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Mathematical Model for Nth Dimensional Space and the Impact of Modified Gravity and Coriolis Effect on Stratified Deep Water Using Series Solution

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A B S T R A C T

Stratified deep water refers to a layer of water that is deep enough that the effects of surface waves are negligible, and the water can be treated as a continuous, incompressible fluid. In such a system, the Coriolis effect and modified gravity can cause the water to stratify, or layer, with different densities and velocities in different regions. In an n-dimensional space, the Coriolis effect and modified gravity was shown to cause the water to stratify in complex ways, with different regions having different densities, velocities, and pressure gradients. The Coriolis effect, which is caused by the rotation of the Earth, causes the water to rotate and form vortices, while the modified gravity can cause the water to have different densities and pressures in different regions. A number of numerical simulations was carried out to understand the impact of modified gravity in deep water stratification and to predict the interactions between the internal waves within the layers. The model shows that stratification in deep water under the influence of the Coriolis effect and modified gravity have important implications for ocean circulation, climate, and marine ecosystems. Importantly, as there is no model presently which has successfully considered the impact of modified gravity and Coriolis effect together in the study of stratification in deep water. Therefore understanding the dynamics of stratified deep water in this context is undoubtedly an important area of research in oceanography and fluid dynamics in general.

1. Introduction

Deep water stratification refers to the layering of water masses in the ocean based on their density, which is determined by temperature and salinity. In the deep ocean, the stratification is mainly controlled by the vertical gradient of the water density, which is influenced by the modified gravity and Coriolis force (Topman & Mbah, 2025). Modified gravity refers to the effective gravity experienced by an object in a rotating frame of reference, such as the Earth.

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Earth's rotation always cause a centrifugal force that affects the mass distribution within the ocean, leading to a variation in the effective gravity. This variation can cause changes in the density stratification of the water leading to formation of distinct layers that needs to be studied to understand its unique properties (Topman *et al*., 2024; Topman *et al*., 2023). The Coriolis force is the apparent deflection of moving objects caused by the rotation of the Earth. In the ocean, the Coriolis force affects the movement of water masses and can contribute to the stratification of the water. The Coriolis force causes water masses to move in circular patterns, which can lead to the formation of distinct layers within the ocean (Topman & Mbah, 2025). The model for the nth dimensional space of stratified deep water under modified gravity and Coriolis effect is a mathematical framework that describes the behavior of fluid motion in a stratified ocean or sea under the influence of gravity and the Coriolis force (Topman & Mbah, 2025). This model is used to study the dynamics of ocean currents, waves, and other phenomena that occur in the ocean or sea (Lui *et al*., 2024). This model is based on the assumption that the fluid motion is incompressible, irrotational, and stratified, meaning that the density of the fluid varies with depth (Topman & Mbah, 2024). The modified gravity is a term that represents the deviation of the local gravitational acceleration from the standard value $9.8m/s^2$(Martin, 2023), it is as result of density variation in deep water that necessitated the alteration in standard gravity.

The Coriolis Effect is a rotational force that arises due to the rotation of the Earth (Roch *et al*., 2023), it causes moving objects to be deflected to the right in the Northern Hemisphere and to the left in the Southern Hemisphere (Chen *et al*., 2023; Topman *et al*., 2023). In deep water, the Coriolis effect causes water to move in a circular pattern, with the direction of the rotation depending on the hemisphere. Durran (2010). The model is typically formulated using the Navier-Stokes equations (Durran, 2010), this describes the motion of an incompressible fluid under the influence of external forces and gravity. The model equations are solved using perturbation method (Escher *et al*., 2011; Chae, 2020), such as the series solution method, to obtain a solution that describes the velocity and height at each layer in the stratified deep water (Chae, 2020). The model can be extended to higher dimensions, such as four-dimensional to nth-dimensional space (Charney, 1948), to study the effects of different factors (Chen *et al*., 2023), such as the shape of the ocean basin, (Constantin, 2019), the presence of obstacles, or the influence of other forces, such as wind or tides (Le Veque, 2004). The model can also be used to study the effects of climate change on ocean circulation (Martin, 2021; Martin, 2022). Overall, the model for the nth dimensional space of stratified deep water under modified gravity and Coriolis effect (Mbah & Udogu, 2015; Joyce *et al*., 2020), is a powerful tool for understanding the complex dynamics of ocean currents and other phenomena that occur in the ocean. Constantin, (2019); Constantin, (2009). Stratification refers to the layering of deep water masses based on their density. Ponte, (2011). In the context of deep water, stratification is primarily driven by differences in deep water temperature and salinity, which always affect the deep water's density (Rippeth *et al*., (2024); Gaspard-Gustave de Coriolis, (1835); Mengwei *et al*., (2024).

2. Methodology

2.1 Model Assumption

We wish to formulate some necessary and basic assumptions to aid our model mathematically.

The following are the assumptions for the model:

i. The fluid is incompressible with continuous density stratification

ii. In deep water flows the depth is infinite, so the vertical length scale (h) and the horizontal length scale (L) guarantee deep water regime when $\frac{L}{h} \ll 1\ not\ \frac{L}{h} \gg 1\ (deep\ water\ assumption)$

iii. The velocity components in the directions of increasing $x\ and\ y$ will be denoted by $u\ and\ v$

iv. The variation in velocity along y- direction is constant because the flow is predominantly in horizontal direction so that partial derivative of velocity with respect to y is zero.

2.2 Mathematical Formulation of the Problem

The model formulation of stratified deep water under modified gravity and coriolis effect can be done using various physical and mathematical principles, particularly by adopting conservation of momentum and continuity equation (Topman *et al* 2025).

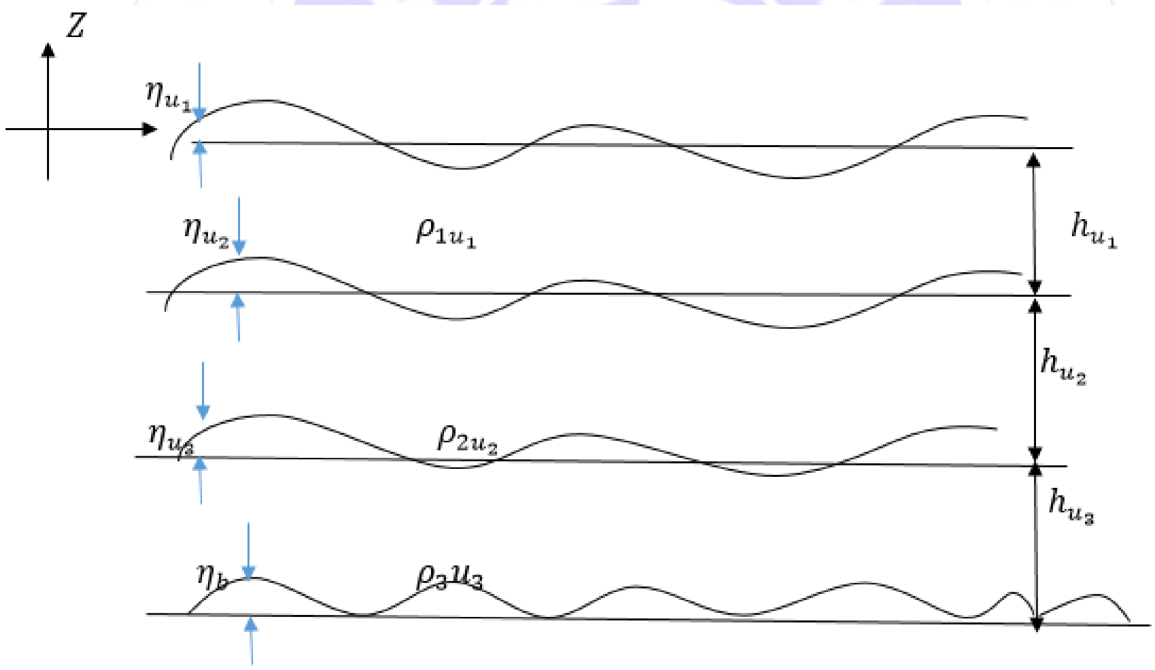

Figure 1:Symmetry of stratified deep water columns.

Consider from Fig 1, layers of incompressible fluid under kinematic and dynamic boundary condition. We shall denote the lower layer 1, the intermediate layer 2 and the upper layer 3, with respective densities $\rho_1, \rho_2\ and\ \rho_3$, the mean horizontal velocities $u_1, u_2 and\ u_3$ and the thicknesses $h_1, h_2 and\ h_3$, with $h_1 + h_2 + h_3 + \cdots = H$, where H is the distance between the mean surface and the overall thermocline position. The pressure P is constant, the unknowns are therefore $h_1, h_2, h_3, u_1, u_2, u_3, \rho_1, \rho_2, \rho_3$ and all are functions of (x, t). Similarly, the further stratified layers can have heights and velocities of $h_{11}, h_{21}, h_{31}, \dots h_{n_1}$ and $u_{11}, u_{21}, u_{31}, \dots u_{n_1}$ to n^{th} dimension.

2.3 Model Equation

By incorporating the effect of rotation and modified gravity (Topman & Mbah, 2025) we obtain our model equations, which are nine set of first order partial differential equations.

$$\frac{\partial(h_1u_1)}{\partial t} + \frac{\partial\left(h_1u_1^2 + \frac{g\frac{\rho_0-\rho_1}{\rho_1}h_1^2}{2}\right)}{\partial x} = -g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial h_2}{\partial x} - g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x} + fu_1 \tag{1}$$

$$\frac{\partial(h_1v_1)}{\partial t} = -g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x} - fu_1 \tag{2}$$

$$\frac{\partial h_1}{\partial t} + \frac{\partial(h_1u_1)}{\partial x} = 0 \tag{3}$$

$$\frac{\partial(h_2u_2)}{\partial t} + \frac{\partial\left(h_2u_2^2 + \frac{g\frac{\rho_2-\rho_1}{\rho_1}h_2^2}{2} + g\frac{\rho_2-\rho_1}{\rho_1}h_2h_1\right)}{\partial x} = -g\frac{\rho_1-\rho_2}{\rho_2}h_1\frac{\partial h_2}{\partial x} - g\frac{\rho_1-\rho_2}{\rho_2}h_2\frac{\partial(\xi)}{\partial x} + fu_2 \tag{4}$$

$$\frac{\partial(h_2v_2)}{\partial t} = -g\frac{\rho_2-\rho_1}{\rho_1}h_1\frac{\partial h_2}{\partial y} - g\frac{\rho_2-\rho_1}{\rho_1}h_2\frac{\partial(\xi)}{\partial y} - fu_2 \tag{5}$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial(h_2u_2)}{\partial x} = 0 \tag{6}$$

$$\frac{\partial(h_3u_3)}{\partial t} + \frac{\partial\left(h_3u_3^2 + \frac{g\frac{\rho_3-\rho_2}{\rho_2}h_2^2}{2} + g\frac{\rho_3-\rho_2}{\rho_2}h_3h_2\right)}{\partial x} = -g\frac{\rho_3-\rho_2}{\rho_2}h_1\frac{\partial h_3}{\partial x} - g\frac{\rho_3-\rho_2}{\rho_2}h_3\frac{\partial(\xi)}{\partial x} + fu_3 \tag{7}$$

$$\frac{\partial(h_3v_3)}{\partial t} = -g\frac{\rho_3-\rho_2}{\rho_2}h_1\frac{\partial h_2}{\partial y} - g\frac{\rho_3-\rho_2}{\rho_2}h_2\frac{\partial(\xi)}{\partial y} - fu_3 \tag{8}$$

$$\frac{\partial h_3}{\partial t} + \frac{\partial(h_3u_3)}{\partial x} = 0 \tag{9}$$

Let us impose necessary boundary condition on our model equations:

$$\left.\begin{array}{lll} u_1(x,0) = u_0, & u_x(0,y,t) = 0, & u_x(l,t) = 0 \\ u_2(x,0) = u_0, & u_2(x,t) = 0, & u_x(x,t) = 0 \\ & u_{(2)x}(0,t) = 0, & u_{2(x)}(l,t) = 0 \end{array}\right\} \tag{10}$$

$h(x,0) = me^{-s(x^2)} - \xi(x), \quad h_x(u,t),$

$\xi(x,y) = \beta sin(\alpha x)$ $0 \le \alpha \le 90$, and the values of β and α which are the measure of stability and measure of strength of the system respectively, depend on the size of the stratified layer at thermocline regime..

$h(x,t=0) = me^{-s(x^2)} - \xi(x)$

Where m represents disturbance in the deep water and the variable s represents the spatial scale.

Let us expand our model into series

$$\frac{\partial h}{\partial t}+\frac{\partial(h_{11}u_{11})}{\partial x}+\frac{\partial(h_{21}u_{21})}{\partial x}+\frac{\partial(h_{31}u_{31})}{\partial x}=0$$

$$\frac{\partial h}{\partial t}+\frac{\partial(h_{11}u_{11}+h_{21}u_{21}+h_{31}u_{31})}{\partial x}=0 \tag{11}$$

That is, $\frac{dh_{11}}{dt}+h_{11}\frac{\partial(u_{11})}{\partial x}=0$ (12)

$$\frac{dh_{21}}{dt}+h_{21}\frac{\partial(u_{21})}{\partial x}=0 \tag{13}$$

$$\frac{dh_{31}}{dt}+h_{31}\frac{\partial(u_{31})}{\partial x}=0 \tag{14}$$

Therefore, $\frac{dh_{11}}{dt}=-h_{11}\frac{\partial(u_{11})}{\partial x}$ (15)

$$\frac{dh_{21}}{dt}=-h_{21}\frac{\partial(u_{21})}{\partial x} \tag{16}$$

$$\frac{dh_{31}}{dt}=-h_{31}\frac{\partial(u_{31})}{\partial x} \tag{17}$$

Equation (11) accounts for total partial derivative in the three layers , while equation (15), (16), and (17) are the equations used for model modification.

Note that if, $f=(x,y,t)\Rightarrow f=(x,t)$, since the variation of velocity in y- direction is very small and then neglected.

$$\frac{df}{dt}=\frac{\partial f}{\partial t}.\frac{dt}{dt}+\frac{\partial f}{\partial x}.\frac{dx}{dt}+\frac{\partial f}{\partial y}.{}^{dy}/_{dt} \tag{18}$$

Equation (18) is a product rule, we can use it to rewrite the model equation in the first layer as,

$$\frac{\partial(h_{11}u_{11})}{\partial t}+\frac{\partial\left(h_{11}u_{11}^2+g(\frac{\rho_1-\rho_0}{\rho_0})^{h_{11}^2}/_2\right.}{\partial x}=-g\frac{\rho_1-\rho_0}{\rho_0}\ \frac{\partial\xi}{\partial x}+fu_{11} \tag{19}$$

$$u_{11}\ \frac{\partial h_{11}}{\partial t}+h_{11}\frac{\partial u_{11}}{\partial t}+u_{11}^2\frac{\partial h_{11}}{\partial x}+2h_{11}u_{11}\frac{\partial u_{11}}{\partial x}+g\frac{\rho_1-\rho_0}{\rho_0}h_{11}\frac{\partial h_{11}}{\partial x}-g\frac{\rho_1-\rho_0}{\rho_0}\frac{\partial\xi}{\partial x}+fu_{11} \tag{20}$$

This is further simplified as:

$$u_{11}\left(\frac{\partial h_{11}}{\partial t}+u_{11}\frac{\partial h_{11}}{\partial x}\right)+h_{11}\left(\frac{\partial u_{11}}{\partial t}+2u_{11}\frac{\partial u_{11}}{\partial x}\right)+g\frac{(\rho_1-\rho_0)}{\rho_0}h_{11}\frac{\partial h_{11}}{\partial x}=-g\frac{(\rho_0-\rho_1)}{\rho_1}\frac{\partial\xi}{\partial x}+fu_{11} \tag{21}$$

From equation (41), recall the continuity equation for each of the stratification.

From the first stratification level,

$$\frac{\partial}{\partial t}(h_{11})+\frac{\partial}{\partial x}(h_{11}u_{11})=0 \tag{22}$$

$$\Rightarrow\frac{\partial}{\partial t}(h_{11})+h_{11}\frac{\partial}{\partial x}(u_{11})+u_{11}\frac{\partial(h_{11})}{\partial x}=0 \tag{23}$$

$$\frac{\partial}{\partial t}(h_{11})+u_{11}\frac{\partial(h_{11})}{\partial x}=-h_{11}\frac{\partial}{\partial x}(u_{11}) \tag{24}$$

Substituting equation (24) into (21), we get

$$-u_{11}h_{11}\frac{\partial u_{11}}{\partial t}+h_{11}\frac{\partial}{\partial x}(u_{11})+2u_{11}h_{11}\frac{\partial u_{11}}{\partial x}+gh_{11}\frac{\rho_1-\rho_0}{\rho_0}\frac{\partial h_{11}}{\partial x}=-gh_{11}\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial\xi}{\partial x}+fu_{11}$$

$$h_{11}\left(\frac{\partial}{\partial t}u_{11}+u_{11}\frac{\partial}{\partial x}u_{11}\right)+gh_{11}\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial h_{11}}{\partial x}=-g\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial\xi}{\partial x}+fu_{11} \tag{25}$$

From equation (25),

$$\left(\frac{\partial}{\partial t}u_{11} + u_{11}\frac{\partial}{\partial x}u_{11}\right) = \frac{du_{11}}{dt} \tag{26}$$

$$h_{11}\frac{du_{11}}{dt} + gh_{11}\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial h_{11}}{\partial x} = -g\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial \xi}{\partial x} + fu_{11} \tag{27}$$

Using (12) in (21), yields;

$$h_{11}\frac{du_{11}}{dt} + g\frac{(\rho_1-\rho_0)}{\rho_0}h_{11}\frac{\partial h_{11}}{\partial x} = -g\frac{(\rho_1-\rho_0)}{\rho_0}\frac{\partial \xi}{\partial x} + fu_{11} \tag{28}$$

Now let $0 < f \ll 1, and\ g^{'} = g\frac{(\rho_{i+_1}-\rho_i)}{\rho_i} = af$

The inclusion of Coriolis parameter on the left-hand side of the equation accounts for the rotation, and the solution obtained will be more accurate. Since the coriolis parameter is approximately $7.29 \times 10^{-5} rad/s$ (Gaspard-Gustave de Coriolis, (1835), which is $\ll 1$, then the series expansion of the velocities and heights can be written in terms of f. Suppose the solution $(u_{11}, u_{21}, u_{31}, h_{11}, h_{21}, h_{31})$, where

$$\left.\begin{aligned} u_{11}(x,t) &= u_{12}(x,t) + fu_{13}(x,t) + f^2u_{14} + \cdots \\ u_{21}(x,t) &= u_{22}(x,t) + fu_{23}(x,t) + f^2u_{24} + \cdots \\ u_{31}(x,t) &= u_{32}(x,t) + fu_{33}(x,t) + f^2u_{34} + \cdots \\ h_{11}(x,t) &= h_{12}(x,t) + fh_{13}(x,t) + f^2u_{14} + \cdots \\ h_{21}(x,t) &= h_{22}(x,t) + fh_{23}(x,t) + f^2u_{24} + \cdots \\ h_{31}(x,t) &= h_{32}(x,t) + fh_{33}(x,t) + f^2u_{34} + \cdots \end{aligned}\right\} \tag{29}$$

Substituting equation (29) in (12), (13), and (14), we get

$$\left.\begin{aligned} \frac{d}{dt}(h_{12} + fh_{13} + \cdots) + (h_{12} + fh_{13} + \cdots)\left(\frac{\partial}{\partial x}(u_{12} + fu_{13} + \cdots) + \frac{\partial}{\partial x}(u_{12} + fu_{13} + \cdots)\right) = 0 \\ \frac{d}{dt}(h_{22} + fh_{23} + \cdots) + (h_{22} + fh_{23} + \cdots)\left(\frac{\partial}{\partial x}(u_{12} + fu_{13} + \cdots) + \frac{\partial}{\partial x}(u_{22} + fu_{23} + \cdots)\right) = 0 \\ \frac{d}{dt}(h_{32} + fh_{33} + \cdots) + (h_{32} + fh_{33} + \cdots)\left(\frac{\partial}{\partial x}(u_{12} + fu_{13} + \cdots) + \frac{\partial}{\partial x}(u_{32} + fu_{33} + \cdots)\right) = 0 \end{aligned}\right\} \tag{30}$$

Which simplifies to;

$$\left.\begin{aligned} \frac{d}{dt}h_{12} + \frac{d}{dt}fh_{13} + \cdots + h_{12}\frac{\partial}{\partial x}u_{12} + h_{12}\frac{\partial}{\partial x}fu_{13} + fh_{13}\frac{\partial}{\partial x}u_{12} + f^2h_{13}\frac{\partial}{\partial x}fu_{13} + \cdots = 0 \\ \frac{d}{dt}h_{22} + \frac{d}{dt}fh_{23} + \cdots + h_{22}\frac{\partial}{\partial x}u_{22} + h_{22}\frac{\partial}{\partial x}fu_{23} + fh_{23}\frac{\partial}{\partial x}u_{22} + f^2h_{23}\frac{\partial}{\partial x}fu_{23} + \cdots += 0 \\ \frac{d}{dt}h_{32} + \frac{d}{dt}fh_{33} + \cdots + h_{32}\frac{\partial}{\partial x}u_{32} + h_{32}\frac{\partial}{\partial x}fu_{33} + fh_{33}\frac{\partial}{\partial x}u_{32} + f^2h_{33}\frac{\partial}{\partial x}fu_{33} + \cdots += 0 \end{aligned}\right\} \tag{31}$$

Substituting equation (29) in (28), we get,

$$(h_{12} + fh_{13} + \cdots)\frac{d}{dt}(u_{12} + fu_{13} + \cdots) + af(h_{12} + fh_{13} + \cdots)\frac{\partial}{\partial x}(h_{12} + fh_{13} + \cdots = -af\frac{\partial \xi}{\partial x} + f(u_{12} + fu_{13} + \cdots)$$

$$\left.\begin{aligned} h_{12}\frac{d}{dt}u_{12} + fh_{12}\frac{d}{dt}u_{13} + fh_{13}\frac{d}{dt}u_{12} + f^2h_{13}\frac{d}{dt}u_{13} + \cdots + afh_{12}\frac{\partial}{\partial x}h_{12} + af^2h_{12}\frac{\partial}{\partial x}h_{13} + \\ af^2h_{13}\frac{\partial}{\partial x}h_{12} + af^3h_{13}\frac{\partial}{\partial x}h_{13} + \cdots += -af\frac{\partial \xi}{\partial x} + fu_{12} + f^2u_{13} + \cdots \end{aligned}\right\} \tag{32}$$

Substituting equation (29) in (32), we obtain;

$(h_{12}+fh_{13}+\cdots)\frac{d}{dt}(u_{22}+fu_{23}+\cdots)+af(h_{22}+fh_{23}+\cdots)\frac{\partial}{\partial x}(h_{22}+fh_{23}+\cdots)=-af\frac{\partial \xi}{\partial x}+f(u_{12}+fu_{13}+\cdots)$(33)

$$\left.\begin{aligned} &h_{22}\frac{d}{dt}u_0+fh_{22}\frac{d}{dt}u_{23}+fh_{23}\frac{d}{dt}u_{22}+f^2h_{23}\frac{d}{dt}u_{23}+\cdots+afh_{22}\frac{\partial}{\partial x}h_0+af^2h_{22}\frac{\partial}{\partial x}h_{23}+\\ &af^2h_{23}\frac{\partial}{\partial x}h_{22}+af^3h_{23}\frac{\partial}{\partial x}h_{23}+\cdots+=-af\frac{\partial \xi}{\partial x}+fu_{22}+f^2u_{23}+\cdots \end{aligned}\right\} \quad (34)$$

Comparing the coefficient of powers of f from (32) and (34), we have;

f^0

$$h_{12}\frac{du_{12}}{dt}=0 \quad ; \qquad u_{13}(x,0)=u_{13} \quad (35)$$

$$h_{22}\frac{du_{22}}{dt}=0 \quad ; \qquad u_{23}(x,0)=u_{23} \quad (36)$$

$$h_{32}\frac{du_{32}}{dt}=0 \quad ; \qquad u_{33}(x,0)=u_{33} \quad (37)$$

$$\left.\begin{aligned} &\frac{dh_{12}}{dt}=-h_{12}\left(\frac{\partial u_{13}}{\partial x}\right);\ h_{12}(x,0)=me^{-s(x^2)}-\xi(x,0)\\ &\frac{dh_{22}}{dt}=-h_{22}\left(\frac{\partial u_{23}}{\partial x}\right);\ h_{22}(x,0)=me^{-s(x^2)}-\xi(x,0)\\ &\frac{dh_{32}}{dt}=-h_{32}\left(\frac{\partial u_{33}}{\partial x}\right);\ h_{32}(x,0)=me^{-s(x^2)}-\xi(x,0) \end{aligned}\right\} \quad (38)$$

f^1:

$$\left.\begin{aligned} &h_{12}\frac{du_{13}}{dt}+h_{13}\frac{du_{12}}{dt}+ah_{12}\frac{\partial h_{12}}{\partial x}=-a\frac{\partial \xi}{\partial x}+u_{12};\ u_{13}(x,0)=0\\ &h_{22}\frac{du_{23}}{dt}+h_{23}\frac{du_{22}}{dt}+ah_{22}\frac{\partial h_{22}}{\partial x}=-a\frac{\partial \xi}{\partial x}+u_{22};\ u_{23}(x,0)=0\\ &h_{32}\frac{du_{33}}{dt}+h_{33}\frac{du_{32}}{dt}+ah_{32}\frac{\partial h_{32}}{\partial x}=-a\frac{\partial \xi}{\partial x}+u_{32};\ u_{33}(x,0)=0 \end{aligned}\right\} \quad (39)$$

$$\frac{dh_{13}}{dt}+h_{12}\frac{\partial u_{13}}{\partial x}+h_{13}\frac{\partial u_{12}}{\partial x}=0 \quad (40)$$

$$\frac{dh_{23}}{dt}+h_{22}\frac{\partial u_{23}}{\partial x}+h_{23}\frac{\partial u_{22}}{\partial x}=0 \quad (41)$$

$$\frac{dh_{33}}{dt}+h_{32}\frac{\partial u_{3}}{\partial x}+h_{33}\frac{\partial u_{32}}{\partial x}=0 \quad (42)$$

f^2:

$$\left.\begin{aligned} &h_{13}\frac{d}{dt}u_{13}+ah_{12}\frac{\partial}{\partial x}h_{13}+ah_{13}\frac{\partial}{\partial x}h_{12}=u_{13};\ u_{13}(x,0)=0\\ &h_{23}\frac{d}{dt}u_{23}+ah_{22}\frac{\partial}{\partial x}h_{23}+ah_{23}\frac{\partial}{\partial x}h_{22}=u_{23};\ u_{23}(x,0)=0\\ &h_{33}\frac{d}{dt}u_{33}+ah_{32}\frac{\partial}{\partial x}h_{33}+ah_{33}\frac{\partial}{\partial x}h_{32}=u_{33};\ u_{33}(x,0)=0 \end{aligned}\right\} \quad (43)$$

f^3:

$$\left.\begin{aligned} &ah_{13}\frac{\partial}{\partial x}h_{13=0}\\ &ah_{23}\frac{\partial}{\partial x}h_{23=0}\\ &ah_{33}\frac{\partial}{\partial x}h_{33=0} \end{aligned}\right\} \quad (44)$$

Now integrating (35) with respect to t, we have;

$u_{12}(x,t) = c_{13} = const\ ant$

$u_{12}(x,0) = c_{13} = u_{12}$

$$u_{12}(x,t) = u_{12} \tag{45}$$

Using (45), equation (39) reduces to

$$\frac{du_{13}}{dt} = 2amsxe^{-s(x^2)} + \alpha\alpha\beta cos\alpha x - \frac{\alpha\alpha\beta cos\alpha x}{me^{-sx^2} - \beta\sin\alpha x} + \frac{u_{12}}{me^{-sx^2} - \beta\sin\alpha x} \tag{46}$$

Integrating with respect to t, we have

$$u_{13}(\,x,t) = \left(2amsxe^{-sx^2} + \alpha\alpha\beta cos\alpha x - \frac{\alpha\alpha\beta cos\alpha x}{me^{-sx^2} - \beta\sin\alpha x} + \frac{u_0}{me^{-sx^2} - \beta\sin\alpha x}\right) t \tag{47}$$

Equation (47) is for speed of wave in first layer.

Now using (70) and (81)), equation (40) becomes;

$$\frac{dh_{13}}{dt} = -h_{12}\left(\frac{\partial u_1}{\partial x}\right) - h_{13}\left(\frac{\partial u_{12}}{\partial x}\right) \tag{48}$$

That is

$$\frac{dh_{13}}{dt} = -h_{12}\left(\frac{\partial u_{13}}{\partial x}\right) \tag{49}$$

This simplifies to;

$$\frac{\partial u_{13}}{\partial x} = \left(\begin{array}{c} 2amsxe^{-sx^2} - 4ams^2x^2e^{-sx^2} + \alpha\alpha^2\beta\sin(\alpha x) - \\ \frac{m\alpha\beta e^{-sx^2}(2cos\alpha x - sin\alpha x) + \alpha\alpha^2\beta^2(\sin^2\alpha x + \cos^2\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} + u_{12}\frac{(2mxe^{-sx^2} + \alpha\beta cos\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2}\end{array}\right) t \tag{50}$$

Integrating with respect to t, we get

$$h_{13} = -t^2 g\frac{\rho_1-\rho_0}{\rho_0}m^2sxe^{-2sx^2} + t^2\beta sin\alpha x g\frac{\rho_1-\rho_0}{\rho_0}msxe^{-sx^2} + 2t^2 g\frac{\rho_1-\rho_0}{\rho_0}m^2s^2x^2e^{-2sx^2} -$$
$$2t^2\beta\sin(\alpha x)\, g\frac{\rho_1-\rho_0}{\rho_0}ms^2x^2e^{-sx^2} - \frac{1}{2}t^2me^{-sx^2} g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta\sin(\alpha x) + \frac{1}{2}t^2 g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^2\sin^2(\alpha x) +$$
$$\frac{1}{2}t^2\frac{m^2\alpha\beta e^{-2sx^2}(2cos\alpha x - sin\alpha x) + me^{-sx^2}g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^2}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} - \frac{1}{2}t^2\frac{m\alpha\beta^2\sin(\alpha x)e^{-sx^2}(2cos\alpha x - sin\alpha x) + g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^3\sin(\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} -$$
$$\frac{1}{2}t^2u_{12}me^{-sx^2}\frac{(2mxe^{-sx^2} + \alpha\beta cos\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} + \frac{1}{2}t^2u_{12\beta\sin(\alpha x)}\frac{(2mxe^{-sx^2} + \alpha\beta cos\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \tag{51}$$

3. The N^{th} Stratified Column

We consider continuously stratified deep water up to nth layer and derive a general equation for the series expansion.

$$h_{10}\frac{du_{11}}{dt} + h_{11}\frac{du_{10}}{dt} + ah_{10}\left(\frac{\partial h_{10}}{\partial x}\right) = -a\frac{\partial\xi}{\partial x} + u_{10} \tag{52}$$

$$h_{20}\frac{du_{21}}{dt} + h_{21}\frac{du_{20}}{dt} + ah_{20}\left(\frac{\partial h_{20}}{\partial x}\right) = -a\frac{\partial\xi}{\partial x} + u_{20} \tag{53}$$

$$h_{30}\frac{du_{31}}{dt} + h_{31}\frac{du_{30}}{dt} + ah_{30}\left(\frac{\partial h_{30}}{\partial x}\right) = -a\frac{\partial\xi}{\partial x} + u_{30} \tag{54}$$

$$\frac{dh_{11}}{dt} = -h_{11}\left(\frac{\partial u_{10}}{\partial x}\right) - h_{10}\left(\frac{\partial u_{11}}{\partial x}\right) \tag{55}$$

$$\frac{dh_{21}}{dt} = -h_{21}\left(\frac{\partial u_{20}}{\partial x}\right) - h_{20}\left(\frac{\partial u_{21}}{\partial x}\right) \tag{56}$$

$$\frac{dh_{31}}{dt} = -h_{31}\left(\frac{\partial u_{30}}{\partial x}\right) - h_{30}\left(\frac{\partial u_{31}}{\partial x}\right) \tag{57}$$

Equation (52-56) are the equations of heights for stratified columns as we have established.

Similarly, extending to nth dimensional space we have;

$$h_{n0}\frac{du_{n1}}{dt} + h_{n1}\frac{du_{n0}}{dt} + ah_{n0}\left(\frac{\partial h_{10}}{\partial x} + \frac{\partial h_{20}}{\partial x} + \frac{\partial h_{30}}{\partial x} + \cdots + \partial u_{n0}\right) + nh_{(n-1)0}a\frac{\partial h_{n0}}{\partial x} = -ah_{n0}\frac{\partial \xi}{\partial x} + u_{n0} \tag{58}$$

$$\frac{dh_{n+1}}{dt} = -h_{n1}\left(\frac{\partial u_{n0}}{\partial x}\right) - h_{n0}\frac{\partial u_{n1}}{\partial x} \tag{59}$$

Equation (57) and (58) are general equations of height for nth stratification.

4. Results and Simulations

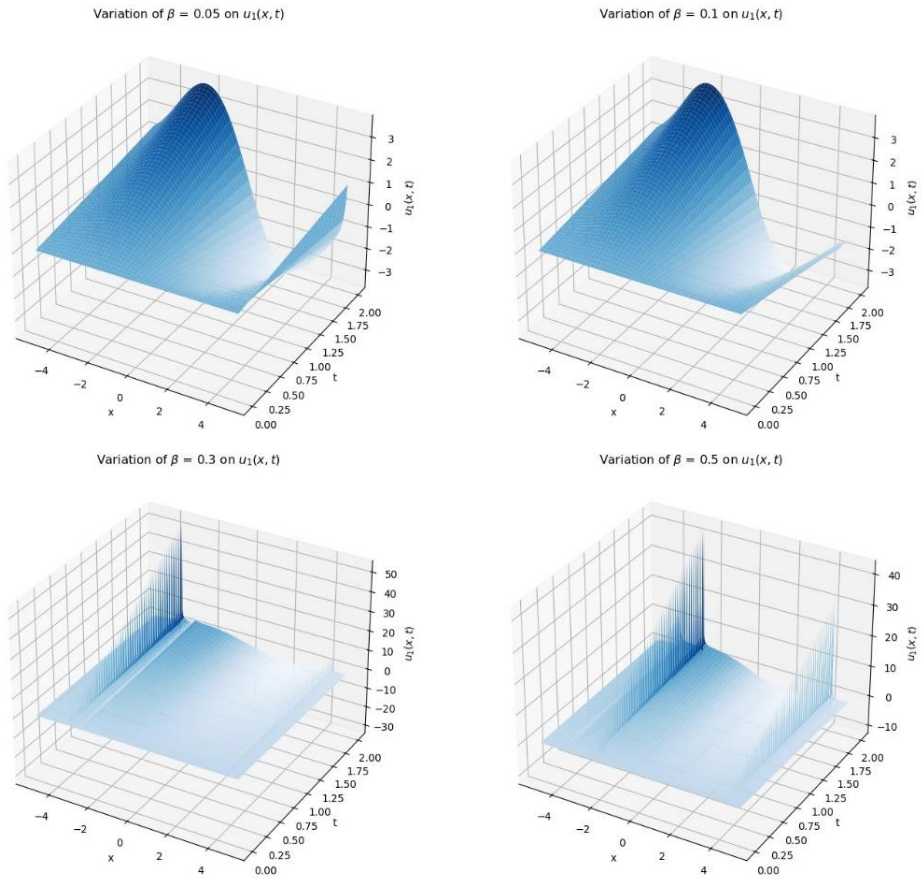

Figure 2: The stratification of deep water in first layer under modified gravity and coriolis effect with time t

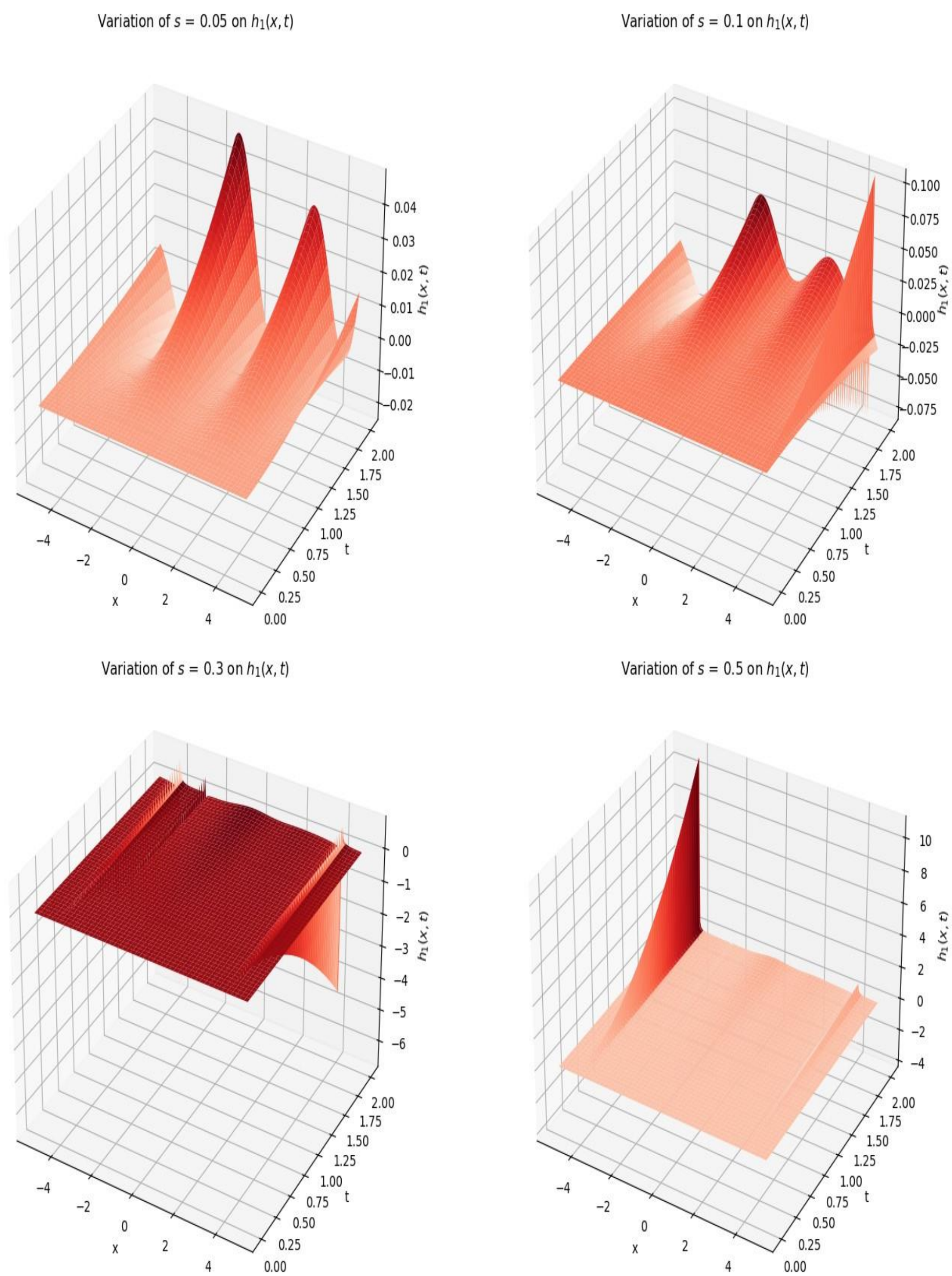

Figure 3: Stratification in second layer showing increase in depth as time varies

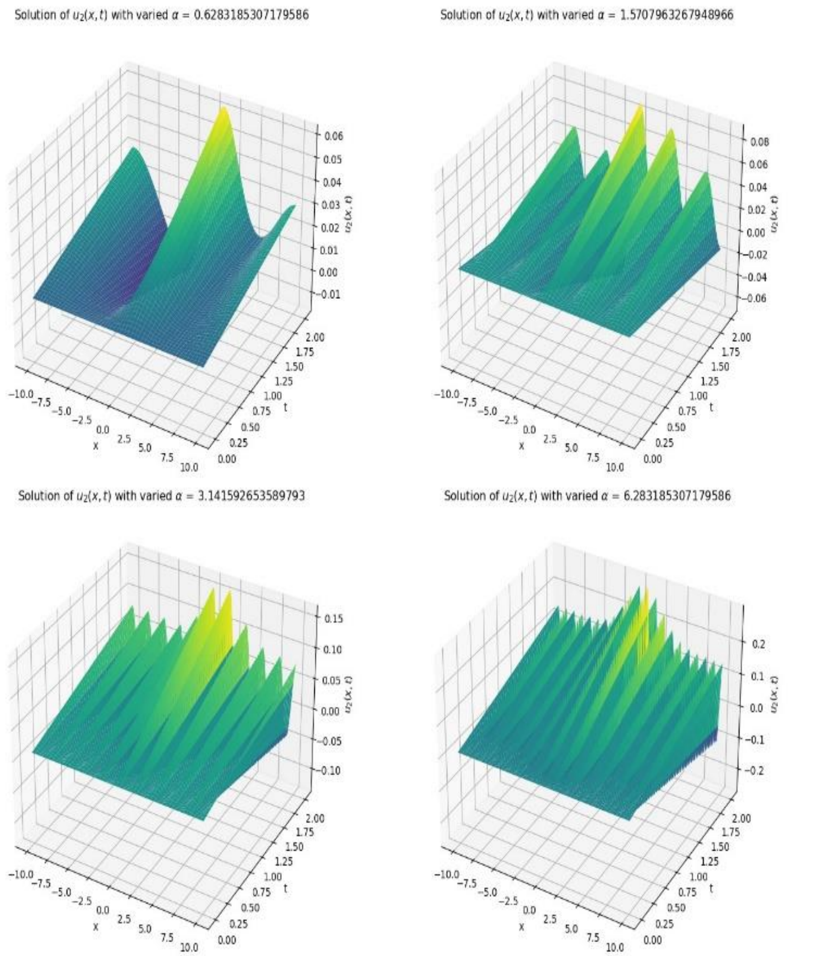

Figure 4: The nth stratified columns as height of deep water increases with varied density for given time

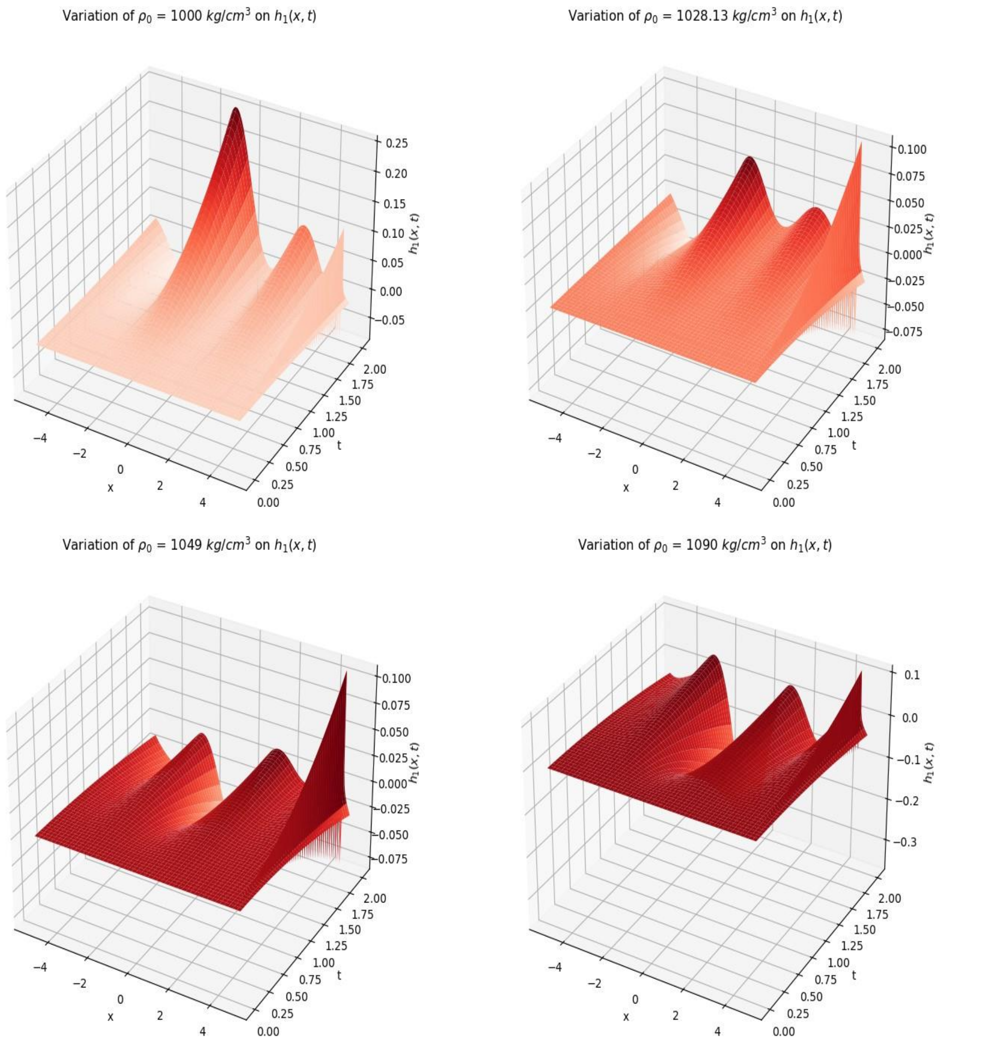

Figure 5: The density variation as result of gravity modification and coriolis effect showing stable stratified columns *at* n^{th} *dimension of the deep water*

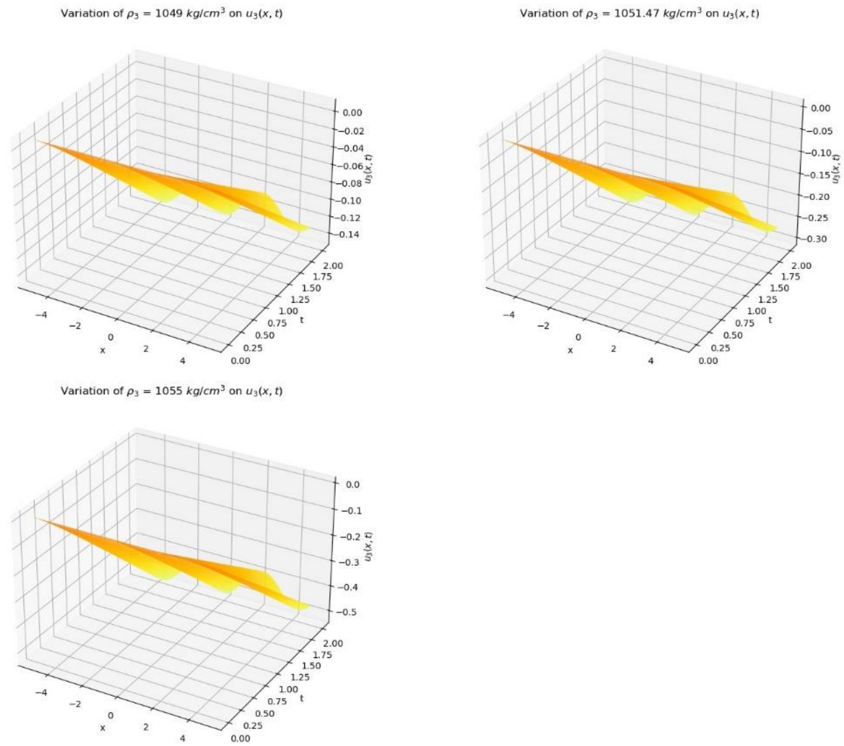

Figure 6: The continuous stratification of deep water column as the densities and velocities increases at nth *dimensional space.*

5. Conclusion

We have deduced general equation for nth stratification in deep water under the influence of modified gravity and coriolis effect and carried out numerical simulations that show the behavior of stratified deep water under modified gravity and coriolis effect at nth dimensional space. This provided better insights in making accurate predictions on the dynamics of fluid motion in stratified systems as our model proved that stratification occurs throughout the depth. The model took into account the effects of gravity modification and Coriolis force which makes it a valuable tool for studying a wide range of phenomena in oceanography, meteorology, and other fields of interest.

Furthermore, the model provides an enhance understanding of stratified deep water environments which are often rich in oil and gas resources. By studying the behavior of fluids in these environments using this model, we can develop new techniques for oil and gas exploration and extraction, which can lead to more efficient and cost-effective methods for extracting these valuable resources and in addition design up-to-date underwater robotic device for future exploration.

List of Symbols

h	Vertical length scale
x	Horizontal, x direction
y	Horizontal, y direction
z	Vertical, z direction
t	Time
u_1	Velocity in the first layer in the $x-$ direction
u_2	Velocity in the second layer in the $x-$ direction
u_2	Velocity in third layer in the $x-$ direction
u_2	Velocity in the nth layer in the $x-$ direction
α	Measure of strength of the system
β	Measure of stability of the system
F	Summation of forces
m	Deep water mass
$u = (u, v, w)$	The three dimensional velocity vector
H	Deep water dept
h	The water height above each stratified column
ρ	The density
g	The gravity constant
$g^{'}$	Modified gravity
f	Coriolis parameter
u	Velocity in the horizontal x direction
v	Velocity in the horizontal y direction
$h(x, y, t)$	The height of water surface from the same reference
$\xi(x, y)$	Denotes the thermocline regime
δx	Width ($x-$direction)
δy	Width ($y-$direction)
a	Acceleration of deep water particle
$\frac{D}{Dt}$	Material derivative

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Ethics Statement:
This research did not involve human participants or animals, and no ethical approval was required.

Conflicts of Interest

The authors declare that there is no conflict of interest.

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