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Numerical Solution of Linear Volterra-Fredholm Integro-Differential Equations using Chebyshev Polynomials

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A B S T R A C T

This paper presents a numerical technique for solving linear Volterra-Fredholm integro-differntial equations using Chebyshev polynomials as basis function. The method involves approximating the unknown function as a finite sum of Chebyshev polynomials. The integral terms, both Volterra-type and Fredholm-type are evaluated using Gauss-Chebyshev quadrature rules, which are particularly effective due to their compatibility with the orthogonality of Chebyshev polynomials. This approach transforms the original Volterra-Fredholm integro-differential equation into a system of algebraic equations. The resultant algebraic system of equations is then solved using symbolic computation capabilities in Maple 18. Some numerical examples are provided to demonstrate the accuracy and efficiency of the proposed method, showing good agreement with known or reference solutions. Consistency, stability and convergence of the method developed were also analyzed. The technique is shown to be a reliable and practical tool for approximating solutions of linear Volterr-Fredholm integro-differential equations encountered in various scientific and engineering applications.

1. Introduction

Mathematical equations referred to as "Integro-Differential Equations" (IDEs) combine both integrals and derivatives. These equations are found across a range of scientific and engineering fields, including physics, biology, economics, and finance, particularly in systems characterized by memory or history-dependent behavior. Integro-Differential Equations (IDEs) account for the influence of past values of the unknown function through their integration component, unlike Ordinary Differential Equations (ODEs), which focus solely on derivatives. Scenarios involving diffusion, wave movement, population dynamics, control theory, and various other subjects often include IDEs. They provide a more precise representation of phenomena that demonstrate memory effects or spatial interactions. (Ajileye *et al.,* 2024).

The mathematical description of many intricate real-world problems gives rise to Integro-Differential Equations. Integro-Differential Equations have been used to express a variety of scientific phenomena. Numerous scientific disciplines, including physics, chemistry, biology, and engineering, use these kinds of equations. The theory of elasticity, biomechanics, electromagnetism, electrodynamics, fluid dynamics, heat and mass transfer, and oscillating magnetic fields are only a few of the numerous applications in which they can be found. (Abbas *et al.,* 2020).

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Groundbreaking studies by Abel, Lotka, Fredholm, Malthus, Verhulst, and Volterra in fields such as Mechanics, Mathematical Biology, and Economics established the groundwork for understanding integral and Integro-Differential Equations (IDEs). IDEs have attracted considerable attention from researchers because of their significance and applicability in various scientific domains. Since most IDEs cannot be solved through analytical means, numerical methods are essential for finding approximate solutions. (Didigwu *et al.,* 2024).

Existing literature presents diverse numerical approaches for tackling IDEs including; the least squares method for obtaining numerical solutions to Volterra-Fredholm integro-differential equations (Adebisi & Okunola, 2025). The Modified Adomian Decomposition Method (MADM) and Variational Iteration Method (VIM) were used by (Abed *et al.,* 2022) to investigate the numerical solution of a nonlinear Volterra-Fredholm integro-differential equation with initial conditions. A method combining the Simplified Reproducing Kernel Method (SRKM) and the Homotopy Perturbation Method (HPM) by (Hou *et al.,* 2021) to solve the nonlinear Volterra-Fredholm Integro-Differential Equations (V-FIDE) was proposed. The Boubaker polynomials method were utilized by (Ali *et al.,* 2020) to find approximate solution for the initial value problem of nonlinear high order Volterra and Fredholm integro-differential equation of the second kind. The least squares method in conjunction with shifted Chebyshev polynomials as a framework were employed by (Bello *et al.,* 2024) to approximate solutions for fourth-order Volterra integro-differential equations. Power series and shifted Chebyshev polynomials as basis function by (Ajileye & Aminu, 2022) were employed to solve first order volterra integro-differential equations using standard collocation method. A numerical approximation of Volterra Integro-Differential Equations (VIDEs) of the second kind using the quadrature rule in the modified block method was presented by (Muhammad *et al.,* 2023). Collocation methods based on the shifted Legendre polynomials was proposed by (Shirani *et al.,* (2022). An iterative method was proposed by (Yassein, 2021) for resolving many types of Volterra - Fredholm Integro - Differential Equations of the second kind with initial conditions. A third derivative trigonometrically fitted Simpson's method was developed and applied to approximate the solution of Volterra Integro-Differential Equations (VIDEs) via the multistep collocation method by (Olowe *et al.,* 2023). Akbari-Ganji's Method (AGM) was applied to solve Volterra Integro-Differential Difference Equations (VIDE) using Legendre polynomials as basis functions by (Uwaheren *et al.,* 2022). Ahsan & Hameeda (2021) proposed a novel approach that integrates the Least-Squares Method (LSM) with Euler polynomials to obtain approximate solutions for Integro-Differential Equations (IDEs) under mixed conditions. This method has a higher computational complexity compared to other approaches, which could lead to longer computational times and greater resource demands, making it less efficient for tackling complex Integro-Differential Equations. The method's sensitivity to initial conditions necessitates careful selection of parameters and starting values to guarantee convergence and accurate results.

In this research, Chebyshev polynomials are being considered due to their ability to approximate a wide variety of functions, making them a flexible tool for addressing Integro-Differential Equations. The application of Chebyshev polynomials has contributed to enhancing the stability of numerical algorithms employed for solving Integro-Differential Equations, thereby minimizing the potential for numerical instability.

The aim of this work is to present a numerical approximation based on Chebyshev polynomials to solve linear Volterra-Fredholm Integro-Differential Equations (VFIDEs) of the form:

$$y'(x)=F(x),+\lambda_1\int_a^x K_1(x,t)G_1(y(t))dt+\lambda_2\int_a^b K_2(x,t)G_2(y(t))dt \tag{1}$$

with the initial conditions

$$y(a)=y_0 \tag{2}$$

where;

$y'(x)$ is the derivative,

$F(x)$ is a known function,

$K_1(x,t)$ is the Voltera integral kernel, integrating over the variable upper limit x,

$K_2(x,t)$ is the Fredholm integral kernel, integrating over the fixed interval $[a,b]$

$G_1(y(t))$ and $G_2(y(t))$ are linear functions,

λ_1, λ_2 a and b are constants.

Assumptions

In this research, $G_1(y(t))=G_2(y(t))$, are linear functions of $y(t)$.

Subject to the following assumptions: (Hamoud *et al*., 2018)

(i) G_1 and G_2 are Lipschitz continuous, that is

$$|G_1(y_1(t))-G_1(y_2(t))|\le L|y_1-y_2| \tag{3}$$

Similarly,

$$|G_2(y_1(t))-G_2(y_2(t))|\le L|y_1-y_2| \tag{4}$$

(ii) There exist functions $K_1^*, K_2^* \in C(D, \mathbb{R}^+)$, the set of all positive continuous function on $D=\{(x,t)\in\mathbb{R}\times\mathbb{R}: a\le t\le x\le b\}$ such that

$$K_1^*=\max_{x\in J}\int_a^x |K_1(x,t)|dt \tag{5}$$

and

$$K_2^*=\max_{x\in J}\int_a^b |K_2(x,t)|dt \tag{6}$$

(iii) $F:[a,b]\to\mathbb{R}$ is a continuous function.

(iv) $k:[a,b]\times[a,b]\to\mathbb{R}$ is a continuous function.

2. Preliminaries

2.1 Integro-Differential equations (Ayinde *et al.,* 2022): An integro-differential equation is an equation in which the unknown function $y(x)$ appears under an integral sign and contains an ordinary derivative $y^{(n)}(x)$ as well. A standard integro-differential equation is of the form:

$$y^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t.)y(x)dt \tag{7}$$

where $f(x)$ is a given function, $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, $K(x,t.)$ is a function of two variables x and t called the kernel or the nucleus of the integral equation and $y^{(n)}(x)$ is the n order derivative.

2.2 Volterra-Fredholm integro-differential equation (Adebisi & Okunola, 2025): Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators. The Volterra-Fredholm integro-differential equations appear in two forms:

$$y^{(n)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt + \lambda_2 \int_a^b K_2(x,t)y(t)dt, \tag{8}$$

and

$$y^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,y(\xi,\tau))d\xi d\tau,\ (x,t) \in \Omega \times [0,T], \tag{9}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,y(\xi,\tau))$ are analytical functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1$, 2, 3. It is interesting to note that (8) contains disjoint Volterra and Fredholm integral equations, whereas (9) contains mixed integrals. The unknown functions $y(x)$ and $y(x,t)$ appears inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown function appears only inside the integral signs, the resulting equations are of first kind.

2.3 Linearity Concept (Falha, 2019): If the exponent of the unknown function $y(x)$ inside the integral sign is one, the integro-differential equation is called linear. If the unknown function $y(x)$ has exponent other than one, or if the equation contains nonlinear functions of $y(x)$, such as e^y, $\sinh y$, $\cos y$, and $\ln(1+x)$, the integro-differential equation is called nonlinear.

2.4 Numerical solution (Oyedepo *et al.*, 2024): A numerical or approximate solution refers to an estimation or an educated guess of a value, quantity, or solution to a problem that is not obtained precisely but is close enough to the actual or true value to be useful for practical purposes. In various fields, including mathematics, science, engineering, and computing, it's common to encounter problems that are difficult or even impossible to solve exactly due to their complexity, nonlinearity, or lack of analytical solutions. In such cases, an approximate solution provides a practical way to gain insights, make predictions, or solve problems within an acceptable level of accuracy.

2.5 Exact solution (Oyedepo *et al.*, 2024)

An exact solution refers to a precise and rigorous mathematical expression or representation that completely satisfies a given problem or equation. In various mathematical, scientific, and engineering contexts, finding an exact solution is highly valued because it provides an unambiguous and complete description of the problem at hand. An exact solution fully adheres to the principles and conditions of the problem, leaving no room for uncertainty or approximation.

2.6 Chebyshev polynomials (Adebisi *et al.,* 2021)

Chebyshev polynomials are sequence of orthogonal polynomials which are related to de-Moivre's formula and which can be defined recursively. One usually distinguishes between Chebyshev polynomials of first kind which are denoted by $T_n(x)$ and Chebyshev polynomials of second kind which are denoted by $U_n(x)$.

3. The Method of Solution

Since Chebyshev polynomials are defined on the interval $[-1,1]$. We first transform the interval $[a,b]$ to $[-1,1]$ as follows

$$x=\frac{b-a}{2}z+\frac{b+a}{2}, \quad z\in[-1,1] \tag{10}$$

Let $y(x)=Y(z)$

Using the Chain rule

$$\frac{dy}{dx}=\frac{2}{b-a}\frac{dY}{dz} \tag{11}$$

Rewriting equations (1) and (2) in terms of z, gives

$$\frac{2}{b-a}\frac{dY}{dz}=F(z),+\lambda_1\int_{-1}^{z}\tilde{K}_1(z,s)G_1(y(s))ds+\lambda_2\int_{-1}^{1}\tilde{K}_2(z,s)G_2(y(s))ds \tag{12}$$

with the initial conditions

$$y(-1)=y_0 \tag{13}$$

The approximate solution of the unknown function in terms of a finite series of Chebyshev polynomials is given by

$$y(z)\approx y_N=\sum_{i=0}^{N}c_iT_i(z) \tag{14}$$

where $T_i(z)=\cos(i\cos^{-1}z)$ are Chebyshev polynomials and c_i are the unknown coefficients to be determined. Then, the derivative is approximated as

$$y_N^{'}(z)=\sum_{i=0}^{N}c_i T_i^{'}(z) \tag{15}$$

where $T_i^{'}(z)$ is the derivative of the Chebyshev polynomial.

Let the number of quadrature points be M, the Volterra Integral term in equation (1) can be approximated using Gauss-Chebyshev quadrature as follows:

$$\int_a^x K_1(x,t)G_1(y(t))dt \rightarrow \int_{-1}^{z_i}\tilde{K}_1(z_i,s)G_1(y(s))ds \rightarrow \int_{-1}^{z_i}G_1(y(s))ds \approx \sum_{k=1}^{M}w_k G_1(s_k) \tag{16}$$

where s_k are mapped Gauss-Chebyshev nodes on $[-1,z_i]$ and w_k are the corresponding weights.

Similarly, Fredholm Integral term in equation (1) can also be approximated using Gauss-Chebyshev quadrature as follows:

$$\int_a^b K_1(x,t)G_1(y(t))dt \rightarrow \int_{-1}^{1}\tilde{K}_2(z_i,s)G_2(y(s))ds \rightarrow \int_{-1}^{1}G_2(y(s))ds \approx \sum_{k=1}^{M}w_k G_2(s_k) \tag{17}$$

Now, select $N+1$ collocation points. The Chebyshev-Gauss-Lobato points are commonly used;

$$z_i = \cos\left(\frac{i\pi}{N}\right),\ i=0,1,\ldots,\ N. \tag{18}$$

Substituting the Chabyshev approximation, its derivative and the integral approximations into the model equation (1) at each collocation point z_i, yields,

$$\frac{2}{b-a}\sum_{i=0}^{N}c_i T_i^{'}(z_i)=F(z_i)+\sum_{k=1}^{M}w_k\,\tilde{K}_1(z_i s_k)\sum_{i=0}^{N}c_i T_i(s_k)+\sum_{k=1}^{M}w_k\,\tilde{K}_2(z_i s_k)\sum_{i=0}^{N}c_i T_i(s_k) \tag{19}$$

This produces a linear system in the unknown coefficients c_i.

Applying the initial condition at $z=-1$ using equation (14), we have

$$\sum_{i=0}^{N}c_i T_i(-1)=y_0$$

This will result into matrix form as follows

$$Ac=f \tag{20}$$

where;

A is the coefficient matrix which includes the derivative and the integral terms,

c is the vector of unknown coefficients, and

f is the right-hand side vector.

Thus, the resulting system of linear algebraic equations can be written in matrix form as:

$$A = \begin{pmatrix} A_{01} & A_{02} & \cdots & A_{0M} \\ A_{11} & A_{12} & \cdots & A_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NM} \\ T_1(0) & T_2(0) & \cdots & T_N(0) \end{pmatrix}, \quad c = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}, \text{ and } \quad f = \begin{pmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_N) \\ y_0 \end{pmatrix} \tag{21}$$

It should be noted that the matrix A has columns up to M which is the number of quadrature points but rows up to N which is the number of Chebyshev nodes, and then an extra row for the initial condition.
Solving equation (20), gives

$$c = A^{-1} f \tag{22}$$

The approximate solution is reconstructed as follows:

$$y(z) \approx y_N = \sum_{i=0}^{N} c_i T_i \left(\frac{2x - (a+b)}{b-a} \right) \tag{23}$$

The solution of the above algorithm is implemented using MAPLE 18.

4.1 Consistency of the method

The Volterra-Fredholm Integro-Differention Equation (1) is consistent if the local truncation error or the residual $R_N(x)$ defined as the difference between the exact solution $y(x)$ and the approximate solution $y_N(x)$ tends to zero as the number of Chebyshev nodes (collocation points) increases.
The method is consistent if

$$\|R_N(x)\|_\infty \to 0 \text{ as } N \to \infty.$$

Define the residual $R_N(x)$ as

$$R_N(x) = y'(x) - F(x), -\lambda_1 \int_a^x K_1(x,t) G_1(y(t)) dt - \lambda_2 \int_a^b K_2(x,t) G_2(y(t)) dt \tag{24}$$

If $y(x)$ is the exact solution, then:

$$R_N(x) = y'(x) - F(x), -\lambda_1 \int_a^x K_1(x,t) G_1(y(t)) dt - \lambda_2 \int_a^b K_2(x,t) G_2(y(t)) dt = 0 \tag{25}$$

For $y_N(x)$ to converge to $y(x)$ as $N \to \infty$, $R_N(x)$ must converge to 0 uniformly on $[a,b]$.

(Akinsanya *et al.,* 2025)

4.2 Stability of the Method

Stability is determined by the properties of the matrix of the linear system obtained from collocation:

$$Ac = f \tag{26}$$

where;

A is the matrix formed by evaluating the derivatives of the basis functions and integral terms at the collocation points,

c is the vector of unknown coefficients, and

f is the vector of the known function $F(x)$.

Stability is ensured if the condition number of A remains bounded as $N \to \infty$.

4.3 Convergence of the Method

A numerical method is convergent if the global error tends to zero as the step size h tends to zero.

The method is convergent if:

$$\lim_{h \to 0} |y_N(x) - y(x)| = 0 \tag{27}$$

Let (X, d) be a metric space and $T: X \to X$ be a continuous mapping and $y_N(x), y_{N-1}(x) \in X$ are approximate solutions of equations (1) and (2).

Let $\Delta_N(x) = |y_N(x) - y_{N-1}(x)|$, if $\lim_{N\to\infty}(\Delta_N(x)) \to 0$, then the method converges to exact solution.

(Ajileye, *et al.,* 2024)

5. Numerical Examples

The following numerical illustrations are provided to assess the applicability and accuracy of the method. Let the approximate and exact solutions be $y_N(x)$ and $y(x)$ respectively. The

$$Error_N = |y_N(x) - y(x)|. \tag{28}$$

Example 1: Consider the first order Volterra-Fredholm Integro-differential equation (Shahooth *et al*., 2016).

$$y'(x) = F(x) + \int_0^x x^2 t y(t) dt + \int_0^1 x(x-t) y(t) dt \tag{29}$$

subject to initial condition

$$y(0) = 1 \tag{30}$$

where

$$F(x)=\sin(x)-x^2\cos(x)-x^3\sin(x)+x^2-x^2\sin(1)+x\cos(1)+x\sin(1)-x \tag{31}$$

Exact solution: $y(x)=\cos(x)$.

Solution 1

Comparing with equations (1) and (2), $\lambda_1=1,\ \lambda_2=1,\ K_1(x,t)=x^2t,\ K_2(x,t)=x(x-t)$ and

$$F(x)=\sin(x)-x^2\cos(x)-x^3\sin(x)+x^2-x^2\sin(1)+x\cos(1)+x\sin(1)-x.$$

The Taylor series expansions of $\sin(x)$ and $\cos(x)$ are given by

$$\sin(x)=\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \tag{32}$$

and

$$\cos(x)=\sum_{m=0}^{N}\frac{(-1)^m}{(2m)!}x^{2m} \tag{33}$$

Substituting equations (32) and (33) into equation (31), gives

$$F(x)=\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x^{2n+1}-\sum_{m=0}^{N}\frac{(-1)^m}{(2m)!}x^{2m+2}-\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x^{2n+4}+x^2-\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x^2+\sum_{m=0}^{N}\frac{(-1)^m}{(2m)!}x$$
$$+\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x-x \tag{34}$$

Substituting equation (34) into equation (29), gives

$$y'(x)=\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x^{2n+1}-\sum_{m=0}^{N}\frac{(-1)^m}{(2m)!}x^{2m+2}-\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x^{2n+4}+x^2-\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x^2+\sum_{m=0}^{N}\frac{(-1)^m}{(2m)!}x$$
$$+\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}x-x+\int_0^x x^2ty(t)dt+\int_0^1 x(x-t)y(t)dt \tag{35}$$

Solving at N = 5 using Maple 18, gives

$$y_5=-0.51943e-5x^5+0.390215680e-1x^4-0.680543e-4x^3-0.4985934464x^2-0.48309e-5x$$
$$+1.0000256770$$

Repeating the same procedures for example 1 at N = 6 and 10, gives

$$y_6=-0.13239953e-2x^6-1.2604785480\times10^{-7}x^5+0.416092160e-1x^4-3.7551845180\times10^{-7}x^3$$
$$-0.4999815072x^2+5.7587898880\times10^{-9}x$$
$$+1.0000000000$$

$$y_{10} = 0.30197e - 5x^{10} - 0.153727e - 4x^{9} + 0.456070e - 4x^{8} - 0.461010e - 4x^{7} - 0.13601077e - 2x^{6}$$
$$- 0.89017e - 5x^{5} + 0.416702407e - 1x^{4} - 6.0217084390 \times 10^{-7} x^{3} - 0.4999999685 x^{2}$$
$$- 2.5747263060 \times 10^{-12} x + 1.0000000000$$

Table 1: **Exact and approximate values for example 1**

x	Exact	N = 5	N = 6	N = 10
0.2	0.9800665778	0.9801428614	0.9800672278	0.9800665784
0.4	0.9210609940	0.9212433367	0.9210627087	0.9210610132
0.6	0.8253356149	0.8255712294	0.8253373521	0.8253355339
0.8	0.6967067093	0.6968686950	0.6967076639	0.6967046259
1.0	0.5403023059	0.5403757191	0.5403032178	0.5402878136

Table 2: Absolute Errors for Example 1

x	Error (N = 5)	Error (N = 6)	Error (N = 10)
0.2	0.0000762836	6.5000000000e-7	6.0000000000e-10
0.4	0.0001823427	1.714700000e-6	1.9200000000e-8
0.6	0.0002356145	1737200000e-6	8.1000000000e-8
0.8	0.0001619857	9.5460000000e-7	1.86044681900e-8
1.0	0.0000734132	9.1190000000e-7	1.06058267750e-8

Example 2: Consider the first order Volterra-Fredholm Integro-differential equation (Ahsan & Hameeda, 2021).

$$y'(x) = F(x) + \int_0^x K_1(x,t)y(t)dt + \int_0^1 K_2(x,t)y(t)dt \tag{36}$$

where $K_1(x,t) = 0,\ K_2(x,t) = x,$ and $F(x) = xe^x + e^x - x$

subject to initial condition

$$y(0) = 0 \tag{37}$$

Exact solution: $y(x) = xe^x$

Solution 2

Comparing with equation (1) and (2), $\lambda_1 = 1,\ \lambda_2 = 1,\ K_1(x,t) = 0,\ K_2(x,t) = x,$ and

$$F(x) = xe^x + e^x - x \tag{38}$$

Using Taylor series expansions, gives

$$F(x) = \sum_{n=0}^{N} \frac{x^{n+1}}{n!} + \sum_{n=0}^{N} \frac{x^n}{n!} - x \tag{39}$$

Substituting equation (39) into equation (36), gives

$$y'(x)=\sum_{n=0}^{N}\frac{x^{n+1}}{n!}+\sum_{n=0}^{N}\frac{x^{n}}{n!}-x+\int_0^1 xy(t)dt \tag{40}$$

Solving at N = 5 using Maple 18, gives

$$y_5 = 0.04517146895x^5 + 0.1835813232x^4 + 0.4972640000x^3 + 0.9905993270x^2 + 1.000632483x - 8.733014310\,10^{-13}$$

Table 3: Exact, approximate and absolute error values for example 2

x	Exact	Our method (N = 5)	Our Error (N = 5)	(Ahsan & Hameeda, 2021) Error (N = 5)
0.2	0.2442805516	0.2440367667	2.437849e-4	1.98842e-1
0.4	0.5967298792	0.5957360192	9.938600e-4	2.6582e-2
0.6	1.093271280	1.091708944	1.562336e-3	2.0478e-2
0.8	1.780432742	1.779085421	1.347321e-3	4.9422e-2
1.0	2.718281828	2.717248602	1.033226e-3	3.18281e-2

6. Conclusion

An enhanced numerical method was developed for the solution of first order linear Volterra-Fredholm integro-differential equations with initial conditions using the Chebyshev polynomial.

The numerical results of example 1 shows that the approximate solution at N = 5 gives

$$y_5 = -0.51943e-5x^5 + 0.390215680e-1x^4 - 0.680543e-4x^3 - 0.4985934464x^2 - 0.48309e-5x + 1.0000256770.$$

Solving for the values of N = 6 and 10, the numerical results converge to an exact solution as the value of N increases as shown in Table 1.

The result obtained for Example 2, as shown in Table 3, is that the approximate solution at N = 5 gives,

$$y_5 = 0.04517146895x^5 + 0.1835813232x^4 + 0.4972640000x^3 + 0.9905993270x^2 + 1.000632483x - 8.733014310\,10^{-13}.$$

Thus, the numerical results give a better result compared to the result obtained by (Ahsan & Hameeda, 2021) at the same values of N = 5.

Thus, the numerical method derived is consistent, efficient, and reliable. Maple code was used to implement the developed method. Solved numerical examples showed that the method is reliable and suitable for such kinds of problems.

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