilde{A}\times\tilde{B}}(\hat{a} \times \hat{b})$, for $x_1,x_2 \in f_{\tilde{A}}(\hat{a})$ and $y_1,y_2 \in g_{\tilde{B}}(\hat{b})$. Since $f_{\tilde{A}}(\hat{a}) \leq G_1$ and $g_{\tilde{B}}(\hat{b}) \leq G_2$ then $x_1x_2 \in f_{\tilde{A}}(\hat{a})$ and $y_1y_2 \in g_{\tilde{B}}(\hat{b})$. Therefore, $(x_1x_2, y_1y_2) \in f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}(\hat{b})$, then $(x_1,y_1)(x_2,y_2) \in f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}(\hat{b}) = h_{\tilde{A}\times\tilde{B}}(\hat{a} \times \hat{b})$. Thus, $f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}(\hat{b})$ is a subgroup of $G_1 \times G_2$ for all $(\hat{a},\hat{b}) \in \tilde{A} \times \tilde{B}$. Hence, $(H, \tilde{A} \times \tilde{B})$ is an FFPS-group over the group $G_1 \times G_2$.

□

Proposition 3.3.3. *Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPS-groups over the groups G_1 and G_2 respectively. Then $F_{\tilde{A}} \times G_{\tilde{B}}$ is a normal FFPS-group over $G_1 \times G_2$.*

Definition 3.3.4. *Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPS-set over the groups G_1 and G_2 respectively. Suppose $\phi: G \to G_2$ and $\theta: \tilde{A} \to \tilde{B}$ are two functions, the (ϕ, θ) is said to be an FFPS-homomorphism from $F_{\tilde{A}}$ to $G_{\tilde{B}}$ if the following conditions hold:*

1. *ϕ is surjective homomorphism from G_1 to G_2*
2. *θ is a surjective function from $\tilde{A}$ to $\tilde{B}$*
3. *$\phi\left(f_{\tilde{A}}(\hat{a})\right) = g_{\tilde{B}}\left(\theta(\hat{a})\right)$, $\hat{a} \le \theta(\hat{a})\ \forall\ \hat{a} \in \tilde{A}$.*

If in definition ϕ is bijective homomorphism and θ is bijective function, then (ϕ, θ) is said to be an FFPS-isomorphism. i.e, (FFPS-isomorphism) from $F_{\tilde{A}} \to G_{\tilde{B}}$. This is denoted by; $F_{\tilde{A}} \cong G_{\tilde{B}}$

Example 3.3.5. *Let $G_1 = D_8$ and $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Define the surjective homomorphism $\phi: G_1 \to G_2$ by; $\phi(r) = (1,0)$ and $\phi(s) = (0,1)$, where $D_8 = \{< r, s >: r^4 = 1, s^2 = 1, sr = r^{-1}s\}$.*
Consider the sets $\tilde{A}$ and $\tilde{B}$ of collection of fuzzy sets over $E = \{e_1, e_2, e_3, e_4\}$, a set of subgroupness property parameters. The e_i for $i = 1,2,3,4$ stands for the parameters 'normal', "cyclic", "commutative", and "center" respectively. Let
$\tilde{A} = \left\{\hat{a}_1 = \left(\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}, \frac{0.4}{e_4}\right), \hat{a}_2 = \left(\frac{1}{e_1}, \frac{0.5}{e_2}, \frac{1}{e_3}, \frac{0.5}{e_4}\right), \hat{a}_3 = \left(\frac{1}{e_1}, \frac{0.2}{e_2}, \frac{0.4}{e_3}, \frac{0.3}{e_4}\right), \hat{a}_4 = \left(\frac{1}{e_1}, \frac{0.5}{e_2}, \frac{0.4}{e_3}, \frac{0.2}{e_4}\right)\right\}$,
$\tilde{B} = \left\{\hat{b}_1 = \left(\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}, \frac{0.4}{e_4}\right), \hat{b}_2 = \left(\frac{1}{e_1}, \frac{0.4}{e_2}, \frac{1}{e_3}, \frac{1}{e_4}\right), \hat{b}_3 = \left(\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}, \frac{0.5}{e_4}\right)\right\}$. *Where $\hat{a}_i$ and $\hat{b}_j$ are fuzzy sets over E. Therefore, with $f_{\tilde{A}}(\hat{a}_1) = \{e, r, r^2, r^3\}$, $f_{\tilde{A}}(\hat{a}_2) = \{e, r^2, sr, sr^3\}$, $f_{\tilde{A}}(\hat{a}_3) = f_{\tilde{A}}(\hat{a}_4) = D_4$ and $g_{\tilde{B}}(\hat{b}_1) = \{(0,0), (1,0)\}$, $g_{\tilde{B}}(\hat{b}_2) = \{(0,0), (1,1)\}$, $g_{\tilde{B}}(\hat{b}_3) = \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\theta(\hat{a}_1) = \hat{b}_1, \theta(\hat{a}_2) = \hat{b}_3$, and $\theta(\hat{a}_3) = \theta(\hat{a}_4) = \hat{b}_2$. We can deduce that $\phi\left(f_{\tilde{A}}(\hat{a})\right) = g_{\tilde{B}}\left(\theta(\hat{a})\right)$ for all $\hat{a} \in \tilde{A}$. Hence, (ϕ, θ) is an FFPS-isomorphism from $F_{\tilde{A}} \to G_{\tilde{B}}$.*

Proposition 3.3.6. *Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPS-set over the groups G_1 and G_2 respectively. Then, $F_{\tilde{A}} \times G_{\tilde{B}}$ is an FFPS-isomorphic to $G_{\tilde{B}} \times F_{\tilde{A}}$.*
Proof. *Let $(\phi, \theta): F_{\tilde{A}} \times G_{\tilde{B}} \to G_{\tilde{B}} \times F_{\tilde{A}}$. Therefore, from the set theory we know that mapping $\phi: G_1 \times G_2 \to G_2 \times G_1$ is given by $\phi(x, y) = (y, x)$,*
clearly, ϕ is bijective with $\phi^{-1} G_2 \times G_1 \to G_1 \times G_2$ given by $\phi^{-1}(y, x) = (x, y)$, for $x \in G_1$ and $y \in G_2$. Now, we show that ϕ is homomorphism. Let $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$, then, $\phi\big((x_1, x_2)(y_1, y_2)\big) = \phi\big((x_1y_1, x_2y_2)\big) = (x_2y_2, x_1y_1) = \big((x_2x_1), (y_2y_1)\big) = \phi(x_1, x_2)\phi(y_1, y_2)$. So ϕ is homomorphism and hence ϕ is an isomorphism. Also, the mapping $\theta, \tilde{A} \times \tilde{B} \to \tilde{B} \times \tilde{A}$ is given by $\theta\left(\hat{a}, \hat{b}\right) = \left(\hat{b}, \hat{a}\right)$ and obviously θ is a bijection with the inverse mapping. $\theta^{-1}: \tilde{B} \times \tilde{A} \to \tilde{A} \times \tilde{B}$ is given by $\theta^{-1}\left(\hat{b}, \hat{a}\right) = \left(\hat{a}, \hat{b}\right)$ for $\hat{a} \in \tilde{A}$ and $\hat{b} \in \tilde{B}$. Lastly, let $F_{\tilde{A}} \times G_{\tilde{B}} = H_{\tilde{A} \times \tilde{B}}$ and $G_{\tilde{B}} \times F_{\tilde{A}} = D_{\tilde{B} \times \tilde{A}}$, so we need to show that $\phi\left(h_{\tilde{B} \times \tilde{B}}\left(\hat{a} \wedge \hat{b}\right)\right) = d_{\tilde{B} \times \tilde{A}}\left(\theta\left(\hat{a} \wedge \hat{b}\right)\right)$.
Therefore, for all $\hat{a} \in \tilde{A}$ and $\hat{b} \in \tilde{B}$,

$$\phi\left(h_{\tilde{B} \times \tilde{B}}\left(\hat{a} \wedge \hat{b}\right)\right) = \theta\left(f_{\tilde{A}}(\hat{a}) \times g_{\tilde{B}}\left(\hat{b}\right)\right) = \phi\left(\left\{(x, y): x \in f_{\tilde{A}}(\hat{a}), y \in g_{\tilde{B}}\left(\hat{b}\right)\right\}\right) = \left\{(x, y): y \in g_{\tilde{B}}\left(\hat{b}\right), x \in f_{\tilde{A}}(\hat{a})\right\} = g_{\tilde{B}}\left(\hat{b}\right) \times h_{\tilde{A}}(\hat{a}) = d_{\tilde{B} \times \tilde{A}}\left(\theta\left(\hat{a} \wedge \hat{b}\right)\right)$$

Hence, $F_{\tilde{A}} \times G_{\tilde{B}}$ is FFPS-isomorphic to $G_{\tilde{B}} \times F_{\tilde{A}}$.

□

4. Conclusion

In this research work, we have seen that the complete fuzzification of elements of sets are studied in full group structures. It is also clear in this context that the fuzzified set of parameters are mapped into the subgroups of the initial universe which itself is a group. Furthermore, the operations of joint-intersection and meet-union are defined in terms of the new concept introduced. The investigation in this work as presented has opened a new direction that will facilitate researchers to a further study in channeling the ideas of fuzzy and soft set theories in algebraic structures.

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