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On the Extreme Kumaraswamy-Epsilon Distribution and Its Application to Exceedances of Flood Peaks

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ABSTRACT

*This study introduces and applies the **Extreme Kumaraswamy-Epsilon (EKE)** distribution to model exceedances of flood peaks data. The EKE distribution, derived from the Kumaraswamy generator with epsilon distribution as baseline, is a flexible five-parameter distribution capable of capturing complex tail behaviour in extreme hydrological events. Using biannual flood exceedance data from the Wheaton River near Carcross, Yukon Territory, the EKE model is evaluated alongside the **Generalized Extreme Value (GEV)** distribution. Goodness-of-fit measures including the log-likelihood, Akaike Information Criterion (AIC), and Kolmogorov–Smirnov (KS) test suggest that the EKE distribution provides a better fit, offering both statistical robustness and practical relevance for modelling flood extremes.*

1. Introduction

The analysis of extreme values, particularly maxima, is essential in fields such as hydrology, meteorology, finance, and reliability engineering. Order statistics, the arrangement of sample values from smallest to largest, are fundamental in extreme value theory (EVT). Historically, work by Karl Pearson (Tippett, 1925), Fisher and Tippett (1928) laid the foundation for modelling extremes. EVT has found particular application in hydrology, such as in estimating extreme flood flows and droughts. The Generalized Extreme Value (GEV) distribution, which encompasses the Gumbel, Fréchet, and Weibull distributions, has become the standard for modelling block maxima of hydrologic and climatological data. The mathematical theory of extremes has been expanded through the development of various parametric distributions to model the tails of empirical datasets (Cordeiro et al., 2019). A particularly effective approach to modelling such data involves **order statistics**, especially the maximum order statistic. Applications in flood flow modelling, for example, often rely on parametric inference about these extreme events (Gumbel, 1958; Jenkinson, 1955). Among the many advancements in this area is the **Kumaraswamy-G** framework, which allows for the construction of flexible distributions using a baseline distribution and the Kumaraswamy generator (Cordeiro & de Castro, 2011). Building on this framework, the **Kumaraswamy-Epsilon (KE)** distribution was recently introduced to enhance tail modelling (Gongsin and Saporu, 2019). This paper extends that work with the objectives of (1) deriving and studying the **Extreme Kumaraswamy-Epsilon (EKE)** distribution, a model tailored to the upper tail of continuous datasets, and (2) applying it to a well-known hydrological dataset of **flood peak exceedances**.

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2. Methodology

2.1 The EKE Distribution

The **Kumaraswamy-Epsilon** distribution (Gongsin and Saporu, 2019) was generated by applying the Kumaraswamy generator (Cordeiro and de Castro, 2011) to the **Epsilon distribution** (Dombi et al, 2018). The distribution was found to possess attractive statistical properties, especially its flexibility and the form of its order statistics which showed the potential in modelling extremes (Gongsin and Saporu, 2019). This potential for modelling extremes attracted the generation of **Extreme Kumaraswamy-Epsilon (EKE)** distribution (see Appendix A), and used for modelling exceedances of flood peaks. The generated EKE distribution has the following probability density function (PDF), cumulative distribution function (CDF) and quantile function, respectively,

$$f_X(x) = \alpha\beta\lambda\varphi\left(\frac{\delta^2}{\delta^2 - x^2}\right)(1 - \mathrm{Z})\mathrm{Z}^{\alpha-1}(1 - \mathrm{Z}^{\alpha})^{\beta-1}\left(1 - (1 - \mathrm{Z}^{\alpha})^{\beta}\right)^{\varphi-1}, \qquad 0 < x < \delta, \tag{1}$$

$$F_X(x) = \left(1 - (1 - \mathrm{Z}^{\alpha})^{\beta}\right)^{\varphi}, \tag{2}$$

and

$$Q_X(p) = \delta\,\frac{\gamma(p) - 1}{\gamma(p) + 1}, \qquad 0 < p < 1, \tag{3}$$

where $\mathrm{Z} = 1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}$, $\gamma(p) = \left\{1 - \left[1 - \left(1 - p^{\frac{1}{\varphi}}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}\right\}^{-\frac{2}{\lambda\delta}}$, and $\alpha, \beta, \delta, \lambda, \varphi > 0$.

The density plots of the new distribution for varying parameter values are given in Figure 1 below. The flexibility of the EKE model makes it suitable for **skewed**, **heavy-tailed**, and **bounded** data, characteristics often found in environmental extremes.

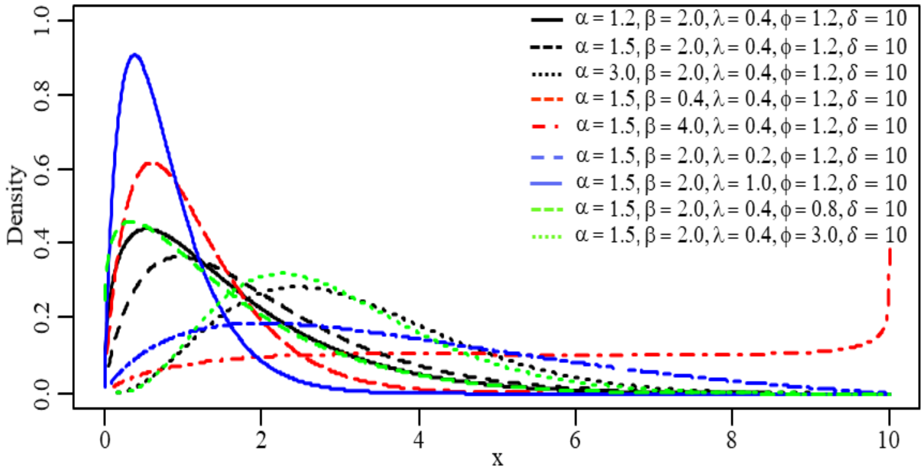

Figure 1 EKE Density Plots at Varying Parameter Values

Proposition

The EKE density function (1) is a true probability density function with unimodal shape property. That is, it suffices to show that

i. $f_X(x)$ is unimodal, and
ii. $\int_0^\delta f_X(x)\,dx = 1$

The proof to this proposition is shown in Appendix B.

2.2 The GEV Distribution

For comparison, the **Generalized Extreme Value (GEV)** distribution is employed, defined by its location μ, scale σ, and shape ξ parameters. The GEV encompasses the Gumbel, Fréchet, and Weibull families, and has been the standard for modelling extremes in hydrological contexts (Jenkinson, 1955; Coles, 2001; Koutsoyiannis, 2004). The density and distribution functions, respectively, are given by

$$f_X(x) = \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}, \tag{4}$$

and

$$F_X(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}, \tag{5}$$

where $x > 0$, μ is the location parameter, $\sigma > 0$ is the scale parameter and $\xi \neq 0$ is the shape parameter. Equation (5) becomes the Gumbel distribution when $\xi = 0$.

2.3 Estimation and Evaluation Criteria

Parameter estimation for both models was performed using **maximum likelihood estimation in R statistical software**. The parameters of the GEV distribution were estimated using the *evd* package, an inbuilt package in R. However, since the EKE distribution was new it was customized into R, introducing the PDF, CDF and quantile functions preceding with the letters d, p, and q, respectively, as

$$\begin{aligned} deke \ = \ & function(x,\alpha,\beta,\lambda,\delta,\varphi)\ \alpha \ast \beta \ast \lambda \ast \varphi \ast (\delta\text{\textasciicircum}2/(\delta\text{\textasciicircum}2 - x\text{\textasciicircum}2)) \ast (((x+\delta)/(\delta - x))\text{\textasciicircum}(-0.5 \ast \lambda \\ & \ast \delta)) \ast ((1-((x+\delta)/(\delta - x))\text{\textasciicircum}(-0.5 \ast \lambda \ast \delta))\text{\textasciicircum}(\alpha - 1)) \ast ((1-(1-((x+\delta)/(\delta \\ & - x))\text{\textasciicircum}(-0.5 \ast \lambda \ast \delta))\text{\textasciicircum}\alpha)\text{\textasciicircum}(\beta - 1)) \ast (1-(1-(1-((x+\delta)/(\delta - x))\text{\textasciicircum}(-0.5 \ast \lambda \\ & \ast \delta))\text{\textasciicircum}\alpha)\text{\textasciicircum}\beta)\text{\textasciicircum}(\varphi - 1) \end{aligned}$$

$$peke \ = \ function(q,\alpha,\beta,\lambda,\delta,\varphi)\ (1-(1-(1-((q+\delta)/(\delta - q))\text{\textasciicircum}(-0.5 \ast \lambda \ast \delta))\text{\textasciicircum}\alpha)\text{\textasciicircum}\beta)\text{\textasciicircum}\varphi$$

$$\begin{aligned} qeke \ = \ & function(p,\alpha,\beta,\lambda,\delta,\varphi)\ \delta \ast ((1-(1-(1-p\text{\textasciicircum}(1/\varphi))\text{\textasciicircum}(1/\beta))\text{\textasciicircum}(1/\alpha))\text{\textasciicircum}(-2/(\delta \ast \lambda)) - 1)/((1 \\ & - (1-(1-p\text{\textasciicircum}(1/\varphi))\text{\textasciicircum}(1/\beta))\text{\textasciicircum}(1/\alpha))\text{\textasciicircum}(-2/(\delta \ast \lambda)) + 1) \end{aligned}$$

Model performance were evaluated using:

i. **Log-Likelihood (LL)**, which measures how well the model explains the observed data;

ii. **Akaike Information Criterion (AIC)**, which balances fit quality with model complexity; and
iii. **Kolmogorov–Smirnov (KS) Test**, which assesses goodness-of-fit with p-value for the hypothesis

$$H_0: ECDF = CDF.$$

where ECDF is the empirical cumulative distribution function, that is, the cumulative distribution of the original data; and CDF is the fitted EKE cumulative distribution of the data.

2.4 Dataset

The dataset used comprises 72 peak discharge measurements (in m³/s) of flood exceedance recorded for the Wheaton River near Carcross, Yukon Territory, over the period 1958–1984. These exceedances have previously been documented and reused in statistical modeling literature (the dataset appears as **"floodpeak"** in the *DataSetsUni* R package). They are useful for extreme-value and flood frequency analyses and provides a credible observational basis for model fitting and comparison. The data are given below:

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0.

3. Results

3.1 Parameter Estimates and Goodness-of-Fit

Estimates of the parameters of the distributions were obtained using the method of maximum likelihood. These estimates possess desirable statistical properties, that is, the estimates are consistent, efficient, minimum variance unbiased and asymptotically normal (Mood *et al*., 1974). **The** parameter estimates, standard errors, and fit statistics for both **the EKE and GEV distributions are presented in Table 1.**

Table 1. Parameter Estimates and Fit Statistics for EKE and GEV Models

Distribution	Parameters (SE)	Log-Likelihood	AIC	KS Statistic (p-value)	Remark
EKE	$\alpha = 1.77\ (0.847)$ $\beta = 0.141\ (0.030)$ $\lambda = 0.426\ (0.005)$ $\varphi = 0.579\ (0.175)$ $\delta = 86.50\ (7.20)$	**-249.5**	**509.1**	**0.1029** **(0.1603)**	Good fit
GEV	$\mu = 4.08\ (0.819)$ $\sigma = 4.98\ (0.916)$ $\xi = 0.85\ (0.233)$	-261.0	528.0	0.1416 (0.1111)	Good fit

3.2 Interpretation of results

The fit results in Table 1 presents the parameter estimates, standard errors, log-likelihood (LL), Akaike Information Criterion (AIC), and Kolmogorov–Smirnov (KS) statistics for the two distributions fitted to exceedances of flood peaks data. The EKE model provides a more reasonable fit to the flood peak exceedances data. Parameter estimates are within 2 times their standard errors, depicting statistical significance. The small KS statistic and relatively high p-value suggest that we cannot reject the null hypothesis that the empirical cumulative distribution function of the original data is equal to the EKE cumulative distribution fit, implying the data follows the EKE distribution. The log-likelihood value is relatively high (less negative), and AIC is small, supporting its comparability. The GEV distribution is also provides a valid and statistically supported fit. However, compared to EKE distribution fit, its log-likelihood value is lower (more negative), the AIC is higher (528.0 > 509.1), and the KS p-value is slightly lower, though still not statistically significant. This suggests the EKE distribution has provided better fit to the flood exceedances data.

3.3 Visual Support from Fitted Plots

The graphical presentations (Figure 2) comparing the **empirical CDF and PDF** of the data with the **theoretical curves from the EKE and GEV distributions** further supports the numeric results. The visual alignment between observed and theoretical distributions indicates a **high-quality fit**, especially in the tails, which is critical in flood modelling. The graphs showed that the EKE distribution aligned better than the GEV.

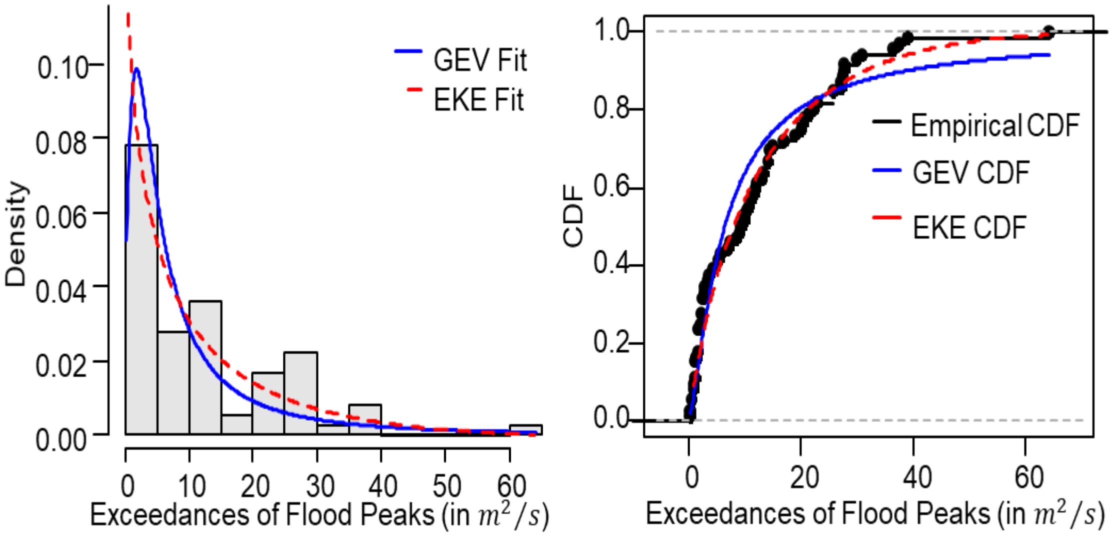

Figure 2 EKE and GEV Distribution Fit to Exceedances of Flood Peaks

4. Discussion

The results suggest that the **Extreme Kumaraswamy-Epsilon (EKE) distribution provides a superior fit** to the Wheaton River flood exceedances data compared to the GEV model. The EKE model's flexibility, with its five parameters, allows it to model **complex tail behaviour** more effectively. The use of **GEV** for flood modelling is well documented (Jenkinson, 1955; Katz et al., 2002; Villarini and Smith, 2010). The application of **Kumaraswamy-G family of distributions**, including EKE, is more recent but gaining attention for **its flexibility** in modelling complex real-world data (Gongsin and Saporu, 2019; Silva et al., 2021).

This study adds to the growing literature by demonstrating the EKE's utility in **extreme hydrologic applications**, aligning with recent efforts to generalize classical Extreme Value Theory (EVT) models (Merz et al., 2022; Oguntunde and Adejumo, 2023). The finding also aligns with recent literature advocating for **flexible parametric models** in environmental applications. For example, Merz et al. (2022), Silva et al. (2021) and Cordeiro et al. (2019) emphasized the benefits of extended distributions when modelling floods, rainfall extremes, and climate-driven processes. Although GEV remains a standard model in hydrology due to its theoretical foundation and simplicity, the EKE model demonstrates that **modern generalizations** can offer **practical improvements** without significant drawbacks, especially when the dataset is of moderate size.

5. Conclusion

This study demonstrates that the **Extreme Kumaraswamy-Epsilon distribution** is a **statistically sound and practically useful** model for flood exceedance data. When applied to the Wheaton River dataset, it performed better than the standard GEV model across all evaluation metrics. Given the growing availability of complex environmental data, the EKE model provides a valuable tool for hydrologists and environmental statisticians seeking accurate risk assessments and probabilistic modelling of extremes.

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APPENDIX

A. Derivation of the EKE distribution

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be ordered random variables from the Kumaraswamy-Epsilon (KE) distribution, then the distribution of the r^{th} order statistic is given (Gongsin and Saporu, 2019) by

$$f_{X_{(r)}}(x) = \alpha\beta\lambda \frac{n!}{(r-1)!\,(n-r)!} \frac{\delta^2}{\delta^2 - x^2} (1-Z) Z^{\alpha-1} (1-Z^\alpha)^{\beta(n-r+1)-1} \left[1-(1-Z^\alpha)^\beta\right]^{r-1}, \quad (A1)$$

where $Z = 1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}$, $\alpha,\ \beta,\ \delta,\ \lambda > 0,\ 0 < x < \delta$, and n is an integer greater than 0.

The distribution of the maxima is obtained when $r = n$. This gives

$$f_{X_{(n)}}(x) = \alpha\beta\lambda \frac{n!}{(n-1)!\,(n-n)!} \frac{\delta^2}{\delta^2 - x^2} (1-Z) Z^{\alpha-1} (1-Z^\alpha)^{\beta(n-n+1)-1} \left[1-(1-Z^\alpha)^\beta\right]^{n-1}$$

$$= \alpha\beta\lambda n \frac{\delta^2}{\delta^2 - x^2} (1-Z) Z^{\alpha-1} (1-Z^\alpha)^{\beta-1} \left[1-(1-Z^\alpha)^\beta\right]^{n-1} \quad (A2)$$

Then, for real $n = \varphi$, the function in (A2) becomes the distribution of the upper extreme value from the KE distribution. This is referred to as the Extreme KE (EKE) distribution in this study. This is given by

$$f_X(x) = \alpha\beta\lambda\varphi \frac{\delta^2}{\delta^2 - x^2} (1-Z) Z^{\alpha-1} (1-Z^\alpha)^{\beta-1} \left[1-(1-Z^\alpha)^\beta\right]^{\varphi-1}, \quad 0 < x < \delta, \quad (A3)$$

where $Z = 1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}$, $\alpha,\ \beta,\ \delta,\ \lambda,\ \varphi > 0$.

To derive the CDF, observe that the density function (A3) can be expressed in the form

$$f_X(x) = \varphi g(x)[G(x)]^{\varphi-1},$$

where $G(x) = 1 - (1 - Z^{\alpha})^{\beta}$ since

$$\frac{d}{dx}G(x) = g(x) = \beta(1 - Z^{\alpha})^{\beta-1}.\alpha Z^{\alpha-1}\frac{d}{dx}Z.$$

Recall that

$$Z = 1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2} = 1 - u(x), \qquad u(x) = \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2},$$

so

$$\frac{d}{dx}Z = -\frac{d}{dx}u(x) = -u'(x).$$

Using logarithmic differentiation,

$$\ln[u(x)] = -\frac{\lambda\delta}{2}\ln\left(\frac{x+\delta}{\delta-x}\right)$$

$$\frac{d}{dx}\ln[u(x)] = \frac{u'(x)}{u(x)} = -\frac{\lambda\delta}{2}\left(\frac{1}{x+\delta} + \frac{1}{\delta-x}\right) = -\frac{\lambda\delta}{2}\frac{2\delta}{\delta^2-x^2} = -\lambda\frac{\delta^2}{\delta^2-x^2}$$

$$\Rightarrow -u'(x) = \lambda\frac{\delta^2}{\delta^2-x^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2} = \lambda\frac{\delta^2}{\delta^2-x^2}(1-Z).$$

$$\therefore g(x) = \beta(1-Z^{\alpha})^{\beta-1}\cdot\alpha Z^{\alpha-1}\cdot\lambda\frac{\delta^2}{\delta^2-x^2}(1-Z)$$

$$= \alpha\beta\lambda\frac{\delta^2}{\delta^2-x^2}(1-Z)Z^{\alpha-1}(1-Z^{\alpha})^{\beta-1} \qquad (A4)$$

Since equation (A4) is non-φ, it implies that the CDF of $f(x)$ is $F(x) = [G(x)]^{\varphi}$. That is,

$$F(x) = \left[1 - (1 - Z^{\alpha})^{\beta}\right]^{\varphi} \qquad (A5)$$

The EKE Quantile Function

Given the EKE CDF (A5), let $p = F(x) \sim U(0,1)$. Then

$$p = \left\{1 - (1 - Z^{\alpha})^{\beta}\right\}^{\varphi} = \left\{1 - \left(1 - \left(1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}\right)^{\alpha}\right)^{\beta}\right\}^{\varphi},$$

$$\Rightarrow p^{\frac{1}{\varphi}} = 1 - \left(1 - \left(1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}\right)^{\alpha}\right)^{\beta}, \qquad \Rightarrow 1 - p^{\frac{1}{\varphi}} = \left(1 - \left(1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}\right)^{\alpha}\right)^{\beta},$$

$$\Rightarrow \left(1 - p^{\frac{1}{\varphi}}\right)^{\frac{1}{\beta}} = 1 - \left(1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}\right)^{\alpha}, \qquad \Rightarrow 1 - \left(1 - p^{\frac{1}{\varphi}}\right)^{\frac{1}{\beta}} = \left(1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}\right)^{\alpha},$$

$$\Rightarrow \left[1-\left(1-p^{\frac{1}{\varphi}}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}} = 1-\left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}, \qquad \Rightarrow 1-\left[1-\left(1-p^{\frac{1}{\varphi}}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}} = \left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2},$$

$$\Rightarrow \left\{1-\left[1-\left(1-p^{\frac{1}{\varphi}}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}\right\}^{-\frac{2}{\lambda\delta}} = \frac{x+\delta}{\delta-x}. \text{ Let } \gamma(p) = \left\{1-\left[1-\left(1-p^{\frac{1}{\varphi}}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}\right\}^{-\frac{2}{\lambda\delta}}, \text{ then } \gamma(p) = \frac{x+\delta}{\delta-x}.$$

$$\Rightarrow \gamma(p)(\delta-x) = x+\delta, \Rightarrow \gamma(p)x + x = \delta\gamma(p) - \delta, \Rightarrow x[\gamma(p)+1] = \delta[\gamma(p)-1], \Rightarrow x = \delta\frac{\gamma(p)-1}{\gamma(p)+1}$$

$$\therefore Q_X(p) = \delta\frac{\gamma(p)-1}{\gamma(p)+1}, 0 < p < 1 \qquad (A6)$$

B. Prove of the unimodal nature and validity of the EKE distribution

(i) $f_X(x)$ is unimodal

Given $f_X(x) = \alpha\beta\lambda\varphi\frac{\delta^2}{\delta^2-x^2}(1-Z)Z^{\alpha-1}(1-Z^\alpha)^{\beta-1}\left[1-(1-Z^\alpha)^\beta\right]^{\varphi-1}$, with $Z = 1-\left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}$,

when $x = 0$, then

$$Z = 1-\left(\frac{0+\delta}{\delta-0}\right)^{-\lambda\delta/2} = 1-\left(\frac{\delta}{\delta}\right)^{-\lambda\delta/2} = 1-1^{-\lambda\delta/2} = 1-1 = 0$$

$$\Rightarrow f_X(0) = \alpha\beta\lambda\varphi\frac{\delta^2}{\delta^2-0^2}(1-0)0^{\alpha-1}(1-0^\alpha)^{\beta-1}\left[1-(1-0^\alpha)^\beta\right]^{\varphi-1}$$

$$= \alpha\beta\lambda\varphi\frac{\delta^2}{\delta^2}\times 1\times 0\times 1\times[1-1]^{\varphi-1} = 0;$$

also when $x = \delta$, then

$$Z = 1-\left(\frac{\delta+\delta}{\delta-\delta}\right)^{-\lambda\delta/2} = 1-\left(\frac{2\delta}{0}\right)^{-\lambda\delta/2} = 1-\infty^{-\lambda\delta/2} = 1-0 = 1$$

$$\Rightarrow f_X(\delta) = \alpha\beta\lambda\varphi\frac{\delta^2}{\delta^2-\delta^2}(1-1)1^{\alpha-1}(1-1^\alpha)^{\beta-1}\left[1-(1-1^\alpha)^\beta\right]^{\varphi-1}$$

$$= \alpha\beta\lambda\varphi\frac{\delta^2}{0}\times 0\times 1\times 0\times[1-0]^{\varphi-1} = \alpha\beta\lambda\varphi\times\infty\times 0\times 1\times 0\times[1-0]^{\varphi-1} = 0.$$

Therefore, since $f_X(x)$ is zero (0) at both end points, it implies the density function is unimodal. Some examples can be observed from Figure 1.

(ii) Validity of $f_X(x)$ as a true probability density function

It suffices to show that $\int_0^{\delta} f_X(x)\,dx = 1$. Let

$$M = \int_0^{\delta} f_X(x)\,dx = \lim_{x\to\delta} \int_0^{x} f_U(u)\,du = \lim_{x\to\delta} F_X(x)$$

$$= \lim_{x\to\delta}\left[1-(1-Z^{\alpha})^{\beta}\right]^{\varphi} \qquad (B1)$$

To evaluate the limit in (B1), if behoves us to evaluate $\lim_{x\to\delta} Z$. Thus

$$\lim_{x\to\delta} Z = \lim_{x\to\delta}\left(1-\left(\frac{x+\delta}{\delta-x}\right)^{-\lambda\delta/2}\right) = 1-\left(\frac{\delta+\delta}{\delta-\delta}\right)^{-\lambda\delta/2} = 1-\left(\frac{2\delta}{0}\right)^{-\lambda\delta/2} = 1-\infty^{-\lambda\delta/2} = 1-0 = 1$$

$$\therefore M = \lim_{x\to\delta}\left[1-(1-Z^{\alpha})^{\beta}\right]^{\varphi} = \left[1-(1-Z^{\alpha})^{\beta}\right]^{\varphi} = \left[1-(1-1^{\alpha})^{\beta}\right]^{\varphi} = \left[1-0^{\beta}\right]^{\varphi}$$

$$= 1^{\varphi} = 1 \qquad \blacksquare$$