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Block Hybrid Numerical Method for the Direct Solutions of Second, Third and Fourth Order Initial Value Problems

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ABSTRACT

A one-step block hybrid method for the direct numerical solution of second, third and fourth-order ordinary differential equations (ODEs) without reducing them to equivalent systems of first-order equations is developed and implemented in this study. Collocation and interpolation techniques are used in the construction of the method to obtain a continuous implicit scheme that is subsequently converted into an explicit block hybrid form. A thorough analysis is conducted of the scheme's fundamental characteristics, such as order, error constant, consistency, zero-stability, convergence and the region of absolute stability. Comparing the suggested method to existing approaches in the literature, numerical experiments on benchmark problems like the mass-spring system, third-order oscillatory models, and fourth-order oscillatory differential equations show that it is more accurate and stable. The findings indicate a computationally efficient and highly accurate framework for the direct solution of higher-order ODEs, significantly advancing numerical analysis in applied mathematics.

1. Introduction

In science and engineering, mathematical models especially those involving differential equations are crucial for comprehending and forecasting physical phenomena (Abdelrahim, 2021, Adewale & Sabo, 2024). In order to analyze systems in motion, heat transfer, fluid flow, and electrical circuits and to produce optimized designs and technological innovation these models frequently produce equations that explain how quantities change over time or space (Kuboye, 2015, Abdulrahim & Omar, 2017). Differential equations are useful in a variety of disciplines outside of the physical sciences, including economics, medicine, psychology, operations research, biology, and anthropology. In these domains, they are used to model intricate processes like disease transmission, market dynamics, learning behavior, logistics, and cultural evolution (Kuboye, 2015; Abolarin et al., 2022). Particularly in the natural sciences and technology, ordinary differential equations (ODEs) are frequently utilized.

This study consider the direct solution of higher order initial value problems for ordinary differential equations of the form:

$$y^{(\lambda)}(\tau)=f\left(\tau,y,y^{(1)},\cdots,y^{(\lambda-1)}\right),\ y(a_0)=\tau_0,\ y'(a_1)=\tau_1,\cdots,y^{(\lambda-1)}(a_\mu)=\tau_\mu \qquad (1.1)$$

Specifically the second, third and fourth order initial value problems of the form

$$\left.\begin{array}{l} y''(\tau)=f(\tau,y,y'),\ y(a_0)=\tau_0,\ y'(a_1)=\tau_1 \\ y'''(\tau)=f(\tau,y,y',y''),\ y(a_0)=\tau_0,\ y'(a_1)=\tau_1,\ y''(a_2)=\tau_2 \\ y''''(\tau)=f(\tau,y,y',y'',y'''),\ y(a_0)=\tau_0,\ y'(a_1)=\tau_1,\ y''(a_2)=\tau_2,\ y'''(a_3)=\tau_3 \end{array}\right\} \qquad (1.2)$$

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Higher-order ordinary differential equations (ODEs) are typically solved using traditional methods that reduce them to systems of first-order ODEs, which are then solved incrementally using numerical techniques. This method is computationally intensive, prone to increased complexity, and may introduce noteworthy errors and implementation challenges. A block hybrid approach is being developed to overcome these constraints by solving higher-order initial value problems directly, without reducing them to first-order systems (Omar & Kuboye, 2017, Skwame et al. 2019, Tumba et al. 2021).

Through creative block approaches designed for higher-order initial value problems, Ramos et al. (2020), Skwame et al. (2020), Sabo et al. (2021), Raymond et al. (2021), and Skwame et al. (2024) all advanced numerical methods for solving ordinary differential equations (ODEs). For second-order IVPs, Ramos et al. (2020) examined k-step linear block methods and found effective formulations that use interpolation and collocation to cut down on computation time without sacrificing accuracy. By introducing a half-step implicit hybrid block method of order four for third-order ODEs, Tumba et al. (2021) overcame the drawbacks of first-order reduction and predictor-corrector schemes and proved its consistency, zero-stability, and convergence.

In a similar vein, Raymond et al. (2021) created a four-step hybrid block method that uses power series-based interpolation and off-grid evaluations to directly solve fourth-order ODEs with improved precision and less computational load. In order to improve efficiency and convergence across a variety of physical applications, Skwame et al. (2024) suggested a novel one-step block method with eight partitions for directly solving second- to fourth-order oscillatory differential equations. When taken as a whole, these studies highlight how block methods are becoming more and more successful at solving the complexity of high-order ODEs in a variety of scientific and engineering fields.

2. Formulation of the Method

Consider the approximate solution of power series in the

$$y(\tau)=\sum_{j=0}^{\eta+\nu}\sigma_j\,\tau^j \tag{2.1}$$

where $\sigma_j's$ are parameters to be determined, $\sigma\in[a,b]$, η and ν are the respective number of distinct collocation and interpolation points.

Let the solution of equation (2.1) be sought on the partition $\pi_N:\sigma=\tau_0<\tau_1<\tau_2<\cdots\tau_n<\tau_{n+1}<\tau_N=b$ on the interval $[a,b]$ with a constant step size h, given by $h=\tau_n-\tau_{n-1}$, where $n=0,1,2,...,N$.

Using (2.1) with $\eta=4$ and $\nu=6$, the polynomial of degree $\eta+\nu-1$ as follows

$$y(\tau)=\sum_{j=0}^{9}\sigma_j\,\tau^j \tag{2.2}$$

Differentiate (3.2) four times, we have

$$y''''(\tau)=\sum_{j=0}^{9}j(j-1)(j-2)(j-3)\sigma_j\,\tau^{j-4} \tag{2.3}$$

Substitute (2.3) in to (1.1), we have

$$\sum_{j=0}^{9}j(j-1)(j-2)(j-3)\sigma_j\,\tau^{j-4}=f(\tau,y,y',y'',y''') \tag{2.4}$$

Now interpolating equation (2.2) at $\sigma_{n+\eta},\eta=0,\frac{1}{5},\frac{2}{5},\frac{3}{5}$ and collocating (2.4) at $\sigma_{n+\nu},\nu=0\left(\frac{1}{5}\right)1$ to give a system of nonlinear equation in a matrix form as

$$\Omega X = Z \tag{2.5}$$

where

$$\Omega = \begin{bmatrix} 1 & \tau_n & \tau_n^2 & \tau_n^3 & \tau_n^4 & \tau_n^5 & \tau_n^6 & \tau_n^7 & \tau_n^8 & \tau_n^9 \\ 1 & \tau_{n+\frac{1}{5}} & \tau^2_{n+\frac{1}{5}} & \tau^3_{n+\frac{1}{5}} & \tau^4_{n+\frac{1}{5}} & \tau^5_{n+\frac{1}{5}} & \tau^6_{n+\frac{1}{5}} & \tau^7_{n+\frac{1}{5}} & \tau^8_{n+\frac{1}{5}} & \tau^9_{n+\frac{1}{5}} \\ 1 & \tau_{n+\frac{2}{5}} & \tau^2_{n+\frac{2}{5}} & \tau^3_{n+\frac{2}{5}} & \tau^4_{n+\frac{2}{5}} & \tau^5_{n+\frac{2}{5}} & \tau^6_{n+\frac{2}{5}} & \tau^7_{n+\frac{2}{5}} & \tau^8_{n+\frac{2}{5}} & \tau^9_{n+\frac{2}{5}} \\ 1 & \tau_{n+\frac{3}{5}} & \tau^2_{n+\frac{3}{5}} & \tau^3_{n+\frac{3}{5}} & \tau^4_{n+\frac{3}{5}} & \tau^5_{n+\frac{3}{5}} & \tau^6_{n+\frac{3}{5}} & \tau^7_{n+\frac{3}{5}} & \tau^8_{n+\frac{3}{5}} & \tau^9_{n+\frac{3}{5}} \\ 0 & 0 & 0 & 0 & 24 & 120\tau_n & 360\tau_n^2 & 840\tau_n^3 & 1680\tau_n^4 & 3024\tau_n^5 \\ 0 & 0 & 0 & 0 & 24 & 120\tau_{n+\frac{1}{5}} & 360\tau^2_{n+\frac{1}{5}} & 840\tau^3_{n+\frac{1}{5}} & 1680\tau^4_{n+\frac{1}{5}} & 3024\tau^5_{n+\frac{1}{5}} \\ 0 & 0 & 0 & 0 & 24 & 120\tau_{n+\frac{2}{5}} & 360\tau^2_{n+\frac{2}{5}} & 840\tau^3_{n+\frac{2}{5}} & 1680\tau^4_{n+\frac{2}{5}} & 3024\tau^5_{n+\frac{2}{5}} \\ 0 & 0 & 0 & 0 & 24 & 120\tau_{n+\frac{3}{5}} & 360\tau^2_{n+\frac{3}{5}} & 840\tau^3_{n+\frac{3}{5}} & 1680\tau^4_{n+\frac{3}{5}} & 3024\tau^5_{n+\frac{3}{5}} \\ 0 & 0 & 0 & 0 & 24 & 120\tau_{n+\frac{4}{5}} & 360\tau^2_{n+\frac{4}{5}} & 840\tau^3_{n+\frac{4}{5}} & 1680\tau^4_{n+\frac{4}{5}} & 3024\tau^5_{n+\frac{4}{5}} \\ 0 & 0 & 0 & 0 & 24 & 120\tau_{n+1} & 360\tau_{n+1}^2 & 840\tau_{n+1}^3 & 1680\tau_{n+1}^4 & 3024\tau_{n+1}^5 \end{bmatrix}$$

$$X = \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 \end{bmatrix}^T$$

$$Z = \begin{bmatrix} y_n & y_{n+\frac{1}{5}} & y_{n+\frac{2}{5}} & y_{n+\frac{3}{5}} & f_n & f_{n+\frac{1}{5}} & f_{n+\frac{2}{5}} & f_{n+\frac{3}{5}} & f_{n+\frac{4}{5}} & f_{n+1} \end{bmatrix}^T$$

The unknown values of σ_j's, $j = 0(1)9$ in (2.5) can be obtained by using Gaussian elimination method and these values are substituted back into equation (2.2) to produce a continuous implicit scheme with derivatives of the form:

$$y(\xi) = \alpha_0(\xi)y_n + \alpha_{\frac{1}{5}}(\xi)y_{n+\frac{1}{5}} + \alpha_{\frac{2}{5}}(\xi)y_{n+\frac{2}{5}} + \alpha_{\frac{3}{5}}(\xi)y_{n+\frac{3}{5}} + h^4\left[\beta_0(\xi)f_n + \beta_{\frac{1}{5}}(\xi)f_{n+\frac{1}{5}} + \beta_{\frac{2}{5}}(\xi)f_{n+\frac{2}{5}} + \beta_{\frac{3}{5}}(\xi)f_{n+\frac{3}{5}} + \beta_{\frac{4}{5}}(\xi)f_{n+\frac{4}{5}} + \beta_1(\xi)f_{n+1}\right] \tag{2.6}$$

The coefficient of $\alpha_0, \alpha_{\frac{1}{5}}, \alpha_{\frac{2}{5}}, \alpha_{\frac{3}{5}}, y_n, y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, f_n, f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1}$ are obtained in terms of ξ as

$$\alpha_0 = 1 - \frac{55}{6}\tau + 25\tau^2 - \frac{125}{6}\tau^3$$

$$\alpha_{\frac{1}{5}} = 15\tau - \frac{125}{2}\tau^2 + \frac{125}{2}\tau^3$$

$$\alpha_{\frac{2}{5}} = -\frac{15}{2}\tau + 50\tau^2 - \frac{125}{2}\tau^3$$

$$\alpha_{\frac{3}{5}} = \frac{3}{5}\tau - \frac{25}{2}\tau^2 + \frac{125}{6}\tau^3$$

$$\beta_0 = -\frac{937}{12600000}\tau + \frac{1411}{1008000}\tau^2 - \frac{19151}{1814400}\tau^3 + \frac{1}{24}\tau^4 - \frac{137}{1440}\tau^5 + \frac{25}{192}\tau^6 - \frac{425}{4032}\tau^7 + \frac{125}{2688}\tau^8 - \frac{625}{1008}\tau^9$$

$$\beta_{\frac{1}{5}} = -\frac{19}{13125}\tau + \frac{3091}{216000}\tau^2 - \frac{73967}{1814400}\tau^3 + \frac{5}{24}\tau^5 - \frac{385}{864}\tau^6 + \frac{1775}{4032}\tau^7 - \frac{125}{576}\tau^8 + \frac{3125}{1008}\tau^9$$

$$\beta_{\frac{2}{5}} = -\frac{599}{1260000}\tau + \frac{2831}{1512000}\tau^2 + \frac{1261}{181440}\tau^3 - \frac{5}{25}\tau^5 + \frac{535}{864}\tau^6 - \frac{1475}{2016}\tau^7 + \frac{1625}{4032}\tau^8 - \frac{3125}{504}\tau^9$$

$$\beta_{\frac{3}{5}} = -\frac{1}{90000}\tau + \frac{143}{126000}\tau^2 - \frac{7439}{90720}\tau^3 + \frac{5}{96}\tau^5 - \frac{65}{144}\tau^6 + \frac{175}{288}\tau^7 - \frac{125}{336}\tau^8 + \frac{3125}{504}\tau^9$$

$$\beta_{\frac{4}{5}} = \frac{3}{280000}\tau - \frac{1391}{3024000}\tau^2 + \frac{5549}{1814400}\tau^3 - \frac{5}{96}\tau^5 + \frac{305}{1728}\tau^6 - \frac{1025}{4032}\tau^7 + \frac{1375}{8064}\tau^8 - \frac{3125}{1008}\tau^9$$

$$\beta_1 = -\frac{1}{450000}\tau + \frac{23}{302400}\tau^2 - \frac{883}{1814400}\tau^3 + \frac{1}{120}\tau^5 - \frac{25}{864}\tau^6 + \frac{25}{576}\tau^7 - \frac{125}{4032}\tau^8 + \frac{625}{1008}\tau^9$$

Evaluating (2.6) at non interpolating point to obtain the continuous form as

$$\left.\begin{aligned} y_{n+\frac{4}{5}} &= -y_n + 4y_{n+\frac{1}{5}} - 6y_{n+\frac{2}{5}} + 4y_{n+\frac{3}{5}} + h^4\left(-\frac{1}{450000}f_n + \frac{31}{112500}f_{n+\frac{1}{5}} + \frac{79}{75000}f_{n+\frac{2}{5}} + \frac{31}{112500}f_{n+\frac{3}{5}} - \frac{1}{450000}f_{n+\frac{4}{5}}\right) \\ y_{n+1} &= -4y_n + 15y_{n+\frac{1}{5}} - 20y_{n+\frac{2}{5}} + 10y_{n+\frac{3}{5}} + h^4\left(-\frac{1}{112500}f_n + \frac{11}{1000}f_{n+\frac{1}{5}} + \frac{101}{22500}f_{n+\frac{2}{5}} + \frac{97}{45000}f_{n+\frac{3}{5}} + \frac{1}{3750}f_{n+\frac{4}{5}} - \frac{1}{450000}f_{n+1}\right) \end{aligned}\right\} \quad (2.7)$$

The first, second and third derivative of (2.6) is given by

$$y'(\xi) = \alpha'_0(\xi)y_n + \alpha'_{\frac{1}{5}}(\xi)y_{n+\frac{1}{5}} + \alpha'_{\frac{2}{5}}(\xi)y_{n+\frac{2}{5}} + \alpha'_{\frac{3}{5}}(\xi)y_{n+\frac{3}{5}} + h^4\left[\beta'_0(\xi)f_n + \beta'_{\frac{1}{5}}(\xi)f_{n+\frac{1}{5}} + \beta'_{\frac{2}{5}}(\xi)f_{n+\frac{2}{5}} + \beta'_{\frac{3}{5}}(\xi)f_{n+\frac{3}{5}} + \beta'_{\frac{4}{5}}(\xi)f_{n+\frac{4}{5}} + \beta'_1(\xi)f_{n+1}\right] \quad (2.8)$$

$$y''(\xi) = \alpha''_0(\xi)y_n + \alpha''_{\frac{1}{5}}(\xi)y_{n+\frac{1}{5}} + \alpha''_{\frac{2}{5}}(\xi)y_{n+\frac{2}{5}} + \alpha''_{\frac{3}{5}}(\xi)y_{n+\frac{3}{5}} + h^4\left[\beta''_0(\xi)f_n + \beta''_{\frac{1}{5}}(\xi)f_{n+\frac{1}{5}} + \beta''_{\frac{2}{5}}(\xi)f_{n+\frac{2}{5}} + \beta''_{\frac{3}{5}}(\xi)f_{n+\frac{3}{5}} + \beta''_{\frac{4}{5}}(\xi)f_{n+\frac{4}{5}} + \beta''_1(\xi)f_{n+1}\right] \quad (2.9)$$

$$y'''(\xi) = \alpha'''_0(\xi)y_n + \alpha'''_{\frac{1}{5}}(\xi)y_{n+\frac{1}{5}} + \alpha'''_{\frac{2}{5}}(\xi)y_{n+\frac{2}{5}} + \alpha'''_{\frac{3}{5}}(\xi)y_{n+\frac{3}{5}} + h^4\left[\beta'''_0(\xi)f_n + \beta'''_{\frac{1}{5}}(\xi)f_{n+\frac{1}{5}} + \beta'''_{\frac{2}{5}}(\xi)f_{n+\frac{2}{5}} + \beta'''_{\frac{3}{5}}(\xi)f_{n+\frac{3}{5}} + \beta'''_{\frac{4}{5}}(\xi)f_{n+\frac{4}{5}} + \beta'''_1(\xi)f_{n+1}\right] \quad (2.10)$$

Evaluating equations (2.8) to (2.10) of continuous scheme at all the points $0\left(\frac{1}{5}\right)1$ to gives discrete scheme and its derivatives can be written explicit in the block form as

$$\left.\begin{aligned}
y'_n &= -\frac{55}{6}y_n + 15y_{n+\frac{1}{5}} - 15y_{n+\frac{2}{5}} + \frac{3}{5}y_{n+\frac{3}{5}} + h^3\left(-\frac{937}{12600000}f_n - \frac{19}{13125}f_{n+\frac{1}{5}} - \frac{599}{1260000}f_{n+\frac{2}{5}} - \frac{1}{90000}f_{n+\frac{3}{5}} + \frac{3}{280000}f_{n+\frac{4}{5}} - \frac{1}{450000}f_{n+1}\right)\\
y'_{n+\frac{1}{5}} &= -\frac{5}{3}y_n - \frac{5}{2}y_{n+\frac{1}{5}} + 5y_{n+\frac{2}{5}} + \frac{5}{6}y_{n+\frac{3}{5}} + h^3\left(-\frac{1}{675000}f_n + \frac{2809}{7560000}f_{n+\frac{1}{5}} + \frac{43}{126000}f_{n+\frac{2}{5}} - \frac{229}{3780000}f_{n+\frac{3}{5}} + \frac{1}{54000}f_{n+\frac{4}{5}} - \frac{11}{4200000}f_{n+1}\right)\\
y'_{n+\frac{2}{5}} &= -\frac{5}{6}y_n - 5y_{n+\frac{1}{5}} + \frac{5}{2}y_{n+\frac{2}{5}} + \frac{5}{3}y_{n+\frac{3}{5}} + h^3\left(\frac{169}{37800000}f_n - \frac{311}{1260000}f_{n+\frac{1}{5}} - \frac{353}{756000}f_{n+\frac{2}{5}} + \frac{1}{16875}f_{n+\frac{3}{5}} - \frac{7}{360000}f_{n+\frac{4}{5}} + \frac{53}{18900000}f_{n+1}\right)\\
y'_{n+\frac{3}{5}} &= -\frac{5}{3}y_n + \frac{15}{2}y_{n+\frac{1}{5}} - 15y_{n+\frac{2}{5}} + \frac{55}{6}y_{n+\frac{3}{5}} + h^3\left(-\frac{41}{6300000}f_n + \frac{173}{360000}f_{n+\frac{1}{5}} + \frac{11}{7500}f_{n+\frac{2}{5}} + \frac{61}{1260000}f_{n+\frac{3}{5}} + \frac{17}{1260000}f_{n+\frac{4}{5}} - \frac{11}{4200000}f_{n+1}\right)\\
y'_{n+\frac{4}{5}} &= -\frac{55}{6}y_n + 35y_{n+\frac{1}{5}} - \frac{95}{2}y_{n+\frac{2}{5}} + \frac{65}{3}y_{n+\frac{3}{5}} + h^3\left(-\frac{671}{37800000}f_n + \frac{169}{67500}f_{n+\frac{1}{5}} + \frac{12821}{1260000}f_{n+\frac{2}{5}} + \frac{7447}{1890000}f_{n+\frac{3}{5}} + \frac{509}{7560000}f_{n+\frac{4}{5}} - \frac{1}{450000}f_{n+1}\right)\\
y'_{n+1} &= -\frac{65}{3}y_n + \frac{155}{2}y_{n+\frac{1}{5}} + 95y_{n+\frac{2}{5}} + \frac{235}{6}y_{n+\frac{3}{5}} + h^3\left(\frac{31}{675000}f_n + \frac{1663}{280000}f_{n+\frac{1}{5}} + \frac{6847}{270000}f_{n+\frac{2}{5}} + \frac{60863}{3780000}f_{n+\frac{3}{5}} + \frac{2473}{630000}f_{n+\frac{4}{5}} + \frac{2041}{37800000}f_{n+1}\right)
\end{aligned}\right\}\quad(2.11)$$

$$\left.\begin{aligned}
y''_n &= 50y_n + 125y_{n+\frac{1}{5}} + 100y_{n+\frac{2}{5}} - 25y_{n+\frac{3}{5}} + h^2\left(\frac{1411}{50400}f_n + \frac{30921}{108000}f_{n+\frac{1}{5}} + \frac{2831}{756000}f_{n+\frac{2}{5}} + \frac{143}{63000}f_{n+\frac{3}{5}} - \frac{1391}{1512000}f_{n+\frac{4}{5}} + \frac{23}{151200}f_{n+1}\right)\\
y''_{n+\frac{1}{5}} &= 25y_n - 50y_{n+\frac{1}{5}} + 25y_{n+\frac{2}{5}} + h^2\left(-\frac{73}{756000}f_n - \frac{1601}{504000}f_{n+\frac{1}{5}} + \frac{1}{189000}f_{n+\frac{2}{5}} - \frac{11}{108000}f_{n+\frac{3}{5}} + \frac{11}{252000}f_{n+\frac{4}{5}} - \frac{11}{1512000}f_{n+1}\right)\\
y''_{n+\frac{2}{5}} &= 25y_{n+\frac{1}{5}} - 50y_{n+\frac{2}{5}} + 25y_{n+\frac{3}{5}} + h^2\left(\frac{11}{151200}f_n - \frac{53}{378000}f_{n+\frac{1}{5}} - \frac{773}{252000}f_{n+\frac{2}{5}} - \frac{53}{37800}f_{n+\frac{3}{5}} + \frac{11}{1512000}f_{n+\frac{4}{5}}\right)\\
y''_{n+\frac{3}{5}} &= 25y_n + 100y_{n+\frac{1}{5}} - 125y_{n+\frac{2}{5}} + 50y_{n+\frac{3}{5}} + h^2\left(-\frac{1}{18000}f_n + \frac{10427}{1512000}f_{n+\frac{1}{5}} + \frac{9901}{378000}f_{n+\frac{2}{5}} + \frac{107}{28000}f_{n+\frac{3}{5}} - \frac{37}{189000}f_{n+\frac{4}{5}} + \frac{11}{1512000}f_{n+1}\right)\\
y''_{n+\frac{4}{5}} &= -50y_n + 175y_{n+\frac{1}{5}} - 200y_{n+\frac{2}{5}} + 75y_{n+\frac{3}{5}} + h^2\left(-\frac{179}{1512000}f_n + \frac{3469}{252000}f_{n+\frac{1}{5}} + \frac{6421}{108000}f_{n+\frac{2}{5}} + \frac{3791}{94500}f_{n+\frac{3}{5}} + \frac{121}{33600}f_{n+\frac{4}{5}} + \frac{23}{151200}f_{n+1}\right)\\
y''_{n+1} &= -75y_n + 250y_{n+\frac{1}{5}} - 275y_{n+\frac{2}{5}} + 100y_{n+\frac{3}{5}} + h^2\left(-\frac{11}{756000}f_n + \frac{29689}{1512000}f_{n+\frac{1}{5}} + \frac{499}{5250}f_{n+\frac{2}{5}} + \frac{58271}{756000}f_{n+\frac{3}{5}} + \frac{4561}{108000}f_{n+\frac{4}{5}} + \frac{271}{100800}f_{n+1}\right)
\end{aligned}\right\}\quad(2.12)$$

$$\left.\begin{aligned}
y'''_n &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(-\frac{19151}{302400}f_n - \frac{73967}{302400}f_{n+\frac{1}{5}} + \frac{1261}{30240}f_{n+\frac{2}{5}} - \frac{7439}{151200}f_{n+\frac{3}{5}} + \frac{5549}{302400}f_{n+\frac{4}{5}} - \frac{883}{302400}f_{n+1}\right)\\
y'''_{n+\frac{1}{5}} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(\frac{799}{302400}f_n - \frac{14033}{302400}f_{n+\frac{1}{5}} - \frac{10453}{151200}f_{n+\frac{2}{5}} + \frac{2683}{151200}f_{n+\frac{3}{5}} + \frac{1717}{302400}f_{n+\frac{4}{5}} + \frac{251}{302400}f_{n+1}\right)\\
y'''_{n+\frac{2}{5}} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(-\frac{67}{60480}f_n + \frac{12721}{302400}f_{n+\frac{1}{5}} + \frac{11009}{151200}f_{n+\frac{2}{5}} - \frac{547}{30240}f_{n+\frac{3}{5}} + \frac{1517}{302400}f_{n+\frac{4}{5}} - \frac{211}{302400}f_{n+1}\right)\\
y'''_{n+\frac{3}{5}} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(-\frac{127}{302400}f_n + \frac{1763}{60480}f_{n+\frac{1}{5}} + \frac{27851}{151200}f_{n+\frac{2}{5}} + \frac{14107}{151200}f_{n+\frac{3}{5}} - \frac{2389}{302400}f_{n+\frac{4}{5}} + \frac{251}{302400}f_{n+1}\right)\\
y'''_{n+\frac{4}{5}} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(-\frac{67}{60480}f_n + \frac{12049}{302400}f_{n+\frac{1}{5}} + \frac{22433}{151200}f_{n+\frac{2}{5}} + \frac{35569}{151200}f_{n+\frac{3}{5}} + \frac{4873}{60480}f_{n+\frac{4}{5}} - \frac{883}{302400}f_{n+1}\right)
\end{aligned}\right\}\quad(2.13)$$

Equations (2.7), (2.11) to (2.13) are combined together in matrix form and by using matrix inversion to gives the following new schemes as explicit block hybrid method as

$$\begin{aligned}
y_{n+\frac{1}{5}} = y_n + \frac{1}{5}h^1 y'_n + \frac{1}{50}h^2 y''_n + \frac{1}{750}h^3 y'''_n + \frac{3509}{81000000}h^4 f_n + \frac{9809}{226800000}h^4 f_{n+\frac{1}{5}}\\
- \frac{251}{7087500}h^4 f_{n+\frac{2}{5}} + \frac{2543}{113400000}h^4 f_{n+\frac{3}{5}} - \frac{133}{16200000}h^4 f_{n+\frac{4}{5}} + \frac{1469}{1134000000}h^4 f_{n+1}
\end{aligned}$$

$$y_{n+\frac{2}{5}} = y_n + \frac{2}{5} h^1 y'_n + \frac{2}{25} h^2 y''_n + \frac{4}{375} h^3 y'''_n + \frac{4264}{8859375} h^4 f_n + \frac{1592}{1771875} h^4 f_{n+\frac{1}{5}}$$
$$- \frac{982}{1771875} h^4 f_{n+\frac{2}{5}} + \frac{88}{253125} h^4 f_{n+\frac{3}{5}} - \frac{32}{253125} h^4 f_{n+\frac{4}{5}} + \frac{176}{8859375} h^4 f_{n+1}$$

$$y_{n+\frac{3}{5}} = y_n + \frac{3}{5} h^1 y'_n + \frac{9}{50} h^2 y''_n + \frac{9}{250} h^3 y'''_n + \frac{1593}{875000} h^4 f_n + \frac{1809}{400000} h^4 f_{n+\frac{1}{5}}$$
$$- \frac{189}{100000} h^4 f_{n+\frac{2}{5}} + \frac{1917}{1400000} h^4 f_{n+\frac{3}{5}} - \frac{351}{700000} h^4 f_{n+\frac{4}{5}} + \frac{1107}{14000000} h^4 f_{n+1}$$

$$y_{n+\frac{4}{5}} = y_n + \frac{4}{5} h^1 y'_n + \frac{8}{25} h^2 y''_n + \frac{32}{375} h^3 y'''_n + \frac{40448}{8859375} h^4 f_n + \frac{3328}{253125} h^4 f_{n+\frac{1}{5}}$$
$$- \frac{5888}{1771875} h^4 f_{n+\frac{2}{5}} + \frac{6656}{1771875} h^4 f_{n+\frac{3}{5}} - \frac{2272}{1771875} h^4 f_{n+\frac{4}{5}} + \frac{256}{1265625} h^4 f_{n+1}$$

$$y_{n+1} = y_n + h^1 y'_n + \frac{1}{2} h^2 y''_n + \frac{1}{6} h^3 y'''_n + \frac{239}{25920} h^4 f_n + \frac{2105}{72576} h^4 f_{n+\frac{1}{5}} - \frac{5}{1296} h^4 f_{n+\frac{2}{5}}$$
$$+ \frac{335}{36288} h^4 f_{n+\frac{3}{5}} - \frac{85}{36288} h^4 f_{n+\frac{4}{5}} + \frac{149}{362880} h^4 f_{n+1}$$

$$y'_{n+\frac{1}{5}} = y'_n + \frac{1}{5} h^1 y''_n + \frac{1}{50} h^2 y'''_n + \frac{3929}{5040000} h^3 f_n + \frac{199}{201600} h^3 f_{n+\frac{1}{5}} - \frac{1931}{2520000} h^3 f_{n+\frac{2}{5}}$$
$$+ \frac{173}{360000} h^3 f_{n+\frac{3}{5}} - \frac{883}{5040000} h^3 f_{n+\frac{4}{5}} + \frac{139}{5040000} h^3 f_{n+1}$$

$$y'_{n+\frac{2}{5}} = y'_n + \frac{2}{5} h^1 y''_n + \frac{2}{25} h^2 y'''_n + \frac{317}{78750} h^3 f_n + \frac{367}{39375} h^3 f_{n+\frac{1}{5}} - \frac{38}{7875} h^3 f_{n+\frac{2}{5}} +$$
$$\frac{122}{39375} h^3 f_{n+\frac{3}{5}} - \frac{89}{78750} h^3 f_{n+\frac{4}{5}} + \frac{1}{5625} h^3 f_{n+1}$$

$$y'_{n+\frac{3}{5}} = y'_n + \frac{3}{5} h^1 y''_n + \frac{9}{50} h^2 y'''_n + \frac{783}{80000} h^3 f_n + \frac{16119}{560000} h^3 f_{n+\frac{1}{5}} - \frac{2187}{280000} h^3 f_{n+\frac{2}{5}}$$
$$+ \frac{423}{56000} h^3 f_{n+\frac{3}{5}} - \frac{1539}{560000} h^3 f_{n+\frac{4}{5}} + \frac{243}{560000} h^3 f_{n+1}$$

$$y'_{n+\frac{4}{5}} = y'_n + \frac{4}{5} h^1 y''_n + \frac{8}{25} h^2 y'''_n + \frac{712}{39375} h^3 f_n + \frac{2336}{39375} h^3 f_{n+\frac{1}{5}} - \frac{32}{5625} h^3 f_{n+\frac{2}{5}}$$
$$+ \frac{704}{39375} h^3 f_{n+\frac{3}{5}} - \frac{8}{1575} h^3 f_{n+\frac{4}{5}} + \frac{32}{39375} h^3 f_{n+1}$$

$$y'_{n+1} = y'_n + 1 h^1 y''_n + \frac{1}{2} h^2 y'''_n + \frac{233}{8064} h^3 f_n + \frac{815}{8064} h^3 f_{n+\frac{1}{5}} + \frac{5}{4032} h^3 f_{n+\frac{2}{5}} + \frac{155}{4032} h^3 f_{n+\frac{3}{5}}$$

$$-\frac{5}{1152}h^3f_{n+\frac{4}{5}}+\frac{11}{8064}h^3f_{n+1}$$

$$y''_{n+\frac{1}{5}}=y''_n+\frac{1}{5}h^1y'''_n+\frac{1231}{126000}h^2f_n+\frac{863}{50400}h^2f_{n+\frac{1}{5}}-\frac{761}{63000}h^2f_{n+\frac{2}{5}}+\frac{941}{126000}h^2f_{n+\frac{3}{5}}$$

$$-\frac{341}{126000}h^2f_{n+\frac{4}{5}}+\frac{107}{252000}h^2f_{n+1}$$

$$y''_{n+\frac{2}{5}}=y''_n+\frac{2}{5}h^1y'''_n+\frac{71}{3150}h^2f_n+\frac{544}{7875}h^2f_{n+\frac{1}{5}}-\frac{37}{1575}h^2f_{n+\frac{2}{5}}+\frac{136}{7875}h^2f_{n+\frac{3}{5}}$$

$$-h^2f_{n+\frac{4}{5}}+\frac{101}{15750}h^2f_{n+1}$$

$$y''_{n+\frac{3}{5}}=y''_n+\frac{3}{5}h^1y'''_n+\frac{123}{3500}h^2f_n+\frac{3501}{28000}h^2f_{n+\frac{1}{5}}-\frac{9}{3500}h^2f_{n+\frac{2}{5}}+\frac{87}{2800}h^2f_{n+\frac{3}{5}}$$

$$-\frac{9}{875}h^2f_{n+\frac{4}{5}}+\frac{9}{5600}h^2f_{n+1}$$

$$y''_{n+\frac{4}{5}}=y''_n+\frac{4}{5}h^1y'''_n+\frac{376}{7875}h^2f_n+\frac{1424}{7875}h^2f_{n+\frac{1}{5}}+\frac{176}{7875}h^2f_{n+\frac{2}{5}}+\frac{608}{7875}h^2f_{n+\frac{3}{5}}$$

$$-\frac{16}{1575}h^2f_{n+\frac{4}{5}}+\frac{16}{7875}h^2f_{n+1}$$

$$y''_{n+1}=y''_n+h^1y'''_n+\frac{61}{1008}h^2f_n+\frac{475}{2016}h^2f_{n+\frac{1}{5}}+\frac{25}{504}h^2f_{n+\frac{2}{5}}+\frac{125}{1008}h^2f_{n+\frac{3}{5}}$$

$$+\frac{25}{1008}h^2f_{n+\frac{4}{5}}+\frac{11}{2016}h^2f_{n+1}$$

$$y'''_{n+\frac{1}{5}}=y'''_n+\frac{19}{288}h^1f_n+\frac{1427}{7200}h^1f_{n+\frac{1}{5}}-\frac{133}{1200}h^1f_{n+\frac{2}{5}}+\frac{241}{3600}h^1f_{n+\frac{3}{5}}-\frac{173}{7200}h^1f_{n+\frac{4}{5}}$$

$$+\frac{3}{800}h^1f_{n+1}$$

$$y'''_{n+\frac{2}{5}}=y'''_n+\frac{14}{225}h^1f_n+\frac{43}{150}h^1f_{n+\frac{1}{5}}+\frac{7}{225}h^1f_{n+\frac{2}{5}}+\frac{7}{225}h^1f_{n+\frac{3}{5}}-\frac{1}{75}h^1f_{n+\frac{4}{5}}$$

$$+\frac{1}{450}h^1f_{n+1}$$

$$y'''_{n+\frac{3}{5}}=y'''_n+\frac{51}{800}h^1f_n+\frac{219}{800}h^1f_{n+\frac{1}{5}}+\frac{57}{400}h^1f_{n+\frac{2}{5}}+\frac{57}{400}h^1f_{n+\frac{3}{5}}-\frac{21}{800}h^1f_{n+\frac{4}{5}}$$

$$+\frac{3}{800}h^1f_{n+1}$$

$$y'''_{n+\frac{4}{5}}=y'''_n+\frac{14}{225}h^1f_n+\frac{64}{225}h^1f_{n+\frac{1}{5}}+\frac{8}{75}h^1f_{n+\frac{2}{5}}+\frac{64}{225}h^1f_{n+\frac{3}{5}}+\frac{14}{225}h^1f_{n+\frac{4}{5}}$$

$$y'''_{n+1} = y'''_n + \frac{19}{288}h^1 f_n + \frac{25}{96}h^1 f_{n+\frac{1}{5}} + \frac{25}{144}h^1 f_{n+\frac{2}{5}} + \frac{25}{144}h^1 f_{n+\frac{3}{5}} + \frac{25}{96}h^1 f_{n+\frac{4}{5}} + \frac{19}{288}h^1 f_{n+1} \tag{2.14}$$

3. Basic Properties of the new Method

Numerical analysis was done on the fundamental characteristics of one-step block hybrid methods. These characteristics, which show how the methods converge, include order, consistency, error constant, and zero-stability. Additionally, the methods' region of absolute stability will be determined.

3.1 Order and Error Constant of the new method

Consider the linear operator associated in L associated with the new method in be defined as

$$L[y(x);h] = \sum_{j=0}^{k}\left\{\alpha_j y(x_n+jh) - \alpha_{vi} y(x_n+vih) - h^d \beta_j y^d(x_n+jh) - h^d \beta_{vi} y^d(x_n+vih)\right\} \tag{3.1}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$. We expand $y(x_n+jh)$ and $y^d(x_n+jh)$ using a Taylor series about x_n and collecting like terms in h and y to obtain the expression;

$$\ell\{y(x);h\} = C_0 y(x) + C_1 y'(x) + \ldots + C_p h^p y^p(x) + C_{p+1}h^{p+1}y^{p+1}(x) + C_{p+2}h^{p+2}y^{p+2}(x) + \ldots \tag{3.2}$$

We consider the linear operator $L[y(t_n);h]$ of the new method with the corollary 3.1 and 3.2 below to determining the order and error constant of the new method (Adewale & Sabo, 2024).

Corollary 3.1

The linear operator $L[y(t_n);h]$ associate with the local truncation error of the method in equation is $C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$.

Therefore, the method is of uniform order six

Proof

The linear difference operators associated with the new method are given

$$\left.\begin{aligned}
L[y(t_n);h] &= y\left(t_n+\frac{1}{5}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n+\frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n+\frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n+\frac{3}{5}h\right) + h^4\sum_{i=0}^{\eta}\left(\beta_i(t)f_{n+i} + \beta_\eta(t)f_{n+\eta}\right)\right)\\
L[y(t_n);h] &= y\left(t_n+\frac{2}{5}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n+\frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n+\frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n+\frac{3}{5}h\right) + h^4\sum_{i=0}^{\eta}\left(\beta_i(t)f_{n+i} + \beta_\eta(t)f_{n+\eta}\right)\right)\\
L[y(t_n);h] &= y\left(t_n+\frac{3}{5}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n+\frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n+\frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n+\frac{3}{5}h\right) + h^4\sum_{i=0}^{\eta}\left(\beta_i(t)f_{n+i} + \beta_\eta(t)f_{n+\eta}\right)\right)\\
L[y(t_n);h] &= y\left(t_n+\frac{4}{5}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n+\frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n+\frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n+\frac{3}{5}h\right) + h^4\sum_{i=0}^{\eta}\left(\beta_i(t)f_{n+i} + \beta_\eta(t)f_{n+\eta}\right)\right)\\
L[y(t_n);h] &= y(t_n+h) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n+\frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n+\frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n+\frac{3}{5}h\right) + h^4\sum_{i=0}^{\eta}\left(\beta_i(t)f_{n+i} + \beta_\eta(t)f_{n+\eta}\right)\right)
\end{aligned}\right\} \tag{3.3}$$

Corollary 3.2

The local truncation error of the new method is assume $y(t)$ to be sufficiently differentiable and expanding $y(t_n + qh)$ and $y(t_n + jh)$ about t_n using Taylor series, have

$$L_{\frac{1}{5}}[y(t_n);h] = (-6.5552\times10^{-11}),\quad L_{\frac{2}{5}}[y(t_n);h] = (-9.8969\times10^{-10}),\quad L_{\frac{3}{5}}[y(t_n);h] = (-3.9703\times10^{-09}),$$

$$L_{\frac{4}{5}}[y(t_n);h] = (-1.0171\times10^{-08}),\quad L_{1}[y(t_n);h] = (-2.0723\times10^{-08})$$

Proof

Expand equation (3.3) using corollary 3.2 and then collect the like terms to the power of h gives

$$L_{\frac{1}{5}}[y(t_n);h] = (-6.5552\times10^{-11})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

$$L_{\frac{2}{5}}[y(t_n);h] = (-9.8969\times10^{-10})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

$$L_{\frac{3}{5}}[y(t_n);h] = (-3.9703\times10^{-09})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

$$L_{\frac{4}{5}}[y(t_n);h] = (-1.0171\times10^{-08})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

$$L_{1}[y(t_n);h] = (-2.0723\times10^{-08})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

3.2 Consistency of the New Method

The numerical method is said to be consistent, if its order is greater than or equal to zero. The new method is consistent since it of uniform order six (Sunday, 2018).

3.3 Zero Stability of the New Method

Definition 3.1: A numerical method is said to be zero-stable if the roots $z_s,\ s = 1,2,\ldots,n$ of the first characteristic polynomial $\overline{\rho}(z)$, defined by

$$\overline{\rho}(z) = \det\left[zA^{(0)} - E\right] \tag{3.4}$$

satisfies $|z_s| \le 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. Moreover, as $h \to 0$, $\rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ *and* E. The main consequence of zero-stability is to control the propagation of the error as the integration proceeds (Adewale & Sabo, 2024).

To determine the zero-stability of the new method, we applying definition 3.1 on the new method, with the first characteristic polynomial given by

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} z & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & -1 \\ 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & z-1 \end{bmatrix} \right| = z^4(z-1)$$

Solving for z in

$$z^4(z-1) \tag{3.5}$$

Gives $z = 0, 0, 0, 1$. Hence, the new method is zero-stable.

3.4 Convergence of the New Method

The new method is convergent, since it satisfied consistent and zero stable according to Dhalquist theorem (Sunday, 2018).

3.5 Region of Absolute Stability (RAS) of the New Method

Definition 3.2: The region of absolute stability is a region in the complex Z plane, where $z = \lambda h$. It is defined as those values of Z such that the numerical solutions of $y^{d} = -\lambda^{d} y$ satisfy $y_j \to 0 \; as \; j \to \infty$ for any initial condition (Adewale & Sabo, 2024). To determine the regions of absolute stability of K-step method, a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. This method is called the Boundary Locus Method (BLM).

Applying the Boundary Locus Method on the new method, we obtained the stability polynomial a

$$\bar{h}(w) = \left(-\frac{1}{56250}w^4 - \frac{4}{84375}w^5\right)h^5 + \left(-\frac{79}{112500}w^4 + \frac{1121}{1012500}w^5\right)h^4 + \left(-\frac{749}{45000}w^4 - \frac{17857}{1215000}w^5\right)h^3 + \left(-\frac{81}{500}w^4 + \frac{781}{6750}w^5\right)h^2 + \left(-\frac{1}{2}w^4 - \frac{1}{2}w^5\right)h - w^4 + w^5 \quad (3.6)$$

Using the stability polynomial (3.6), the region of absolute stability of the new method is obtained as

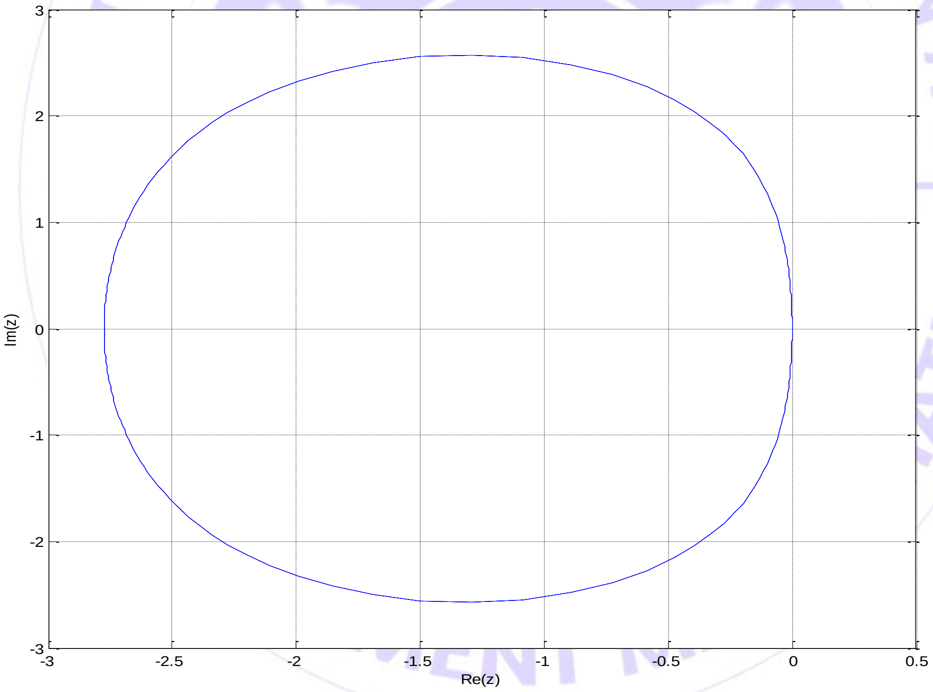

Figure 3.1: Showing an A-stable region of absolute stability of the new method

4. Numerical Examples

Ordinary differential equations of the form (1.1) with second, third, and fourth order initial value problems were implemented using the new block hybrid method without requiring a reduction to an equivalent system of first order ODEs.

Every application's absolute error of the approximate solutions is calculated and compared to the outcomes of previous approaches, especially those put forth by Fasasi (2018), Adoghe and Omole (2019), Skwame et al. (2019), Skwame et al. (2020), Sabo et al. (2021), Atabo & Adee (2021), and Raymond et al. (2023). The following acronym were used in the tables and figures below:

ES: Means Exact Solution
CS: Means Computed Solution in the new method
ENM: Means Error in new method
ESe17: Means Error in Skwame et al. (2017),
ESe21: Means Error in Sabo et al. (2021),
EAO19: Means Error in Adoghe and Omole, (2019),
ETe21: Means Error in Tumba et al. (2021),
EAOm18: Means Error in Adeyeye & Omar, (2019),
ETe21: Means Error in Tumba et al. (2021),

Example 4.1: Consider the second order mass spring system, where a $128lb$ weight is attached to a spring having a spring constant of $64lb/ft$. The weight is started in motion with no initial velocity by displacing it $6inches$ above the equilibrium position and by simultaneously applying to the weight an external force $F_4(\tau) = 8\sin 4t$. Assuming no air resistance, compute the subsequent motion of the weight at $\tau : 0.01 \le \tau \le 0.10$.

Now, we model this problem into a mathematical model and then apply our method to compute the motion on the weight attached to the spring. Here,

$$m = 4, k = 64, b = 0, \text{ and } F_4(\tau) = 8\sin 4\tau$$

Thus, example 4.1 boils down to

$$y''(\tau) + 16y = 2\sin 4\tau,\ y(0) = -\frac{1}{2},\ y'(0) = 0 \tag{4.1}$$

with the exact solution of (4.1) is given by,

$$y(\tau) = -\frac{1}{2}\cos 4\tau + \frac{1}{16}\sin 4\tau - \frac{1}{4}\tau\cos 4\tau \tag{4.2}$$

Source: [Skwame et al., (2017) Sabo, et al., (2021)].

Table 4.1: Numerical Results for Example 4.1

T	**ES**	**CS**	**ENM**	**ESe17**	**ESe21**
0.1	-4.996×10^{-1}	-4.996×10^{-1}	3.0000(-20)	1.6621(-09)	1.0000(-19)
0.2	-4.984×10^{-1}	-4.984×10^{-1}	2.0000(-20)	1.1586(-09)	4.1000(-19)
0.3	-4.964×10^{-1}	-4.964×10^{-1}	8.0000(-20)	2.9743(-08)	9.1000(-19)
0.4	-4.935×10^{-1}	-4.935×10^{-1}	6.0000(-20)	5.6076(-08)	1.6600(-18)
0.5	-4.899×10^{-1}	-4.899×10^{-1}	1.1000(-19)	9.0504(-08)	2.6200)-18)
0.6	-4.854×10^{-1}	-4.854×10^{-1}	7.6000(-19)	1.3291(-07)	3.8000(-18)
0.7	-4.801×10^{-1}	-4.801×10^{-1}	1.0000(-19)	1.8317(-07)	5.2000(-18)
0.8	-4.739×10^{-1}	-4.739×10^{-1}	1.3000(-19)	2.4110(-07)	6.8500(-18)
0.9	-4.670×10^{-1}	-4.670×10^{-1}	1.4000(-19)	3.0653(-07)	8.7500(-18)
1.0	-4.592×10^{-1}	-4.592×10^{-1}	7.4000(-19)	1.6621(-09)	1.0850(-17)

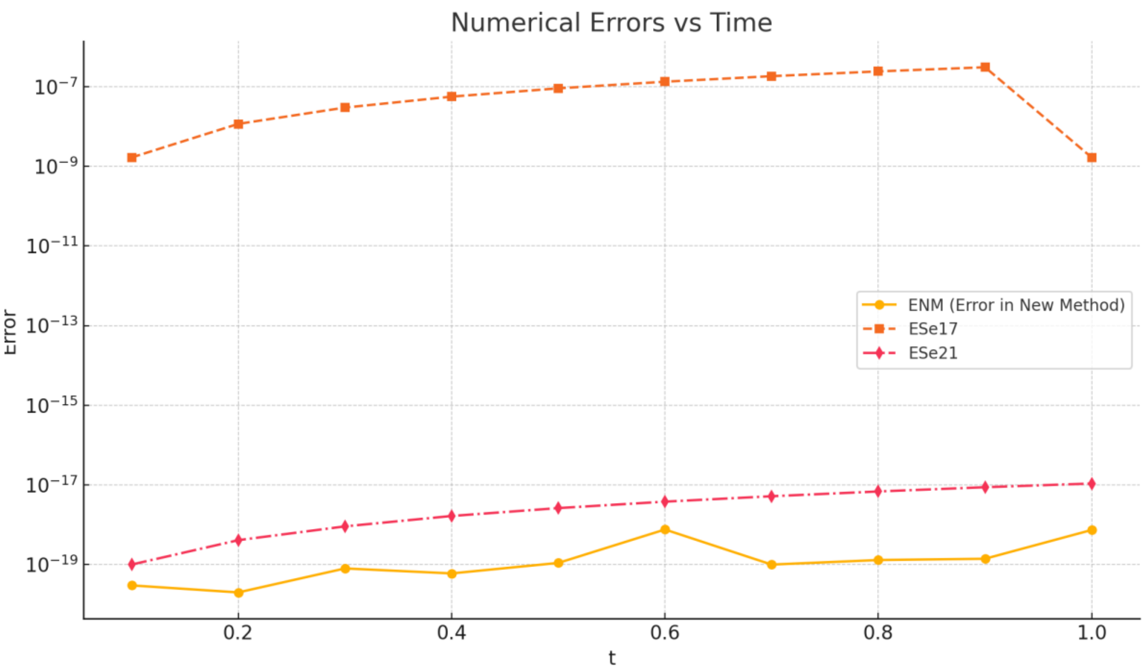

Figure 4.1: Graphical curve of Table 4.1.

Example 4.2: Consider the highly non- stiff third order linear problem describing oscillatory behavior up to the third order of the form

$$y'''(\tau)=y''(\tau)-y'(\tau)+y,\ y(0)=1,\ y'(0)=0,\ y''(0)=-1,\ h=\frac{1}{100} \qquad (4.3)$$

Whose exact solution is

$$y(\tau)=\cos\tau \qquad (4.4)$$

Source: [Adoghe and Omole (2019), Tumba et al. (2021)]

Table 4.2: Numerical Results for Example 4.2

t	**ES**	**CS**	**ENM**	**EAO19**	**ETe21**
0.01	9.9995×10^{-1}	9.9995×10^{-1}	1.0000(-20)	0.0000(00)	6.1000(-20)
0.02	9.9980×10^{-1}	9.9980×10^{-1}	3.0000(-20)	1.1102(-16)	1.2000(-19)
0.03	9.9955×10^{-1}	9.9955×10^{-1}	4.0000(-20)	4.4409(-16)	1.9000(-19)
0.04	9.9920×10^{-1}	9.9920×10^{-1}	6.0000(-20)	5.8842(-15)	2.5000(-19)
0.05	9.9875×10^{-1}	9.9875×10^{-1}	6.0000(-20)	2.6201(-14)	3.2000(-19)
0.06	9.9820×10^{-1}	9.9820×10^{-1}	7.0000(-20)	8.3822(-14)	3.9000(-19)
0.07	9.9755×10^{-1}	9.9755×10^{-1}	8.0000(-	2.0750(-13)	4.5000(-19)

			20)		
0.08	9.9680×10^{-1}	9.9680×10^{-1}	8.0000(-20)	4.4142(-13)	5.1000(-19)
0.09	9.9595×10^{-1}	9.9595×10^{-1}	9.0000(-20)	4.4743(-13)	5.6000(-19)
0.10	9.9500×10^{-1}	9.9500×10^{-1}	9.0000(-20)	1.5086(-12)	6.3000(-19)

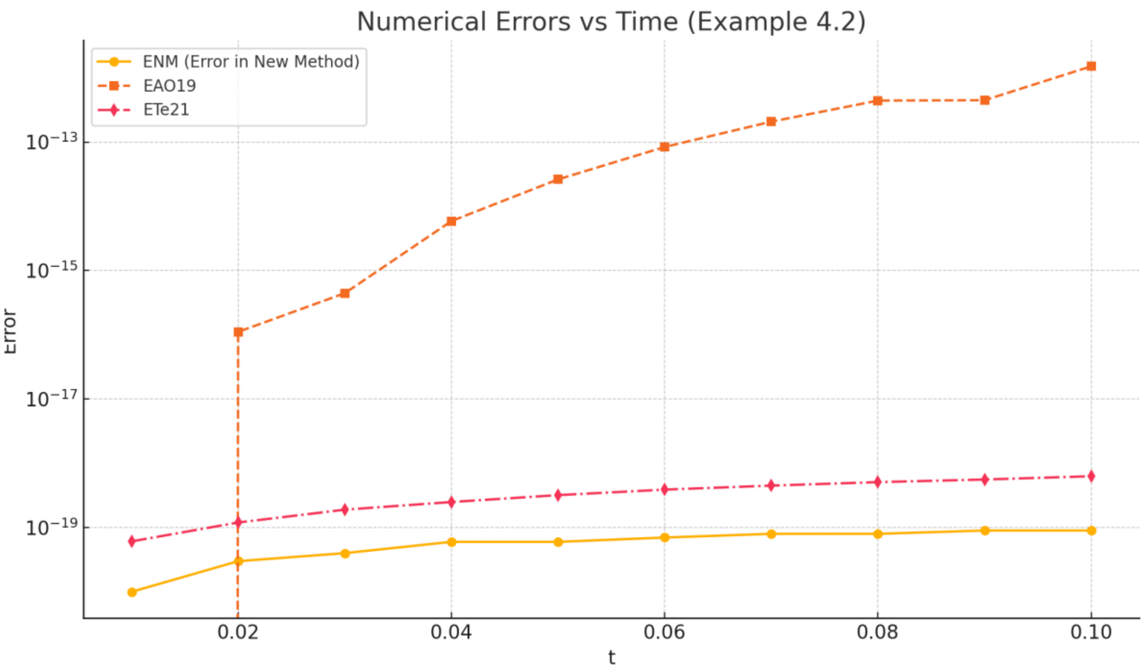

Figure 4.2: Graphical curve of table 4.2.

Example 4.3: Consider the fourth order oscillatory differential equation of the form

$$y''''(\tau) = -y'',\ y(0) = 0,\ y'(0) = \frac{-1.1}{72 - 50\pi},\ y''(0) = \frac{1}{144 - 50\pi},\ y'''(0) = \frac{1.2}{144 - 100\pi} \tag{4.5}$$

with the exact solution,

$$y(\tau) = \frac{1 - \tau\cos\tau - 1.2\sin\tau}{144 - 100\pi} \tag{4.6}$$

Where $v \in \left[0, \frac{\pi}{2}\right]$. This problem was also solved by [Adoghe and Omole (2019), Adeyeye and Omar (2019)]

Table 4.3: Numerical Results for Example 4.3

t	ES	CS	ENM	EAe19	EAO19
0.003125	1.2998×10^{-3}	1.2998×10^{-3}	0.0000(00)	2.1149(-18)	5.4210(-18)
0.003125	2.5299×10^{-3}	2.5299×10^{-3}	0.0000(00)	1.0576(-17)	6.2884(-17)
0.009375	3.6835×10^{-3}	3.6835×10^{-3}	0.0000(00)	1.2683(-17)	2.3289(-16)
0.001250	4.7550×10^{-3}	4.7550×10^{-3}	0.0000(00)	2.1963(-17)	6.3578(-16)
0.015625	5.7392×10^{-3}	5.7392×10^{-3}	0.0000(00)	2.5623(-17)	1.4554(-15)
0.018750	6.6322×10^{-3}	6.6322×10^{-3}	0.0000(00)	3.7297(-17)	2.8970(-15)
0.021875	7.4310×10^{-3}	7.4310×10^{-3}	0.0000(00)	4.4098(-17)	5.2623(-15)
0.025000	8.1335×10^{-3}	8.1335×10^{-3}	0.0000(00)	5.9762(-17)	8.8714(-15)
0.028125	8.7386×10^{-3}	8.7386×10^{-3}	0.0000(00)	7.1313(-17)	1.4107(-14)
0.031250	9.2464×10^{-3}	9.2464×10^{-3}	0.0000(00)	9.2590(-17)	2.1415(-14)

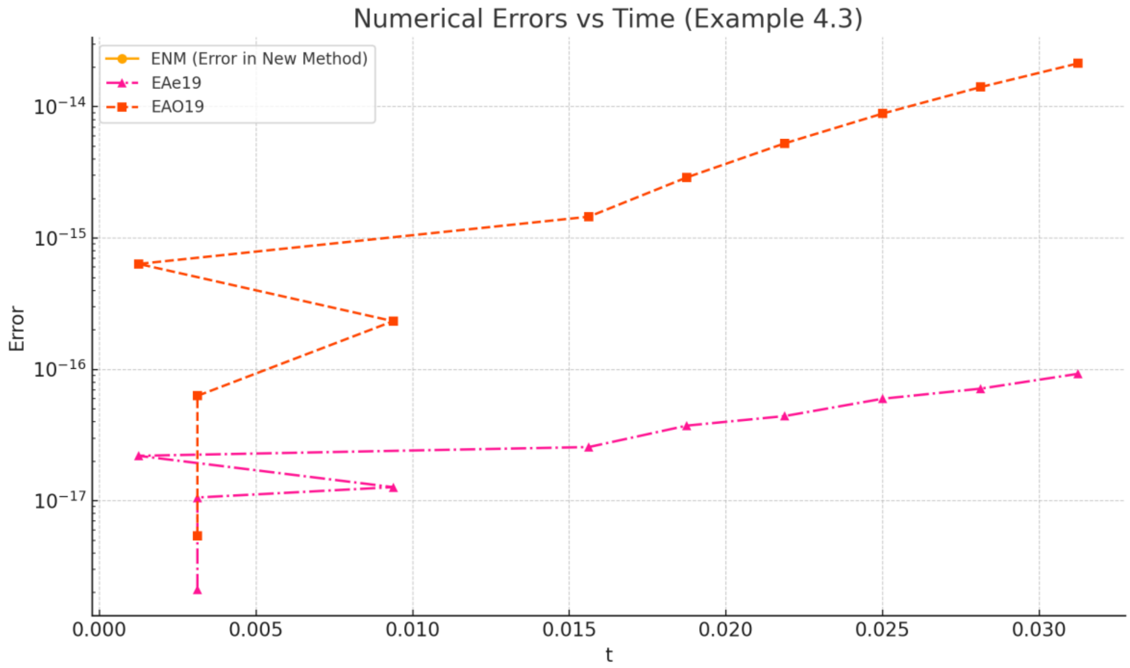

Figure 4.3: Graphical curve of table 4.3.

5. Discussion of Results

When applied to a classical second-order mass–spring system, the results shown in Table 4.1 and Figure 4.1 for Example 4.1 demonstrate the exceptional accuracy of the suggested numerical method. Throughout the time steps, the calculated solutions and the exact solutions are nearly identical. The table values and figure clearly show that these errors are smaller than those reported by Skwame et al. (2017) and Sabo et al. (2021). The figure's flat ENM line illustrates how this approach achieves higher stability and precision in comparison to earlier studies' larger error curves. As the complexity of the problem increases to a non-stiff, third-order oscillatory system in Example 4.2, the trends in Figure 4.2 and the data in Table 4.2 verify that the new algorithm maintains very small ENM values. The ENM errors for the current method stay extremely close to zero, in contrast to previous approaches like those of Adoghe and Omole (2019) and Tumba et al. (2021), which show errors that increase steadily with increasing time. Even for higher-order systems, the method effectively controls error accumulation, as seen by the figure, which displays the ENM curve lying flat and significantly below those of other approaches.

The method's exceptional accuracy for a fourth-order oscillatory differential equation is demonstrated by the data in Table 4.3 and Figure 4.3. In contrast to the reference methods of Adoghe and Omole (2019) and Adeyeye and Omar (2019), whose errors (EAO19 and EAe19) rise with time, the ENM values remain constant at zero for every step. For more complex oscillatory systems, the algorithm can produce results that are almost exact without error growth, as demonstrated by the plotted figure, where the ENM curve stays perfectly flat along the horizontal axis.

The tables and figures from these three examples make it abundantly evident that the suggested method performs noticeably better than current techniques. This numerical scheme is very accurate and stable, as evidenced by the consistent pattern of nearly zero ENM values in Tables 4.1, 4.2, and 4.3 and the corresponding flat error curves in Figures 4.1, 4.2, and 4.3. These findings show that it has a distinct advantage over the traditional approaches described in the literature and has great potential as a reliable tool for resolving oscillatory problems across various orders of differential equations.

6. Conclusion

In order to directly solve second-, third-, and fourth-order ordinary differential equations without reducing them to systems of first-order equations, this study created a one-step block hybrid numerical method. The approach was developed into a continuous implicit scheme and then expressed in block form using collocation and interpolation techniques. The region of absolute stability was determined through the analysis of its theoretical properties, encompassing order, error constant, consistency, zero-stability, and convergence The developed approach yields incredibly small errors in comparison to current methods, according to numerical experiments conducted on benchmark problems like a mass-spring system, third-order oscillatory models, and fourth-order oscillatory equations. The method maintains stability and high precision across different orders of differential equations, as demonstrated by graphic results that validated the error curves' flatness. The results show that, by avoiding the complexity and error accumulation of conventional reduction techniques, the developed block hybrid method provides a more accurate and efficient way to solve high-order initial value problems. Its outstanding accuracy and stability demonstrate its great potential for use in oscillatory system-related scientific and engineering problems. Future developments could investigate adaptive step sizes and applications to nonlinear and stochastic differential equations. The method is a substantial improvement over classical approaches.

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