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Numerical Solution of Fractional Burger-KdV Equation Using Double Natural-Aboodh Transform Method

Aminu Audu[a,b*], Solomon O. Adee[b] and Alhaji Tahir[b]

[a]*Department of Mathematical Sciences, Gombe State University, Tudun-Wada Gombe, Nigeria*

[b]*Department of Mathematics, Faculty of Physical Sciences, Modibbo Adama University, Yola, Nigeria*

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ABSTRACT

In this research, the fractional Burger-KdV partial differential equation is studied; two integral transform; the Natural Transform and Aboodh Transform were used to develop a new Double integral transform called Double Natural-Aboodh transform; the Double Natural-Aboodh Transform, combined with the Adomian Decomposition method, was employed to obtain the series-form solution when fractional order equal one and numerical solution of fractional Burger-KdV equation. The Caputo fractional operator was adopted; existence and uniqueness of the method were investigated in Caputo fractional derivative. The solution was found to be unique and convergent. Two example problems were provided to illustrate the efficiency, simplicity and accuracy of the developed method when compared with the existence literatures.

1. Introduction

The fractional Burger-KdV equation plays an important role in modelling nonlinear wave propagation, fluid mechanic and dispersive phenomena. Numerous scholars in various scientific disciplines, including Biology, Chemistry, fluid mechanics, Physics, and others, have focused on Fractional partial differential equation (FPDEs) and classical partial differential equation (CPDEs). Integral transform methods can solved partial differential equation of both integer and fractional orders, these methods transform problems into algebraic or ordinary forms, depending on whether single or double integral transformation are used (Davies, 2002). Also fractional solutions of partial differential equations (PDEs)have drawn a lot of attention because they can accurately simulate complicated systems and processes those traditional PDEs frequently find difficult to depict. Jasim *et al.* (2023) reviewed integral transforms which includes single, double and mixed double transforms. Mixed double transforms combine two different single integral transforms to form a double integral transform. Many have developed various double integral transform for solving partial differential equation of integer and fractional order including double ARA transform (Rania, 2022), double Natural transform (Kilicman and Omran, 2017), double Laplace and double Sumudu transforms (Hassan and Kilicman, 2010), two-dimensional Aboodh transform (Aboodh et al., 2017), double Shehu transform (Suliman and Emine, 2020), double Rangaig integral transform (Derte *et al.*, 2022), two-dimensional Emad-Falih transform (Turg and Kuffi, 2022), double Mahgoup transform (Patil, 2020) and many more.

Most approaches mentioned in the literature solved linear PDEs and FPDEs problems. It is therefore necessary to develop a double integral transform to address nonlinear fractional PDEs of Burger-KdV equation form. In this research, the Natural transform and Aboodh transform will be combined to form double Natural-Aboodh transform with Adomian decomposition method which will be implemented to solve fractional Burger-KdV equation problem of the form.

$$ {}^{C}D_t^{\sigma}\Phi(x,t) + \alpha\Phi\Phi_x(x,t) + \beta\Phi_{xx}(x,t) + \gamma\Phi_{xxx}(x,t) = 0 \quad (1)$$

* Corresponding author. Tel.:+2348061215024
E-mail address: hanaffbb@gmail.com (Aminu Audu)
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Equation (1) is fractional Burger-KdV equation with Caputo operator, σ the fractional order, α, β, γ, are constant variables.

2. Preliminaries

Definition 1 (Rasool et al., 2019) The Natural transform of $\Phi(x)$ for $x > 0$, $s, u > 0$ is denoted by $N[\Phi(x)]$ is

$$N[\Phi(x)] = \frac{1}{u}\int_0^\infty e^{\frac{-sx}{u}}\Phi(x)dx \qquad (2)$$

Definition 2 (Range and Gade, 2020) The Aboodh transform of $\Phi(t)$for $t > 0$, $p > 0$ is denoted by A $[\Phi(t)]$ is

$$\text{A}\,[\Phi(t)] = \frac{1}{p}\int_0^\infty e^{-pt}\,\Phi(t)dt \qquad (3)$$

Definition 3 (Khalouta, 2023) The Caputo fractional operator of $\Phi(\tau)$ is ${}^{c}D_\tau^\sigma[\Phi(\tau)] = \frac{1}{\Gamma(n-\sigma)}\int_0^\tau(\tau - \vartheta)^{n-\sigma-1}\,\Phi^n(\vartheta)d\vartheta$ (4)

Definition 4 The double Natural-Aboodh transforms of $\Phi(x,t)$ for $x, t > 0$ is denoted by $N_x\text{A}_\text{t}\Phi(x,t)]$ is define as

$$N_x\text{A}_\text{t}\Phi(x,t)] = \xi(s,u,p) = \frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}\Phi(x,t)dxdt \qquad (5)$$

Provided that the integral exists and (s, u, p) are transform variable.

Definition 5 The inverse double Natural-Aboodh transform is denoted by $N^-{}_xA^-{}_t[\xi(s,u,p)]$ is define as

$$N^-{}_xA^-{}_t[\xi(s,u,p)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\frac{sx}{u}}\left\{\int_{\beta-i\infty}^{\beta+i\infty} pe^{pt}\xi(s,u,p)dp\right\}ds$$

By applying double Natural-Aboodh transform on some functions

1. $N_x\text{A}_\text{t}[x^{\text{m}}t^n] = \frac{m!n!u^m}{s^{m+1}p^{n+2}}$ m, n are positive integer
2. $N_x\text{A}_\text{t}[x^m\text{e}^{nt}] = \frac{\Gamma(m+1)u^m}{s^{m+1}(p^2+np)}$
3. $N_x\text{A}_\text{t}\left[\text{e}^{(ax+bt)}\right] = \frac{1}{(s-au)(p^2-bp)}$
4. $N_x\text{A}_\text{t}\left[\text{e}^{i(ax+bt)}\right] = \frac{1}{(s-au)(p^2-ibp)}$
5. $N_x\text{A}_\text{t}[sin(ax+bt)] = \frac{(ap^2u+bsp)}{(s^2+a^2u^2)(p^4+b^2p^2)}$
6. $N_x\text{A}_\text{t}[cos(ax+bt)] = \frac{(sp^2-aup^2)}{(s^2+a^2u^2)(p^4+b^2p^2)}$
7. $N_x\text{A}_\text{t}[sinh(ax+bt)] = \frac{(ap^2u+bsp)}{(s+au)(s-au)(p^2+bp)(p^2-bp)}$
8. $N_x\text{A}_\text{t}[cosh(ax+bt)] = \frac{(sp^2-aubp)}{(s+au)(s-au)(p^2+bp)(p^2-bp)}$

3. Methodology

Consider equation (1) for the formulation of the method for solution of fractional Burger-KdV equation of initial value problem with Caputo fractional operator using double Natural-Aboodh transform. Applying double Natural-Aboodh transform we have,

$$N_x\text{A}_\text{t}[\text{D}_\text{t}^\sigma\Phi(x,t) + \alpha\Phi\Phi_x(x,t) + \beta\Phi_{xx}(x,t) + \gamma\Phi_{xxx}(x,t)] = 0 \qquad (6)$$

$$\Phi(x,0) = \vartheta(x) \qquad (7)$$

By theorem (4), equation (6) becomes,

$$p^\sigma\xi(s,u,p) - \sum_{i=0}^{n-1}\frac{1}{p^{2-\sigma+i}}N_x\left[\frac{\partial^i\Phi(x,0)}{\partial t^i}\right] = -N_x\text{A}_\text{t}[\alpha\Phi\Phi_x(x,t) + \beta\Phi_{xx}(x,t) + \gamma\Phi_{xxx}(x,t)] \quad (8)$$

$$\xi(s,u,p) = \frac{1}{p^2} N_x[\Phi(x,0)] - \frac{1}{p^\sigma} N_x \mathrm{A_t}[\alpha\Phi\Phi_x(x,t) + \beta\Phi_{xx}(x,t) + \gamma\Phi_{xxx}(x,t)] \quad (9)$$

By taking double inverse Natural-Aboodh transform on equation (9),

$$\Phi(x,t) = N^-{}_x A^-{}_t\left\{\frac{1}{p^2} N_x[\vartheta(x)]\right\} - N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x \mathrm{A_t}[\alpha\Phi\Phi_x(x,t) + \beta\Phi_{xx}(x,t) + \gamma\Phi_{xxx}(x,t)]\right\} (10)$$

$\Phi(x,t)$is express as an infinite series

$$\Phi(x,t) = \sum_{n=0}^{\infty} \Phi_n(x,t) \quad (11)$$

The nonlinear term $\Phi\Phi_x(x,t)$ can be generated by Adomian polynomial define by

$$\Phi\Phi_x(x,t) = \sum_{n=0}^{\infty} \mathrm{A}_n \quad (12)$$

Where A_n is also defined as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left\{N\left[\sum_{k=0}^{n} \lambda^k \Phi_k \Phi_{k,x}(x,t)\right]\right\}_{\lambda=0}, n = 0,1,2,\cdots \quad (13)$$

Substituting equation (11) and (12) into equation (10)

$$\sum_{n=0}^{\infty} \Phi_n(x,t) = N^-{}_x A^-{}_t\left\{\frac{1}{p^2} N_x[\vartheta(x)]\right\} - N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x \mathrm{A_t}\left[\gamma \sum_{n=0}^{\infty} \Phi_{n,xxx}(x,t) + \beta \sum_{n=0}^{\infty} \Phi_{n,xx}(x,t) + \alpha \sum_{n=0}^{\infty} \mathrm{A}_n\right]\right\} \quad (14)$$

$$\Phi_0(x,t) = \vartheta(x) \quad (15)$$

$$\sum_{n=0}^{\infty} \Phi_{n+1}(x,t) = -N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x \mathrm{A_t}\left[\gamma \sum_{n=0}^{\infty} \Phi_{n,xxx}(x,t) + \beta \sum_{n=0}^{\infty} \Phi_{n,xx}(x,t) + \alpha \sum_{n=0}^{\infty} \mathrm{A}_n\right]\right\} (16)$$

This also gives

$$\Phi_1(x,t) = -N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x \mathrm{A_t}\left[\gamma\Phi_{0,xxx}(x,t) + \beta\Phi_{0,xx}(x,t) + \alpha A_0\right]\right\}$$

$$\Phi_2(x,t) = -N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x \mathrm{A_t}\left[\gamma\Phi_{1,xxx}(x,t) + \beta\Phi_{1,xx}(x,t) + \alpha A_1\right]\right\}$$

$$\Phi_3(x,t) = -N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x \mathrm{A_t}\left[\gamma\Phi_{2,xxx}(x,t) + \beta\Phi_{2,xx}(x,t) + \alpha A_2\right]\right\} \vdots \quad (17)$$

The series can be express as $\Phi(x,t) = \sum_{n=0}^{\infty} \Phi_n(x,t)$, as the terms of the series $\Phi_0(x,t), \Phi_1(x,t), \Phi_2(x,t), \Phi_3(x,t), \cdots$ are obtained. It is really important to note that the $\Phi_0(x,t)$ is defined by the terms that arises from initial conditions and the proceeding terms $\Phi_1(x,t), \Phi_2(x,t) \cdots$, can be generated by the results of the previous terms. The series will contain a fractional operator in Caputo sense σ where as $0 \le \sigma < 1$, therefore, $\sigma = 1$ the series-form solution will be obtain.

Existence and Uniqueness of Double Natural-Aboodh Transform

A function of $\Phi(x,t)$ is said to be exponential order $\alpha, \beta > 0$, on $0 \le x, t < \infty$, if there exist a positive constant, M, X and Y such that,

$$|\Phi(x,t)| \le e^{(\alpha x + \beta t)}, \qquad \text{for all } x > X, t > Y \quad (18)$$

Therefore, we write

$$\Phi(x,t) = 0(e^{(\alpha x + \beta t)}), \qquad \text{as } x, t \to \infty$$

Theorem 1: *Let $\Phi(x,t)$ be a continous function in every finite interval$(0,X)$, $(0,Y)$ and of exponential order $e^{(\alpha x + \beta t)}$, then, the double Natural-Aboodh transform of $\Phi(x,t)$ exist for all $s, u > \alpha, p > \beta$.*

Proof: Let $\Phi(x,t)$ be exponential order $e^{(\alpha x + bt)}$ such that,

$$|\Phi(x,t)| \le e^{(\alpha x + \beta t)}, \qquad \text{for all } x > X, t > Y$$

Then, we have

$$|\xi(s,u,p)| = \left|\frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}\Phi(x,t)dxdt\right|$$
$$\leq \frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}|\Phi(x,t)|dxdt$$
$$\leq \frac{M}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}e^{(\alpha x+\beta t)}dxdt$$
$$= \frac{M}{up}\int_0^\infty\int_0^\infty e^{-(\frac{s}{u}+\alpha)x}e^{-(p+\beta)t}dxdt$$
$$= \frac{M}{u(p^2-p\beta)}\frac{1}{(\frac{s}{u}-\alpha)}$$
$$= \frac{M}{(p^2-p\beta)(s-\alpha u)}$$

Properties ofDouble Natural-Aboodh Transform and its Partial Derivatives

If the double Natural Aboodh transform of function $\Phi_1(x,t)$ and $\Phi_2(x,t)$ are $\xi_1(s,u,p)$ and $\xi_2(s,u,p)$ respectively and the double Natural-Abbodh transform exist, by definition of the double transform for any positive constant α,β such that

1. Linearity Property
$$N_x\mathrm{A_t}[\alpha\Phi_1(x,t)+\beta\Phi_2(x,t)] = \alpha\xi_1(s,u,p)+\beta\xi_2(s,u,p) \quad (19)$$
2. Shifting Property
$$N_x\mathrm{A_t}\left[e^{(\alpha x+\beta t)}\Phi(x,t)\right] = \frac{1}{p(s-\alpha u)(p-\beta)}\xi\left(\frac{s-\alpha u}{u},p-\beta\right) \quad (20)$$
3. Change of Scale Property
$$N_x\mathrm{A_t}[\Phi(x\alpha,t\beta)] = \frac{1}{\alpha\beta}\xi\left(\frac{s}{u\alpha},\frac{p}{\beta}\right) \quad (21)$$
4. If $\Phi(x,t)$ is a continous function and $N_x\mathrm{A_t}[\Phi(x,t)] = \xi(s,u,p)$. Then the following derivative are
$$N_x\mathrm{A_t}[\Phi_t(x,t)] = p\xi(s,u,p)-\frac{1}{p}N_x[\Phi(x,0)] \quad (22a)$$
$$N_x\mathrm{A_t}[\Phi_{tt}(x,t)] = p^2\xi(s,u,p)-N_x[\Phi(x,0)]-\frac{1}{p}N_x[\Phi_t(x,0)] \quad (22b)$$
$$N_x\mathrm{A_t}[\Phi_x(x,t)] = \frac{s}{u}\xi(s,u,p)-\frac{1}{u}A_t[\Phi(0,t)] \quad (22c)$$
$$N_x\mathrm{A_t}[\Phi_{xx}(x,t)] = \frac{s^2}{u^2}\xi(s,u,p)-\frac{s}{u^2}A_t[\Phi(0,t)]-\frac{1}{u}A_t[\Phi_x(0,t)] \quad (22d)$$
$$N_x\mathrm{A_t}[\Phi_{xxx}(x,t)] = \frac{s^3}{u^3}\xi(s,u,p)-\frac{s^2}{u^3}A_t[\Phi(0,t)]-\frac{s}{u^2}A_t[\Phi_x(0,t)]-\frac{1}{u}A_t[\Phi_{xx}(0,t)] \quad (22e)$$
$$N_x\mathrm{A_t}\left[\frac{\partial^2\Phi(x,t)}{\partial x\partial t}\right] = \frac{sp}{u}\xi(s,u,p)+\frac{sp}{u}N_x[\Phi(x,0)]-\frac{p}{u}A_t[\Phi(0,t)]-\frac{1}{up}[\Phi(0,0)] \quad (22f)$$

Convolution Theorem of Double Natural-Aboodh Transform: The convolution of the function $\Phi(x,t)$ and $g(x,t)$ is denoted by $\Phi ** g(x,t)$ and defined by

$$N_x\mathrm{A_t}[\Phi ** g(x,t)] = \int_0^x\int_0^t \Phi(x-\rho,t-\lambda)g(\rho,\lambda)d\rho d\lambda$$
$$= N_x\mathrm{A_t}[\Phi(x-\rho,t-\lambda)] * N_x\mathrm{A_t}[g(\rho,\lambda)] \quad (23)$$

Theorem 2: *If* $\xi(s,u,p) = N_x\mathrm{A_t}[\Phi(x,t)]$, *then for constants* ρ,λ*we have*

$$N_x\mathrm{A_t}[\Phi(x-\rho,t-\lambda)\mathrm{H}(x-\rho,t-\lambda)] = e^{-(\frac{s\rho}{u}+p\lambda)}\xi(s,u,p)$$

Where $H(x,t)$ *is the Heaviside unit function defined by*

$$\{H(x-\rho,t-\lambda)=\begin{cases}1, & x>\rho, t>\lambda\\ 0, & Otherwise\end{cases}$$

Proof:

$$N_x\mathrm{A_t}[\Phi(x-\rho,t-\lambda)\mathrm{H}(x-\rho,t-\lambda)]=\frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}\Phi(x-\rho,t-\lambda)\mathrm{H}(x-\rho,t-\lambda)dxdt$$

$$=\frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}\Phi(x-\rho,t-\lambda)dxdt$$

Put $m=x-\rho\Rightarrow x=m+\rho$ and $n=t-\lambda\Rightarrow t=n+\lambda$

$$=\frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{s(m+\rho)}{u}+p(n+\lambda))}\Phi(m,n)dmdn$$

$$=e^{-(\frac{s\rho}{u}+p\lambda)}\frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sm}{u}+pn)}\Phi(m,n)dmdn$$

$$=e^{-\left(\frac{s\rho}{u}+p\lambda\right)}\xi(s,u,p)$$

Theorem 3: *Let $\xi(s,u,p)$ and $G(s,u,p)$ be double Natural-Aboodh transform of a function $\Phi(x,t)$ and $g(x,t)$ respectively,*

$$N_x\mathrm{A_t}[\Phi**g(x,t)]=up\xi(s,u,p)G(s,u,p) \tag{24}$$

Proof:

$$N_x\mathrm{A_t}[\Phi(x,t)]=\frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}\Phi(x,t)dxdt$$

$$N_x\mathrm{A_t}[\Phi**g(x,t)]=\frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}\left\{\int_0^x\int_0^t\Phi(x-\rho,t-\lambda)g(\rho,\lambda)d\rho d\lambda\right\}dxdt$$

By Heaviside unit step function,

$$=\frac{1}{up}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+pt)}\left\{\int_0^\infty\int_0^\infty\Phi(x-\rho,t-\lambda)\mathrm{H}(x-\rho,t-\lambda)g(\rho,\lambda)d\rho d\lambda\right\}dxdt$$

$$=\int_0^\infty\int_0^\infty g(\rho,\lambda)d\rho d\lambda\left\{\frac{1}{up}\int_0^x\int_0^t e^{-(\frac{sx}{u}+pt)}\Phi(x-\rho,t-\lambda)\mathrm{H}(x-\rho,t-\lambda)\right\}dxdt$$

By theorem (2)

$$=\int_0^\infty\int_0^\infty g(\rho,\lambda)d\rho d\lambda\left\{e^{-\left(\frac{s\rho}{u}+p\lambda\right)}\xi(s,u,p)\right\}$$

$$=\xi(s,u,p)\int_0^\infty\int_0^\infty e^{-\left(\frac{s\rho}{u}+p\lambda\right)}g(\rho,\lambda)d\rho d\lambda$$

$$=up\xi(s,u,p)G(s,u,p)$$

Theorem 4: *If $0\le\sigma<1$, the double Natural-Aboodh transform of Caputo fractional derivatives is*

$$N_x\mathrm{A_t}\left[{}^{c}D_t^{\sigma}\Phi(x,t)\right]=p^{\sigma}\xi(s,u,p)-\sum_{i=0}^{n-1}\frac{1}{p^{2-\sigma+i}}N_x[\frac{\partial^i\Phi(x,0)}{\partial t^i}] \tag{25}$$

Proof:

$$N_x\mathrm{A_t}\left[{}^{c}D_t^{\sigma}\Phi(x,t)\right]=N_x\mathrm{A_t}\left\{\frac{1}{\Gamma(n-\sigma)}\int_0^t(t-\vartheta)^{n-\sigma-1}\frac{\partial^n\vartheta(x,\vartheta)}{\partial\vartheta^n}d\vartheta\right\}$$

By convolution theorem,

$$N_x A_t\left[{}^{c}D_t^{\sigma}\Phi(x,t)\right] = N_x A_t\left[\frac{1}{\Gamma(n-\sigma)}\left\{t^{n-\sigma-1} * \frac{\partial^n \Phi(x,t)}{\partial t^n}\right\}\right]$$
$$= \frac{1}{\Gamma(n-\sigma)} N_x\left[A_t\{t^{n-\sigma-1}\} * A_t\left\{\frac{\partial^n \Phi(x,t)}{\partial t^n}\right\}\right]$$

$A\{t^{n-\sigma-1}\} = \frac{\Gamma(n-\sigma)}{p^{n-\sigma-1}}$,

$$A\left\{\frac{\partial^n \Phi(x,t)}{\partial t^n}\right\} = p^n A[\Phi(x,t)] - \sum_{i=0}^{n-1}\frac{1}{p^{2-n+i}}\frac{\partial^i \Phi(x,0)}{\partial t^i}$$

Therefore,

$$= \frac{1}{\Gamma(n-\sigma)} N_x\left[\frac{\Gamma(n-\sigma)}{p^{n-\sigma-1}}\left\{p^n A[\Phi(x,t)] - \sum_{i=0}^{n-1}\frac{1}{p^{2-n+i}}\frac{\partial^i \Phi(x,0)}{\partial t^i}\right\}\right]$$
$$= \frac{p^n}{p^{n-\sigma-1}} N_x A_t[\Phi(x,t)] - \sum_{i=0}^{n-1}\frac{1}{p^{2-n+i}}\frac{1}{p^{n-\sigma-1}}\left[\frac{\partial^i \Phi(x,0)}{\partial t^i}\right]$$
$$N_x A_t\left[{}^{c}D_t^{\sigma}\Phi(x,t)\right] = p^{\sigma}\xi(s,u,p) - \sum_{i=0}^{n-1}\frac{1}{p^{2-\sigma+i}} N_x\left[\frac{\partial^i \Phi(x,0)}{\partial t^i}\right]$$

Numerical Examples

In this work, two numerical examples are used to test the accuracy and efficiency of the developed method. The error is obtained as $Error = exact - Approximate$.

Problem 1

Helal and Mehanna (2006) examine the fractional Burger-KdV equation

$$D^{\sigma}{}_t\Phi(x,t) + 2(\Phi^3)_x + \Phi_{xx} - \Phi_{xxx} = 0, \quad 0 \le \sigma < 1 \tag{26}$$

Subject to initial condition

$$\Phi(x,0) = \frac{1}{6} + \frac{1}{6}\tanh(\frac{1}{6}) \tag{27}$$

Exact solution

$$\Phi(x,t) = \frac{1}{6}\left\{1 + \tanh\left(\frac{1}{6}\left(x - \frac{2}{9}t\right)\right)\right\} \tag{28}$$

Solution

Applying double Natural-Aboodh transform on equation (26)

$$N_x A_t[D^{\sigma}{}_t\Phi(x,t)] = N_x A_t[\Phi_{xxx} - \Phi_{xx} - 2(\Phi^3)_x] \tag{29}$$

By double Natural-Aboodh transform property of Caputo operator,

$$p^{\sigma}\xi(s,u,p) - \sum_{i=0}^{n-1}\frac{1}{p^{2-\sigma+i}} N_x\left[\frac{\partial^i \Phi(x,0)}{\partial t^i}\right] = N_x A_t[\Phi_{xxx} - \Phi_{xx} - 2(\Phi^3)_x] \tag{30}$$

$$\xi(s,u,p) = \frac{1}{p^2} N_x[\Phi(x,0)] + \frac{1}{p^{\sigma}} N_x A_t[\Phi_{xxx} - \Phi_{xx} - 2(\Phi^3)_x] \tag{31}$$

Substituting the initial condition and taking double inverse Natural-Aboodh transform on equation (31),

$$\Phi(x,t) = N^-{}_x A^-{}_t\left\{\frac{1}{p^2} N_x\left[\frac{1}{6} + \frac{1}{6}\tanh(\frac{1}{6})\right]\right\} + N^-{}_x A^-{}_t\left\{\frac{1}{p^{\sigma}} N_x A_t[\Phi_{xxx} - \Phi_{xx} - 2(\Phi^3)_x]\right\} \tag{32}$$

$\Phi(x,t)$is express as an infinite series

$$\Phi(x,t) = \sum_{n=0}^{\infty}\Phi_n(x,t) \tag{33}$$

The nonlinear term $(\Phi^3)_x(x,t)$ can be generated by Adomian polynomial define by

$$(\Phi^3)_x = \sum_{n=0}^{\infty} A_n \tag{34}$$

Where the Adomian polynomial are;

$$A_0 = \Phi_{0,x}^3$$

$$A_1 = 3\Phi_{0,x}^2\Phi_{1,x}$$
$$A_2 = 3\Phi_{0,x}^2\Phi_{2,x} + 3\Phi_{0,x}\Phi_{1,x}^2$$
$$A_3 = 3\Phi_{0,x}^2\Phi_{3,x} + 6\Phi_{0,x}\Phi_{1,x}\Phi_{2,x} + \Phi_{1,x}^3$$
$$\vdots \tag{35}$$

By substituting equation (33) and (34) into equation (32), we have

$$\sum_{n=0}^{\infty}\Phi_{n+1}(x,t) = N^-{}_xA^-{}_t\left\{\frac{1}{p^2}N_x\left[\frac{1}{6}+\frac{1}{6}\tanh(\frac{1}{6})\right]\right\} + N^-{}_xA^-{}_t\left\{\frac{1}{p^\sigma}N_x\mathrm{A_t}[\sum_{n=0}^{\infty}\Phi_{n,xxx} - \sum_{n=0}^{\infty}\Phi_{n,xx} - 2\sum_{n=0}^{\infty}A_n]\right\} \tag{36}$$

We have Φ_n as follows;

$$\Phi_0(x,t) = \frac{1}{6}+\frac{1}{6}\tanh(\frac{1}{6})$$

$$\sum_{n=0}^{\infty}\Phi_{n+1}(x,t) = N^-{}_xA^-{}_t\left\{\frac{1}{p^\sigma}N_x\mathrm{A_t}[\sum_{n=0}^{\infty}\Phi_{n,xxx} - \sum_{n=0}^{\infty}\Phi_{n,xx} - 2\sum_{n=0}^{\infty}A_n]\right\} \tag{37}$$

For $n = 0$

$$\Phi_1(x,t) = N^-{}_xA^-{}_t\left\{\frac{1}{p^\sigma}N_x\mathrm{A_t}[\Phi_{0,xxx} - \Phi_{0,xx} - 2\mathrm{A_0}]\right\} \tag{38}$$

$$\Phi_1(x,t) = N^-{}_xA^-{}_t\left\{\frac{1}{p^\sigma}N_x\mathrm{A_t}[\left\{\frac{1}{108}\left[\frac{1}{3}sech^2\left(\frac{x}{6}\right)tanh^2\left(\frac{x}{6}\right) - \frac{1}{6}sech^4\left(\frac{x}{6}\right)\right] + \frac{1}{108}sech^2\left(\frac{x}{6}\right)tanh\left(\frac{x}{6}\right) - \frac{1}{23328}sech^6\left(\frac{x}{6}\right)\right\}]\right\} \tag{39}$$

$$= \left[\left\{\frac{1}{108}\left[\frac{1}{3}sech^2\left(\frac{x}{6}\right)tanh^2\left(\frac{x}{6}\right) - \frac{1}{6}sech^4\left(\frac{x}{6}\right)\right] + \frac{1}{108}sech^2\left(\frac{x}{6}\right)tanh\left(\frac{x}{6}\right) - \frac{1}{23328}sech^6\left(\frac{x}{6}\right)\right\}\right]A^-{}_t\left[\frac{1}{p^{\sigma+2}}\right] \tag{40}$$

$$\Phi_1(x,t) = \left[\left\{\frac{1}{108}\left[\frac{1}{3}sech^2\left(\frac{x}{6}\right)tanh^2\left(\frac{x}{6}\right) - \frac{1}{6}sech^4\left(\frac{x}{6}\right)\right] + \frac{1}{108}sech^2\left(\frac{x}{6}\right)tanh\left(\frac{x}{6}\right) - \frac{1}{23328}sech^6\left(\frac{x}{6}\right)\right\}\right]\left\{\frac{t^\sigma}{\sigma!}\right\} \tag{41}$$

For $n = 1$

$$\Phi_2(x,t) = N^-{}_xA^-{}_t\left\{\frac{1}{p^\sigma}N_x\mathrm{A_t}[\Phi_{1,xxx} - \Phi_{1,xx} - 2\mathrm{A_1}]\right\} \tag{42}$$

$$\Phi_2(x,t) = N^-{}_x\left\{N_x\left[\frac{1}{23328}sech^8\left(\frac{x}{6}\right)\tanh(\frac{x}{6})\right]\right\}A^-{}_t\left\{\frac{1}{p^\sigma}A_t\left[\frac{t^\sigma}{\sigma!}\right]\right\} \tag{43}$$

$$\Phi_2(x,t) = \frac{1}{23328}sech^8\left(\frac{x}{6}\right)\tanh(\frac{x}{6})\left\{\frac{t^{2\sigma}}{2\sigma!}\right\} \tag{44}$$

The series solution is therefore,

$$\Phi(x,t) = \sum_{n=0}^{\infty}\Phi_n(x,t) = \Phi_0(x,t) + \Phi_1(x,t) + \Phi_2(x,t) + \cdots \tag{45}$$

$$\Phi(x,t) = \frac{1}{6}+\frac{1}{6}\tanh\left(\frac{1}{6}\right) + \left[\left\{\frac{1}{108}\left[\frac{1}{3}sech^2\left(\frac{x}{6}\right)tanh^2\left(\frac{x}{6}\right) - \frac{1}{6}sech^4\left(\frac{x}{6}\right)\right] + \frac{1}{108}sech^2\left(\frac{x}{6}\right)tanh\left(\frac{x}{6}\right) - \frac{1}{23328}sech^6\left(\frac{x}{6}\right)\right\}\right]\left\{\frac{t^\sigma}{\sigma!}\right\} + \frac{1}{23328}sech^8\left(\frac{x}{6}\right)\tanh\left(\frac{x}{6}\right)\left\{\frac{t^{2\sigma}}{2\sigma!}\right\} + \cdots \tag{46}$$

Table 1: Numerical results and Error of Problem 1

$t=0.02$, x	exact	Our method $\sigma=1$	Our method $\sigma=0.9$	Helal and Mehanna, (2006) Error	Our method $\sigma=1$ Error
0.1	0.1693207633	0.1694155861	0.1694002119	0.0889801	-9.48228 e-5
0.3	0.1748699093	0.1749708838	0.1749589621	0.123026	-1.009745e-4
0.5	0.1804008835	0.1805079302	0.1804995643	0.0399017	-1.070467e-4
0.7	0.1859015669	0.1860145480	0.1860098073	-0.0795711	-1.129811e-4
0.9	0.1913601055	0.1914788257	0.1914777443	0.0545362	-1.187202e-4

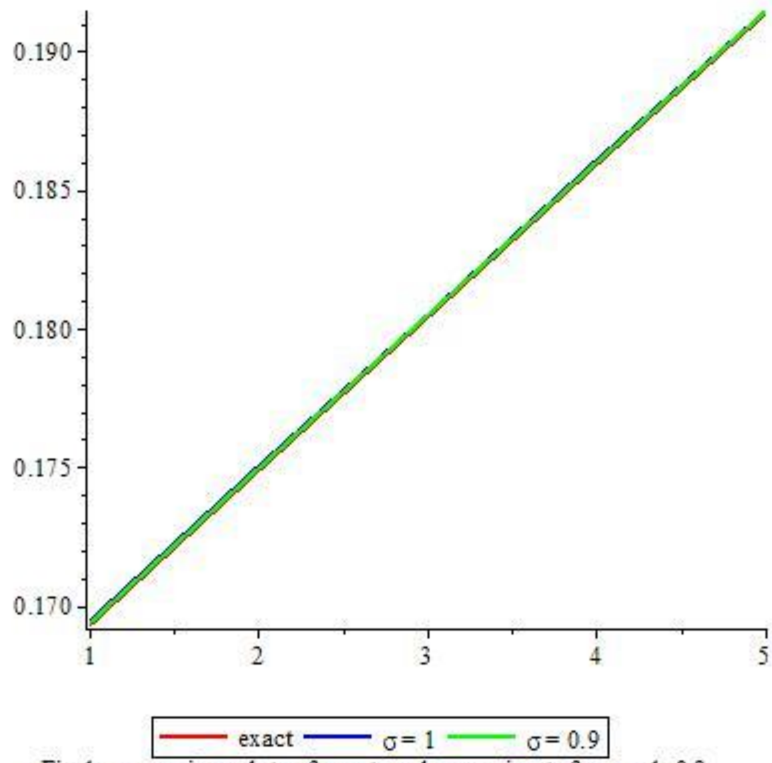

Fig 1. comparison plots of exact, and approximate for σ = 1, 0.9

Problem 2

Khan *et al* (2022) consider the fractional Burger-KdV equation

$$D^{\sigma}{}_{t}\Phi(x,t)+\alpha\Phi_{xxx}+\beta\Phi_{xx}+\gamma\Phi\Phi_{x}=0 \quad (47)$$

Subject to the condition

$$\Phi(x,0) = a_0 + \frac{3\beta^2 tanh^2(\frac{\beta x}{10\alpha})}{25\alpha\gamma} + \frac{6\beta^2 \tanh(\frac{\beta x}{10\alpha})}{25\alpha\gamma} \tag{48}$$

Exact solution

$$\Phi(x,t) = a_0 + \frac{6\beta^2}{25\alpha\gamma}\tanh\left\{\frac{\beta}{10\alpha}\left\{x + \frac{(3\beta^2-25a_0\alpha\gamma)t}{25\alpha}\right\}\right\} - \frac{3\beta^2}{25\alpha\gamma}tanh^2\left\{\frac{\beta}{10\alpha}\left\{x + \frac{(3\beta^2-25a_0\alpha\gamma)t}{25\alpha}\right\}\right\} \tag{49}$$

Solution

Applying double Natural-Aboodh transform on equation (47)

$$N_x A_t[D^\sigma{}_t \Phi(x,t)] = -N_x A_t[\alpha\Phi_{xxx} + \beta\Phi_{xx} + \gamma\Phi\Phi_x] \tag{50}$$

$$p^\sigma \xi(s,u,p) - \sum_{i=0}^{n-1} \frac{1}{p^{2-\sigma+i}} N_x\left[\frac{\partial^i \Phi(x,0)}{\partial t^i}\right] = -N_x A_t[\alpha\Phi_{xxx} + \beta\Phi_{xx} + \gamma\Phi\Phi_x] \tag{51}$$

$$\xi(s,u,p) = \frac{1}{p^2} N_x[\Phi(x,0)] - \frac{1}{p^\sigma} N_x A_t[\alpha\Phi_{xxx} + \beta\Phi_{xx} + \gamma\Phi\Phi_x] \tag{52}$$

Taking double inverse Natural-Aboodh transform on equation (52) and substituting the initial condition,

$$\Phi(x,t) = N^-{}_x A^-{}_t\left\{\frac{1}{p^2} N_x\left[a_0 + \frac{3\beta^2 tanh^2(\frac{\beta x}{10\alpha})}{25\alpha\gamma} + \frac{6\beta^2 \tanh(\frac{\beta x}{10\alpha})}{25\alpha\gamma}\right]\right\} - N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x A_t[\alpha\Phi_{xxx} + \beta\Phi_{xx} + \gamma\Phi\Phi_x]\right\} \tag{53}$$

$\Phi(x,t)$ Express as an infinite series

$$\Phi(x,t) = \sum_{n=0}^{\infty} \Phi_n(x,t) \tag{54}$$

The nonlinear term $\Phi\Phi_x$ can be calculated by Adomian polynomials which define by,

$$\Phi\Phi_x(x,t) = \sum_{n=0}^{\infty} A_n(x,t) \tag{55}$$

Where Adomian polynomial are;

$$\begin{aligned} A_0 &= \Phi_0\Phi_{0,x} \\ A_1 &= \Phi_0\Phi_{1,x} + \Phi_1\Phi_{0,x} \\ A_2 &= \Phi_0\Phi_{2,x} + \Phi_1\Phi_{1,x} + \Phi_2\Phi_{0,x} \\ A_3 &= \Phi_0\Phi_{3,x} + \Phi_1\Phi_{2,x} + \Phi_2\Phi_{1,x} + \Phi_3\Phi_{0,x} \\ &\vdots \end{aligned} \tag{56}$$

By substituting equation (54) and (55) into equation (53), we have

$$\sum_{n=0}^{\infty} \Phi_{n+1}(x,t) = N^-{}_x A^-{}_t\left\{\frac{1}{p^2} N_x\left[a_0 + \frac{3\beta^2 tanh^2(\frac{\beta x}{10\alpha})}{25\alpha\gamma} + \frac{6\beta^2 \tanh(\frac{\beta x}{10\alpha})}{25\alpha\gamma}\right]\right\} - N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x A_t\left[\alpha\sum_{n=0}^{\infty}\Phi_{n,xxx} + \beta\sum_{n=0}^{\infty}\Phi_{n,xx} + \gamma\sum_{n=0}^{\infty}A_n\right]\right\} \tag{57}$$

Therefore, Φ_n are obtain as follows;

$$\Phi_0 = a_0 + \frac{3\beta^2 tanh^2(\frac{\beta x}{10\alpha})}{25\alpha\gamma} + \frac{6\beta^2 \tanh(\frac{\beta x}{10\alpha})}{25\alpha\gamma} \tag{58}$$

$$\sum_{n=0}^{\infty} \Phi_{n+1}(x,t) = -N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x A_t\left[\alpha\sum_{n=0}^{\infty}\Phi_{n,xxx} + \beta\sum_{n=0}^{\infty}\Phi_{n,xx} + \gamma\sum_{n=0}^{\infty}A_n\right]\right\} \tag{59}$$

For $n = 0$,

$$\Phi_1(x,t) = -N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma} N_x A_t\left[\alpha\Phi_{0,xxx} + \beta\Phi_{0xx} + \gamma A_0\right]\right\} \tag{60}$$

$$\Phi_1(x,t) = \frac{3}{3125}\frac{\left[\sinh\left(\frac{\beta x}{10\alpha}\right)-\cosh\left(\frac{\beta x}{10\alpha}\right)\right]\{25a_0\alpha\gamma-3\beta^2\}\beta^3}{cosh^3\left(\frac{\beta x}{10\alpha}\right)\alpha^3\gamma} A^-{}_t\left\{\frac{1}{p^{\sigma+2}}\right\} \quad (61)$$

$$\Phi_1(x,t) = \frac{3\beta^3\left[\tanh\left(\frac{\beta x}{10\alpha}\right)sech^2\left(\frac{\beta x}{10\alpha}\right)-sech^2\left(\frac{\beta x}{10\alpha}\right)\right]\{25a_0\alpha\gamma-3\beta^2\}}{3125\alpha^3\gamma}\left\{\frac{t^\sigma}{\sigma!}\right\} \quad (62)$$

For $n = 1$

$$\Phi_2(x,t) = -N^-{}_x A^-{}_t\left\{\frac{1}{p^\sigma}N_x A_t\left[\alpha\Phi_{1,xxx} + \beta\Phi_{1xx} + \gamma A_1\right]\right\} \quad (63)$$

$$\Phi_2(x,t) = \frac{3\beta^4(3\beta^2-25a_0\alpha\gamma)^2\left\{\left(\tanh\left(\frac{\beta x}{10\alpha}\right)-1\right)\left(3\tanh\left(\frac{\beta x}{10\alpha}\right)+1\right)\right\}sech^2\left(\frac{\beta x}{10\alpha}\right)}{1562500\alpha^5\gamma}A^-{}_t\left\{\frac{1}{p^{2\sigma+2}}\right\} \quad (64)$$

$$\Phi_2(x,t) = \frac{3\beta^4(3\beta^2-25a_0\alpha\gamma)^2\left\{\left(\tanh\left(\frac{\beta x}{10\alpha}\right)-1\right)\left(3\tanh\left(\frac{\beta x}{10\alpha}\right)+1\right)\right\}sech^2\left(\frac{\beta x}{10\alpha}\right)}{1562500\alpha^5\gamma}\left\{\frac{t^{2\sigma}}{2\sigma!}\right\} \quad (65)$$

The series solution is therefore;

$$\Phi(x,t) = \sum_{n=0}^{\infty}\Phi_n(x,t) = \Phi_0(x,t) + \Phi_1(x,t) + \Phi_2(x,t) + \cdots \quad (66)$$

$$\Phi(x,t) = a_0 + \frac{3\beta^2 tanh^2\left(\frac{\beta x}{10\alpha}\right)}{25\alpha\gamma} + \frac{6\beta^2\tanh\left(\frac{\beta x}{10\alpha}\right)}{25\alpha\gamma} + \frac{3\beta^3\left[\tanh\left(\frac{\beta x}{10\alpha}\right)sech^2\left(\frac{\beta x}{10\alpha}\right)-sech^2\left(\frac{\beta x}{10\alpha}\right)\right]\{25a_0\alpha\gamma-3\beta^2\}}{3125\alpha^3\gamma}\left\{\frac{t^\sigma}{\sigma!}\right\} + \frac{3\beta^4(3\beta^2-25a_0\alpha\gamma)^2\left\{\left(\tanh\left(\frac{\beta x}{10\alpha}\right)-1\right)\left(3\tanh\left(\frac{\beta x}{10\alpha}\right)+1\right)\right\}sech^2\left(\frac{\beta x}{10\alpha}\right)}{1562500\alpha^5\gamma}\left\{\frac{t^{2\sigma}}{2\sigma!}\right\} \quad (67)$$

Table 2: Numerical results of Problem 2, $a_0 = 0, \alpha = \beta = 0.1, \gamma = 1$

x/t	Exact	Khan *et al.*(2022)	Our method $\sigma = 1$	Our method $\sigma = 0.9$
-4/0.01	-0.01085076961	-3.5967	-0.007386100474	-0.007385880155
-1/0.01	-0.002510922824	-2.50239	-0.002272513804	-0.002272310650
1/0.01	0.002273084079	1.13462	0.002511493100	0.002511659426
4/0.01	0.007386593321	1.2000	0.01085126246	0.01085136145
-2/0.1	-0.005201177870	-3.3840	-0.004266209422	-0.004265185491
0/0.1	0.000002879827186	0.2693	0.000002879913600	0.000003769678599
2/0.1	0.004271744779	1.1990	0.005206713291	0.005207399581
-4/0.05	-0.01084940939	-3.5963	-0.007384740260	-0.007384055183

-1/0.05	-0.002509668622	-2.4131	-0.002271259585	-0.002270627907
1/0.05	0.002274110910	1.1448	0.002512519955	0.002513037117
4/0.05	0.007387204456	1.200	0.01085187362	0.01085218141
-2/0.2	-0.005197864015	-3.3282	-0.004262895443	-0.004261428629
0/0.2	0.000005759308690	0.4958	0.000005759654400	0.000007034204989
2/0.2	0.004273965647	1.1994	0.005208934480	0.005209917511

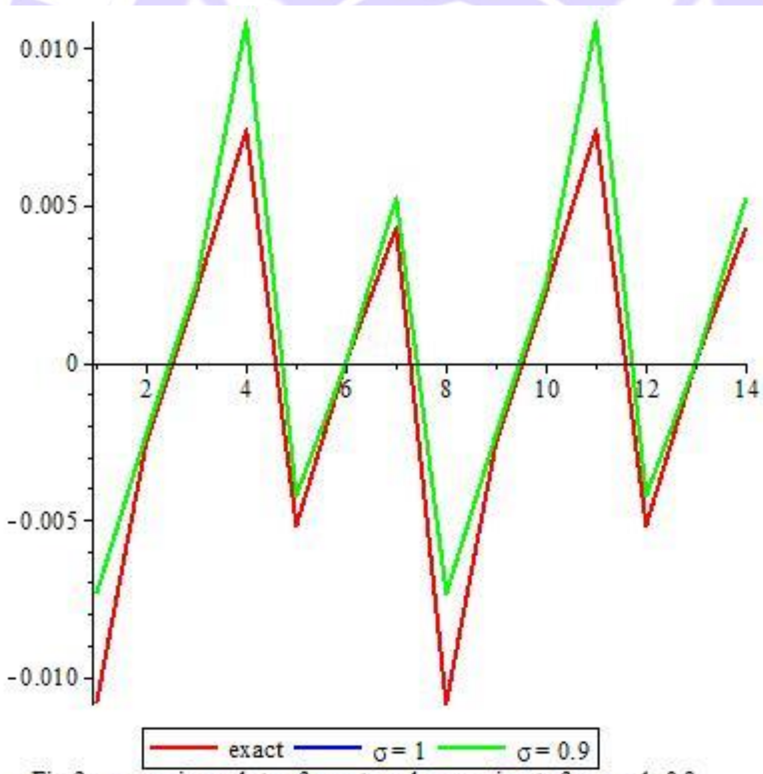

Fig 2. comparison plots of exact, and approximate for σ = 1, 0.9

4. Discussion

Numerical findings and the series-form solution is obtained when the fractional order is one from the solved problems using the developed method are presented. The exact solution were compared with those of the numerical results of the series-form solution at $\sigma = 1$, fractional series solution for fractional order $\sigma = 0.9$ and Helal and Mehanna (2006) and Khan *et al* (2022) from table 1 and table 2 respectively while considering different values of the variables x and t. From problem 1, the error obtained performed favourably than the results presented by Helal and Mehanna (2006), also from problem 2 numerical results obtained outperform the results presented by Khan et al (2022). This results show that our method demonstrate consistency and reliable.

5. Conclusion

In this work, the Double Natural-Aboodh Transform method was employed to solve fractional Burger-KdV equation problems, the fundamental properties, and existence and uniqueness theorems under Caputo fractional derivative was established. The series-form solution can be obtained when the fractional order is one; numerical solution and graphs for different fractional order were presented. The developed method is simple, reliable and effective. Its accuracy and convergence were demonstrated through some numerical examples highlighting its potential for solving similar fractional order problems. Maple 18 software is used for all the computations in this work. This research can extended to other fractional PDEs problem, integral equation problem, integro-differential equation problem and different fractional operator.

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