INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals


INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS (IJDM)

On the Computation of First-Order Differential Systems using Refined Partitioning Algorithm

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ARTICLE INFO

Article history:
Received 03 January 2026
Received in revised form 10 March 2026
Accepted 20 March 2026

Keywords:
Algorithm, Adams method, block backward differentiation formula, systems of differential equations, partitioning technique, stiffness
MSC 2020 Subject classification:
65L05, 65L06

ABSTRACT

In this study, a Refined Partitioning Algorithm (RPA) is formulated for the computation of differential systems of first-order. The algorithm is composed of two sub-algorithms namely the Block Hybrid Adams Algorithm (BHAA) and Block Backward Differentiation Hybrid Algorithm (BBDHA) formulated using integration and differentiation techniques respectively. In computing the solutions of a first-order differential system, the RPA initially treats the systems as non-stiff and computes it solutions using BHAA. However, if as a result of stiffness, a failure step is encountered, then the RPA automatically switches to BBDHA in order to handle the stiffness of such systems. Analyses of basic properties of the sub-algorithms that made up the RPA were established. Furthermore, the results obtained showed that the RPA is more accurate and efficient than some existing methods.

1. Introduction

First-order differential systems, which may be stiff or non-stiff in nature, are applied in several fields of human endeavours. Of the two, the solutions of the non-stiff differential systems are easier to compute than stiff differential systems, which often exhibit varying timescales. The stiff systems possess components that are fast-changing requiring very small time-steps to be accurately captured, alongside slow-changing components that evolve more gradually. This disparity causes severe numerical instability when using traditional explicit methods to compute their solutions (Lambert, 1973). First-order differential systems arise naturally in numerous applications. In Physics, they describe mechanical systems, electrical circuits, fluid flow and quantum dynamics. In engineering, they are used in control theory, signal processing and robotics to model and regulate dynamic behaviours. In biology and medicine, they appear in population dynamics, epidemiological models and neural networks. Economics and social sciences employ these systems to study growth models, market dynamics and interacting agents (Sunday, Chigozie, Omole & Gwong, 2021).

In this study, a RPA is formulated for computing the solutions of first-order differential systems of the form,

$$y_j' = f_j\left(t,\overline{Y}\right), \quad j = 1,2,\dots,r \tag{1}$$

subject to the initial conditions $\overline{Y}(a) = \overline{\eta}$, where $\overline{Y}^T(t) = \left(y_1, y_2, \dots, y_r\right)$ and $\overline{\eta}^T(t) = \left(\eta_1, \eta_2, \dots, \eta_r\right)$. The RPA was derived using partitioning technique, where a differential system of the form (1) is initially treated as non-stiff and its solution computed using an Adams-type algorithm. If stiffness is encountered, then the RPA automatically computes the solution of the system using a backward differentiation algorithm. This approach has been proven to be effective as it addresses some of the setbacks associated with non-partitioning method, such as low rate of convergence, smaller stability regions and inefficiency in terms of computer time (Sunday *et al.,* 2025).

Over the years, scholars have proposed different partitioning techniques for solving problems of the form (1).

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https://doi.org/10.62054/ijdm/0301.02

Partitioning technique was first introduced by Enright & Kamel (1979) and latter improved by Watkins & Hausonsmith (1983). Other authors that developed partitioning techniques were Weiner, Arnold, Rentrop & Strehmel (1993) and Othman, Ibrahim, Suleiman & Majid (2007). Mahayadin, Othman & Ibrahim (2017) solved system of linear ordinary differential equations by formulating intervalwise block partitioning method. Sunday *et al.* (2025) proposed a two-step interval partitioning technique for the solution of systems of the form (1). In a bid to get a more robust and accurate algorithm, we were motivated to extend the work of Sunday *et al.* (2025) by developing a refined partitioning algorithm (within a three-step integration interval with one off-grid point) that will be more computationally reliable. For more details on other algorithms for the solution of stiff and non-stiff differential systems, see are Onyekonwu *et al.* (2024), Sunday, Chigozie, Omole & Gwong (2021), Nasarudin, Ibrahim & Rosali (2020) and Shateyi (2023).

The remaining parts of the paper are organised as follows; the RPA was formulated in Section 2, the basic properties of the RPA was presented in the third section. In Section 4, the strategy for the implementation of the RPA was presented while numerical results were presented in Section 5. The concluding remarks were highlighted in the sixth section.

2. Method

2.1 Formulation of the Refined Partitioning Algorithm

Consider the Linear Multistep Method (LMM) of step number k given by,

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j}, \tag{2}$$

where α_j and β_j are real coefficients and h is the step-size. The proposed RPA shall be executed in hybrid block form by formulating systems of LMMs of the form (2) at the discrete points t_{n+1}, t_{n+2}, $t_{n+5/2}$ and t_{n+3}. Let r be the step-size ratio and let the step-size between t_n and t_{n+1} be defined as rh. In a bid to formulate accurate and efficient algorithm, the step-size ratio of the two sub-algorithms is chosen at three different values; $r=1$ (maintaining the stepsize), $r=2$ (halving the stepsize) and $r=1/2$ (doubling the stepsize), see Suleiman, Ibrahim & Othman (2008), Anuar *et al.* (2011), Othman *et al.* (2013), Sunday, Shokri, Kwanamu & Nonlaopon (2022) and Sunday, Shokri & Marian (2022). The Lagrange interpolating polynomial,

$$\begin{aligned}P(t) = {} & \frac{(t-t_n)(t-t_{n+1})(t-t_{n+2})(t-t_{n+5/2})}{(t_{n+3}-t_n)(t_{n+3}-t_{n+1})(t_{n+3}-t_{n+2})(t_{n+3}-t_{n+5/2})}y_{n+3} + \frac{(t-t_n)(t-t_{n+1})(t-t_{n+2})(t-t_{n+3})}{(t_{n+5/2}-t_n)(t_{n+5/2}-t_{n+1})(t_{n+5/2}-t_{n+2})(t_{n+5/2}-t_{n+3})}y_{n+\frac{5}{2}} \\ & + \frac{(t-t_n)(t-t_{n+1})(t-t_{n+5/2})(t-t_{n+3})}{(t_{n+2}-t_n)(t_{n+2}-t_{n+1})(t_{n+2}-t_{n+5/2})(t_{n+2}-t_{n+3})}y_{n+2} + \frac{(t-t_n)(t-t_{n+2})(t-t_{n+5/2})(t-t_{n+3})}{(t_{n+1}-t_n)(t_{n+1}-t_{n+2})(t_{n+1}-t_{n+5/2})(t_{n+1}-t_{n+3})}y_{n+1} \\ & + \frac{(t-t_{n+1})(t-t_{n+2})(t-t_{n+5/2})(t-t_{n+3})}{(t_n-t_{n+1})(t_n-t_{n+2})(t_n-t_{n+5/2})(t_n-t_{n+3})}y_n.\end{aligned} \tag{3}$$

is adopted as basis function in approximating the function $f_i\left(t,\overline{Y}(t)\right)$ in the first-order differential system (1) at the points of interpolation (t_n, y_n), (t_{n+1}, y_{n+1}), (t_{n+2}, y_{n+2}), $(t_{n+5/2}, y_{n+5/2})$ and (t_{n+3}, y_{n+3}).

2.1.1 Formulation of the Block Hybrid Adams Algorithm (BHAA)

The BHAM is formulated using integration technique. The first-order differential system (1) is integrated over the interval $(t_n, t_{n+\upsilon})$, for $\upsilon = 1, 2, 5/2$ and 3. That is,

$$\int_{t_n}^{t_{n+\upsilon}} y_i'(t)dt = \int_{t_n}^{t_{n+\upsilon}} f_i\left(t,\overline{Y}(t)\right)dt. \tag{4}$$

The integral equation (4) is evaluated with respect to s where $s=(t-t_{n+3})/h$. The values (-3, -2), (-3, -1), (-3, -1/2) and (-3, 0) were meticulously chosen as the limits of integration and further substituting hds for dt produces the BHAA,

$$y_{n+1}=y_n+\left[\left(\frac{599h}{60r(2r+3)(r+1)(r+2)}\right)\right]f_n+\left[\left(\frac{h}{360}\right)\left(\frac{960r-599}{r}\right)\right]f_{n+1}-\left[\left(\frac{h}{60}\right)\left(\frac{350r-249}{r+1}\right)\right]f_{n+2}+\left[\left(\frac{4h}{45}\right)\left(\frac{135r-97}{2r+3}\right)\right]f_{n+\frac{5}{2}}-\left[\left(\frac{h}{120}\right)\left(\frac{220r-159}{r+2}\right)\right]f_{n+3}, \tag{5}$$

$$y_{n+2}=y_n+\left[\left(\frac{142h}{15r(2r+3)(r+1)(r+2)}\right)\right]f_n+\left[\left(\frac{h}{45}\right)\left(\frac{135r-71}{r}\right)\right]f_{n+1}-\left[\left(\frac{2h}{15}\right)\left(\frac{35r-36}{r+1}\right)\right]f_{n+2}+\left[\left(\frac{32h}{45}\right)\left(\frac{15r-13}{2r+3}\right)\right]f_{n+\frac{5}{2}}-\left[\left(\frac{h}{15}\right)\left(\frac{25r-21}{r+2}\right)\right]f_{n+3}, \tag{6}$$

$$y_{n+\frac{5}{2}}=y_n+\left[\left(\frac{1825h}{192r(2r+3)(r+1)(r+2)}\right)\right]f_n+\left[\left(\frac{25h}{1152}\right)\left(\frac{138r-73}{r}\right)\right]f_{n+1}-\left[\left(\frac{25h}{192}\right)\left(\frac{34r-39}{r+1}\right)\right]f_{n+2}+\left[\left(\frac{5h}{36}\right)\left(\frac{81r-61}{2r+3}\right)\right]f_{n+\frac{5}{2}}-\left[\left(\frac{25h}{384}\right)\left(\frac{26r-21}{r+2}\right)\right]f_{n+3}, \tag{7}$$

$$y_{n+3}=y_n+\left[\left(\frac{189h}{20r(2r+3)(r+1)(r+2)}\right)\right]f_n+\left[\left(\frac{3h}{40}\right)\left(\frac{40r-21}{r}\right)\right]f_{n+1}-\left[\left(\frac{9h}{20}\right)\left(\frac{10r-11}{r+1}\right)\right]f_{n+2}+\left[\left(\frac{12h}{5}\right)\left(\frac{5r-3}{2r+3}\right)\right]f_{n+\frac{5}{2}}-\left[\left(\frac{3h}{40}\right)\left(\frac{20r-23}{r+2}\right)\right]f_{n+3}, \tag{8}$$

Substituting the step-size ratio $r=1$, equations (5)-(8) reduce to,

$$\left.\begin{aligned}
y_{n+1}&=y_n+h\left(\frac{599}{1800}f_n+\frac{361}{360}f_{n+1}-\frac{101}{120}f_{n+2}+\frac{152}{225}f_{n+\frac{5}{2}}-\frac{61}{360}f_{n+3}\right),\\
y_{n+2}&=y_n+h\left(\frac{71}{225}f_n+\frac{64}{45}f_{n+1}+\frac{1}{15}f_{n+2}+\frac{64}{225}f_{n+\frac{5}{2}}-\frac{4}{45}f_{n+3}\right),\\
y_{n+\frac{5}{2}}&=y_n+h\left(\frac{365}{1152}f_n+\frac{1625}{1152}f_{n+1}+\frac{125}{384}f_{n+2}+\frac{5}{9}f_{n+\frac{5}{2}}-\frac{125}{1152}f_{n+3}\right),\\
y_{n+3}&=y_n+h\left(\frac{63}{200}f_n+\frac{57}{40}f_{n+1}+\frac{9}{40}f_{n+2}+\frac{24}{25}f_{n+\frac{5}{2}}+\frac{3}{40}f_{n+3}\right).
\end{aligned}\right\} \tag{9}$$

At $r=2$, we obtain,

$$\left.\begin{aligned}
y_{n+1}&=y_n+h\left(\frac{599}{10080}f_n+\frac{1321}{720}f_{n+1}-\frac{451}{180}f_{n+2}+\frac{692}{315}f_{n+\frac{5}{2}}-\frac{281}{480}f_{n+3}\right),\\
y_{n+2}&=y_n+h\left(\frac{71}{1260}f_n+\frac{199}{90}f_{n+1}-\frac{68}{45}f_{n+2}+\frac{544}{315}f_{n+\frac{5}{2}}-\frac{29}{60}f_{n+3}\right),\\
y_{n+\frac{5}{2}}&=y_n+h\left(\frac{1825}{32256}f_n+\frac{5075}{2304}f_{n+1}-\frac{725}{576}f_{n+2}+\frac{505}{252}f_{n+\frac{5}{2}}-\frac{775}{1536}f_{n+3}\right),\\
y_{n+3}&=y_n+h\left(\frac{9}{160}f_n+\frac{177}{80}f_{n+1}-\frac{27}{20}f_{n+2}+\frac{12}{5}f_{n+\frac{5}{2}}-\frac{51}{160}f_{n+3}\right).
\end{aligned}\right\} \tag{10}$$

While at $r=1/2$, equations (5)-(8) give,

$$\left.\begin{aligned}
y_{n+1} &= y_n + h\left(\frac{599}{450}f_n - \frac{119}{180}f_{n+1} + \frac{37}{45}f_{n+2} - \frac{59}{90}f_{n+\frac{5}{2}} + \frac{49}{300}f_{n+3}\right),\\
y_{n+2} &= y_n + h\left(\frac{284}{225}f_n - \frac{7}{45}f_{n+1} + \frac{74}{45}f_{n+2} + \frac{44}{45}f_{n+\frac{5}{2}} - \frac{17}{75}f_{n+3}\right),\\
y_{n+\frac{5}{2}} &= y_n + h\left(\frac{365}{288}f_n - \frac{25}{144}f_{n+1} + \frac{275}{144}f_{n+2} - \frac{205}{288}f_{n+\frac{5}{2}} + \frac{5}{24}f_{n+3}\right),\\
y_{n+3} &= y_n + h\left(\frac{63}{50}f_n - \frac{3}{20}f_{n+1} + \frac{9}{5}f_{n+2} - \frac{3}{10}f_{n+\frac{5}{2}} + \frac{39}{100}f_{n+3}\right).
\end{aligned}\right\} \tag{11}$$

Equations (9)-(11) collectively produce the BHAA. Generally, the BHAA is expressed as,

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+5/2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}+h\begin{bmatrix}\beta_1&\beta_2&\beta_3&\beta_4\\ \overrightarrow{\beta_1}&\overrightarrow{\beta_2}&\overrightarrow{\beta_3}&\overrightarrow{\beta_4}\\ \overline{\beta}_1&\overline{\beta}_2&\overline{\beta}_3&\overline{\beta}_4\\ \hat{\beta}_1&\hat{\beta}_2&\hat{\beta}_3&\hat{\beta}_4\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+5/2}\\f_{n+3}\end{bmatrix}. \tag{12}$$

Equation (12) is then expressed in matrix finite difference form as,

$$I\hat{Y}_m = A\hat{Y}_{m-1} + hB\hat{F}_m \tag{13}$$

$$I=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad \hat{Y}_m=\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+5/2}\\y_{n+2}\end{bmatrix},\quad \hat{Y}_{m-1}=\begin{bmatrix}y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\end{bmatrix},\quad A=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix},\quad B=\begin{bmatrix}\beta_1&\beta_2&\beta_3&\beta_4\\ \overrightarrow{\beta_1}&\overrightarrow{\beta_2}&\overrightarrow{\beta_3}&\overrightarrow{\beta_4}\\ \overline{\beta}_1&\overline{\beta}_2&\overline{\beta}_3&\overline{\beta}_4\\ \hat{\beta}_1&\hat{\beta}_2&\hat{\beta}_3&\hat{\beta}_4\end{bmatrix},$$

$$\hat{F}_m=\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+5/2}\\f_{n+2}\end{bmatrix}.$$

2.1.2 Formulation of the Block Backward Differentiation Hybrid Algorithm (BBDHA)

The BBDHA is formulated using differentiation technique. Let $s=(t-t_{n+3})/h$, so that $t=t_{n+3}+sh$ is substituted into the Lagrange polynomial (3),

$$\begin{aligned}
P(t)=P(t_{n+3}+sh)&=(s+1)(s+2)(2s+1)\left(\frac{1}{2}\right)\left(\frac{r+s+2}{r+2}\right)y_{n+3}-s(s+1)(s+2)\left(\frac{16}{3}\right)\left(\frac{r+s+2}{2r+3}\right)y_{n+\frac{5}{2}}+s(s+2)(2s+1)\left(\frac{r+s+2}{r+1}\right)y_{n+2}\\
&-s(s+1)(2s+1)\left(\frac{1}{6}\right)\left(\frac{r+s+2}{r}\right)y_{n+1}+s(s+1)(s+2)\left(\frac{2s+1}{r(r+1)(r+2)(2r+3)}\right)y_n.
\end{aligned} \tag{14}$$

Equation (14) is differentiated with respect to s to obtain

$$P'(t)=P'\left(t_{n+3}+sh\right)=hP'\left(t_{n+3}+sh\right)=\left(\frac{7r+42s+6rs^2+14rs+33s^2+8s^3+16}{2(r+2)}\right)y_{n+3}-\left(\frac{16\left(2r+16s+3rs^2+6rs+15s^2+4s^3+4\right)}{3(2r+3)}\right)y_{n+\frac{5}{2}}$$
$$+\left(\frac{2r+24s+6rs^2+10rs+27s^2+8s^3+4}{r+1}\right)y_{n+2}-\left(\frac{r+14s+6rs^2+6rs+21s^2+8s^3+2}{6r}\right)y_{n+1}+\left(\frac{\left(8s^3+12s^2+14s+2\right)}{r\left(2r^3+9r^2+13r+6\right)}\right)y_n. \tag{15}$$

The following equations are obtained on substituting $s=-2,-1,-1/2$ and 0 in equation (15),

$$\left.\begin{aligned}
hf_{n+1}&=\left(\frac{3r}{2(r+2)}\right)y_{n+3}-\left(\frac{32r}{3(2r+3)}\right)y_{n+\frac{5}{2}}+\left(\frac{6r}{r+1}\right)y_{n+2}-\left(\frac{13r-6}{6r}\right)y_{n+1}-\left(\frac{6}{r\left(2r^3+9r^2+13r+6\right)}\right)y_n,\\
hf_{n+2}&=-\left(\frac{r+1}{2(r+2)}\right)y_{n+3}+\left(\frac{16(r+1)}{3(2r+3)}\right)y_{n+\frac{5}{2}}-\left(\frac{2r+1}{r+1}\right)y_{n+2}-\left(\frac{r+1}{6r}\right)y_{n+1}+\left(\frac{1}{r\left(2r^3+9r^2+13r+6\right)}\right)y_n,\\
hf_{n+\frac{5}{2}}&=\left(\frac{\left(\frac{3}{2}r+\frac{9}{4}\right)}{2(r+2)}\right)y_{n+3}+\left(\frac{16\left(\frac{1}{4}r+\frac{3}{4}\right)}{3(2r+3)}\right)y_{n+\frac{5}{2}}-\left(\frac{\left(\frac{3}{2}r+\frac{9}{4}\right)}{r+1}\right)y_{n+2}-\left(\frac{\left(\frac{1}{2}r+\frac{3}{4}\right)}{6r}\right)y_{n+1}-\left(\frac{3}{4r\left(2r^3+9r^2+13r+6\right)}\right)y_n,\\
hf_{n+3}&=\left(\frac{7r+16}{2(r+2)}\right)y_{n+3}-\left(\frac{16(2r+4)}{3(2r+3)}\right)y_{n+\frac{5}{2}}+\left(\frac{2r+4}{r+1}\right)y_{n+2}-\left(\frac{r+2}{6r}\right)y_{n+1}+\left(\frac{2}{r\left(2r^3+9r^2+13r+6\right)}\right)y_n.
\end{aligned}\right\} \tag{16}$$

Equation (16) is solved for y_{n+1}, y_{n+2}, $y_{n+5/2}$ and y_{n+3} to obtain,

$$\left.\begin{aligned}
y_{n+1} &= \frac{\left(\left(\frac{3r}{2(r+2)}\right)y_{n+3}-\left(\frac{32r}{3(2r+3)}\right)y_{n+\frac{5}{2}}+\left(\frac{6r}{r+1}\right)y_{n+2}-\left(\frac{6}{r\left(2r^3+9r^2+13r+6\right)}\right)y_n-hf_{n+1}\right)}{\left(\frac{13r-6}{6r}\right)},\\
y_{n+2} &= \frac{\left(-\left(\frac{r+1}{2(r+2)}\right)y_{n+3}+\left(\frac{16(r+1)}{3(2r+3)}\right)y_{n+\frac{5}{2}}-\left(\frac{r+1}{6r}\right)y_{n+1}+\left(\frac{1}{r\left(2r^3+9r^2+13r+6\right)}\right)y_n-hf_{n+2}\right)}{\left(\frac{2r+1}{r+1}\right)},\\
y_{n+\frac{5}{2}} &= \frac{\left(\left(\frac{\left(\frac{3}{2}r+\frac{9}{4}\right)}{2(r+1)}\right)y_{n+3}-\left(\frac{\left(\frac{3}{2}r+\frac{9}{4}\right)}{r+1}\right)y_{n+2}+\left(\frac{\left(\frac{1}{2}r+\frac{3}{4}\right)}{6r}\right)y_{n+1}-\left(\frac{3}{4r\left(2r^3+9r^2+13r+6\right)}\right)y_n-hf_{n+\frac{5}{2}}\right)}{-\left(\frac{16}{3(2r+3)}\right)\left(\frac{1}{4}r+\frac{3}{4}\right)},\\
y_{n+3} &= \frac{\left(\left(\frac{-16(2r+4)}{3(2r+3)}\right)y_{n+\frac{5}{2}}+\left(\frac{2r+4}{r+1}\right)y_{n+2}-\left(\frac{r+2}{6r}\right)y_{n+1}+\left(\frac{2}{r\left(2r^3+9r^2+13r+6\right)}\right)y_n-hf_{n+3}\right)}{-\left(\frac{7r+16}{2(r+2)}\right)}.
\end{aligned}\right\} \tag{17}$$

Substituting $r=1$ in equation (17) gives,

$$\left.\begin{aligned}
y_{n+1} &= -\frac{6}{35}y_n+\frac{18}{7}y_{n+2}-\frac{64}{35}y_{n+\frac{5}{2}}+\frac{3}{7}y_{n+3}-\frac{6}{7}hf_{n+1},\\
y_{n+2} &= \frac{1}{45}y_n-\frac{2}{9}y_{n+1}+\frac{64}{45}y_{n+\frac{5}{2}}-\frac{2}{9}y_{n+3}-\frac{2}{3}hf_{n+2},\\
y_{n+\frac{5}{2}} &= \frac{3}{128}y_n-\frac{25}{128}y_{n+1}+\frac{225}{128}y_{n+2}-\frac{75}{128}y_{n+3}+\frac{15}{16}hf_{n+\frac{5}{2}},\\
y_{n+3} &= -\frac{2}{115}y_n+\frac{3}{23}y_{n+1}-\frac{18}{23}y_{n+2}+\frac{192}{115}y_{n+\frac{5}{2}}+\frac{6}{23}hf_{n+3}.
\end{aligned}\right\} \tag{18}$$

At $r=2$, equation (17) produces,

$$\left.\begin{aligned}
y_{n+1} &= -\frac{3}{140}y_n + \frac{12}{5}y_{n+2} - \frac{64}{35}y_{n+\frac{5}{2}} + \frac{9}{20}y_{n+3} - \frac{3}{5}hf_{n+1},\\
y_{n+2} &= \frac{1}{280}y_n - \frac{3}{20}y_{n+1} + \frac{48}{35}y_{n+\frac{5}{2}} - \frac{9}{40}y_{n+3} - \frac{3}{5}hf_{n+2},\\
y_{n+\frac{5}{2}} &= \frac{3}{640}y_n - \frac{49}{320}y_{n+1} + \frac{147}{80}y_{n+2} - \frac{441}{640}y_{n+3} + \frac{21}{20}hf_{n+\frac{5}{2}},\\
y_{n+3} &= -\frac{1}{315}y_n + \frac{4}{45}y_{n+1} - \frac{32}{45}y_{n+2} + \frac{512}{315}y_{n+\frac{5}{2}} + \frac{4}{15}hf_{n+3}.
\end{aligned}\right\} \tag{19}$$

On the other hand, at $r = 1/2$ equation (17) produces,

$$\left.\begin{aligned}
y_{n+1} &= -\frac{24}{5}y_n + 12y_{n+2} - 8y_{n+\frac{5}{2}} + \frac{9}{5}y_{n+3} - 6hf_{n+1},\\
y_{n+2} &= \frac{1}{10}y_n - \frac{3}{8}y_{n+1} + \frac{3}{2}y_{n+\frac{5}{2}} - \frac{9}{40}y_{n+3} - \frac{3}{4}hf_{n+2},\\
y_{n+\frac{5}{2}} &= \frac{3}{35}y_n - \frac{2}{7}y_{n+1} + \frac{12}{7}y_{n+2} - \frac{18}{35}y_{n+3} + \frac{6}{7}hf_{n+\frac{5}{2}},\\
y_{n+3} &= -\frac{8}{117}y_n + \frac{25}{117}y_{n+1} - \frac{100}{117}y_{n+2} + \frac{200}{117}y_{n+\frac{5}{2}} + \frac{10}{39}hf_{n+3}.
\end{aligned}\right\} \tag{20}$$

Thus, the BBDHA is given by equations (18)-(20). This method can be expressed as,

$$\left.\begin{aligned}
y_{n+1} &= \theta_1 y_{n+2} + \varphi_1 y_{n+\frac{5}{2}} + \tau_1 y_{n+3} + \alpha_1 hf_{n+1} + \omega_1,\\
y_{n+2} &= \theta_2 y_{n+1} + \varphi_2 y_{n+\frac{5}{2}} + \tau_2 y_{n+3} + \alpha_2 hf_{n+2} + \omega_2,\\
y_{n+\frac{5}{2}} &= \theta_3 y_{n+1} + \varphi_3 y_{n+2} + \tau_3 y_{n+3} + \alpha_3 hf_{n+\frac{5}{2}} + \omega_{\frac{5}{2}},\\
y_{n+3} &= \theta_4 y_{n+1} + \varphi_4 y_{n+2} + \tau_4 y_{n+\frac{5}{2}} + \alpha_4 hf_{n+3} + \omega_3.
\end{aligned}\right\} \tag{21}$$

where ω_1, ω_2, $\omega_{5/2}$ and ω_3 are previous of back values. Equation (21) is expressed in matrix-vector form as,

$$\left(I - A\right)Y_m = hBF_m + \sigma, \tag{22}$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \theta_1 & \varphi_1 & \tau_1 \\ \theta_2 & 0 & \varphi_2 & \tau_2 \\ \theta_3 & \varphi_3 & 0 & \tau_3 \\ \theta_4 & \varphi_4 & \tau_4 & 0 \end{bmatrix}, \quad Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+5/2} \\ y_{n+3} \end{bmatrix}, \quad B = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{bmatrix}, \quad F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+5/2} \\ f_{n+3} \end{bmatrix},$$

$$\sigma = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_{5/2} \\ \omega_3 \end{bmatrix}.$$

The BHAA in equations (9)-(11) and BBDHA in equations (18)-(20) collectively form the RPA.

3 Analyses of Basic Properties of the RPA

Basic properties of the two algorithms that made up the RPA were analysed in this section. The summary of results of the analyses is presented herein.

Definition 3.1: The LMM (2) and its associated linear difference operator L defined by,

$$L\{y(t);h\} = \sum_{j=0}^{k}\left[\alpha_j y(t+jh) - h\beta_j y'(t+jh)\right] \tag{23}$$

are of order p if $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \ldots = \bar{c}_p = 0,\ \bar{c}_{p+1} \neq 0$, (Fatunla, 1980). The parameter $\bar{c}_{p+1} \neq 0$ is called the error constant. The parameters c_p are defined as

$$\left.\begin{aligned} c_0 &= \sum_{j=0}^{k}\alpha_j \\ c_1 &= \sum_{j=0}^{k}\left(j\alpha_j - \beta_j\right) \\ &\ . \\ &\ . \\ &\ . \\ c_p &= \sum_{j=0}^{k}\left[\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!} j^{p-1}\beta_j\right], p = 2,3,\ldots \end{aligned}\right\} \tag{24}$$

For the BHAA, at $r = 1$, the coefficients in equation (24) are obtained as

$$\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \ldots = \bar{c}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{25}$$

with the corresponding error constant

$$\bar{c}_6 = \begin{bmatrix} 1.0833\times10^{-2} & 7.7778\times10^{-3} & 8.1380\times10^{-3} & 7.5000\times10^{-3} \end{bmatrix}^T. \tag{26}$$

While for the BBDHA, we obtain

$$\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{27}$$

at $r = 1$ with the error constant

$$\bar{c}_5 = \begin{bmatrix} 2.1429\times10^{-2} & 5.5556\times10^{-3} & 7.3242\times10^{-3} & -6.5217\times10^{-3} \end{bmatrix}^T. \tag{28}$$

These results imply that the BHAA is of order five while the BBDHA is of order four order, and by implication, the two algorithms are said to be consistent since their orders are greater than or equal to one.

For the BHAA, the stability polynomials at $r = 1$, 2 and 1/2 are given respectively as,

$$R_1(t,H)=t^4\left(\frac{1}{8}H^4-\frac{67}{120}H^3+\frac{13}{10}H^2-\frac{17}{10}H+1\right)-t^3\left(\frac{1}{40}H^4+\frac{23}{120}H^3+\frac{7}{10}H^2+\frac{13}{10}H+1\right) \tag{29}$$

$$R_2(t,H)=t^4\left(\frac{199}{672}H^4-\frac{20233}{20160}H^3+\frac{12517}{6720}H^2-\frac{20249}{10080}H+1\right)-t^3\left(\frac{1}{224}H^4+\frac{221}{2880}H^3+\frac{879}{2240}H^2+\frac{9991}{10080}H+1\right) \tag{30}$$

$$R_{1/2}(t,H)=-t^4\left(\frac{47}{480}H^4-\frac{3809}{14400}H^3+\frac{29}{240}H^2+\frac{4763}{7200}H-1\right)-t^3\left(\frac{1}{10}H^4+\frac{4267}{7200}H^3+\frac{847}{480}H^2+\frac{16837}{7200}H+1\right) \tag{31}$$

On the substitution of $H=0$ in equations (29)-(31), we obtain the following equations

$$R_1(t,0)=R_2(t,0)=R_{1/2}(t,0)=t^4-t^3. \tag{32}$$

Solving the equations in (32) gives $0,0$, 0 and 1. Thus, the BHAA is clearly zero-stable. On the other hand, the stability polynomial of the BBDHA at $r=1$, 2 and 1/2 are given respectively by

$$R_1(t,H)=t^4\left(\frac{45}{322}H^4-\frac{201}{644}H^3+\frac{78}{161}H^2-\frac{153}{322}H+\frac{36}{161}\right)-t^3\left(\frac{3}{322}H^3+\frac{3}{46}H^2+\frac{9}{46}H+\frac{36}{161}\right) \tag{33}$$

$$R_2(t,H)=t^4\left(\frac{63}{625}H^4-\frac{69}{500}H^3+\frac{69}{500}H^2-\frac{9}{100}H+\frac{18}{625}\right)-t^3\left(\frac{3}{2500}H^3+\frac{21}{2500}H^2+\frac{63}{2500}H+\frac{18}{625}\right) \tag{34}$$

$$R_{1/2}(t,H)=t^4\left(\frac{90}{91}H^4-\frac{321}{91}H^3+\frac{708}{91}H^2-\frac{72}{7}H+\frac{576}{91}\right)-t^3\left(\frac{24}{91}H^3+\frac{24}{13}H^2+\frac{72}{13}H+\frac{576}{91}\right) \tag{35}$$

Substituting $H=0$ into equations (33)-(35), we obtain

$$R_1(t,0)=\frac{36}{161}t^4-\frac{36}{161}t^3, \tag{36}$$

$$R_2(t,0)=\frac{18}{625}t^4-\frac{18}{625}t^3, \tag{37}$$

$$R_{1/2}(t,0)=\frac{576}{91}t^4-\frac{576}{91}t^3, \tag{38}$$

Solving equations (36)-(38) gives $0,0$, 0 and 1, which implies that the BBDHA is zero-stable, (Lambert, 1973).

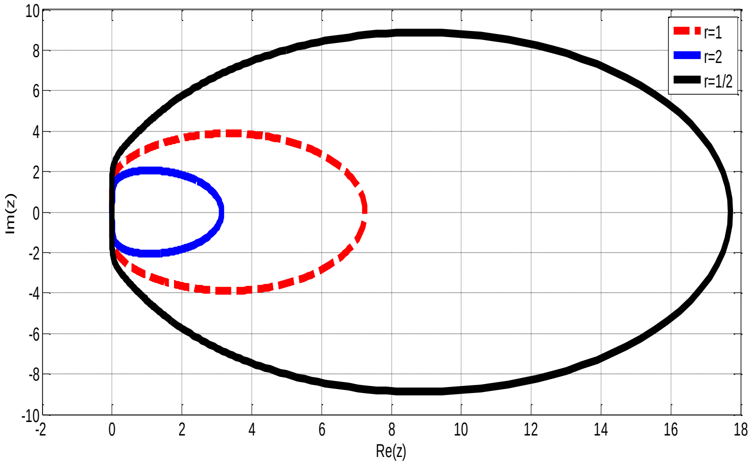
The basic properties of the RPA consisting of the BHAA and BBDHA are summarised in Tables 1 and 2. In the same vein, the regions of absolute stability of the RPA were plotted in Figure 1.

Table 1. Summary of basic properties of BHAA and BBDHA

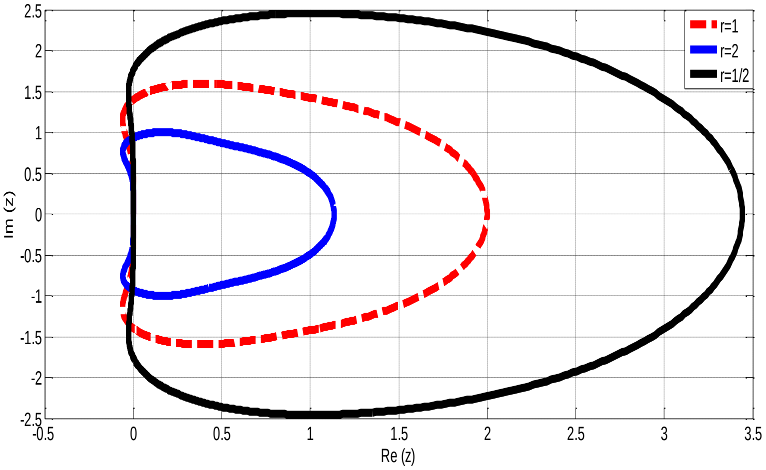
Method	**Order**	**Zero-stability**	**Consistency**	**Convergence**
BHAA	5	Zero-stable	Consistent	Convergent
BBDHA	4	Zero-stable	Consistent	Convergent

Table 2. Roots of stability polynomials and instability intervals of BHAA and BBDHA

Method	**Step-size Ratio**	**Roots of Stability Polynomial**	**Instability Intervals**
BHAA	$r = 1$	$0, 0, 0, 1$	(0,7.2434)
	$r = 2$	$0, 0, 0, 1$	(0,3.0095)
	$r = 1/2$	$0, 0, 0, 1$	(0,17.7164)
BBDHA	$r = 1$	$0, 0, 0, 1$	(0,2.0000)
	$r = 2$	$0, 0, 0, 1$	(0,1.1352)
	$r = 1/2$	$0, 0, 0, 1$	(0,3.4410)



(a) Regions of absolute stability of the BHAA



(b) Regions of absolute stability of the BBDHA

Figure 1. Regions of absolute stability of the RPA

Figure 1 shows the regions of absolute stability of the RPA, with those of the BHAA shown in Figure 1(a) and those of BBDHA shown in Figure 1(b). For the BHAA, the regions of absolute stability, which are the exterior of the contour, are clearly A-stable for all the step-size ratios while those of the BBDHA are $A(\alpha)$-stable. Note that for both algorithms, the largest region of absolute stability was obtained at $r = 2$ (halved step-size), followed by $r = 1$

(maintained step-size) and then $r = 1/2$ (doubled step-size) has the smallest region of absolute stability. The intervals of instability of the algorithms were also presented in Table 2.

4. Strategy for Implementation of the RPA

The strategy for implementing the RPA involves treating the first-order differential system (1) as non-stiff and computing its solution using the BHAA in equations (9)-(11). If a failure step is encountered as a result of the stiffness, then a stronger test is carried out. This test involves computing the trace of the Jacobian, that is $\partial f / \partial y$. The system is treated as stiff (if the trace has negative value) and its solution computed with the aid of BBDHA in equations (18)-(20). However, the computation is continued using the BHAA (with halved step-size) if the trace is positive. According to Suleiman, Ibrahim & Othman (2008), the algorithm of the error control is executed as follows:

i. if the error control is less or equal to the tolerance level, then the step-size is doubled in order improve computational speed (efficiency),
ii. if the step fails, then the step-size is halved (reduced) and the step is repeated.

See the works of Mahayadin, Othman & Ibrahim (2017), Onyekonwu et al. (2024) and Sunday et al. (2025) for more information on strategies for the implementation of the RPA.

5. Numerical Results

The RPA is employed in computing the solutions of some first-order differential systems of the form (1) in order to test its accuracy and efficiency. The following notations and abbreviations are used in the result presentations.

h_{init} : Initial step-size
TLev: Tolerance level
NS: Number of steps
NAS: Number of accepted steps
NFS: Number of failure steps
MErr: Maximum error
TExec: Time of execution (seconds)
ode15s: Inbuilt Matlab stiff solver
IBP: Intervalwise block partitioning method by Mahayadin, Othman & Ibrahim (2017)
IPT: Intervalwise partitioning technique by Sunday *et al.* (2025)
RPA: Newly formulated refined partitioning algorithm

Problem 5.1

Consider the first-order differential system

$$\left.\begin{aligned} y_1' &= -21y_1 + 19y_2 - 20y_3, y_1(0) = 1, \\ y_2' &= 19y_1 - 21y_2 + 20y_3, y_2(0) = 0, \\ y_3' &= 40y_1 - 40y_2 - 40y_3, y_3(0) = -1, \end{aligned}\right\} \tag{39}$$

defined over $[0,10]$. The exact solution of the system is,

$$\left.\begin{aligned} y_1(t) &= (1/2)\left[e^{-2t} + e^{-40t}\left(\sin 40t + \cos 40t\right)\right], \\ y_2(t) &= (1/2)\left[e^{-2t} - e^{-40t}\left(\sin 40t + \cos 40t\right)\right], \\ y_3(t) &= (1/2)\left[2e^{-40t}\left(\sin 40t - \cos 40t\right)\right]. \end{aligned}\right\} \tag{40}$$

The Eigen values of the Jacobian of the system are $\lambda_1 = -2$, $\lambda_2 = -40 + 40i$ and $\lambda_3 = -40 - 40i$. On applying the RPA on Problem 5.1, we obtain the numerical and graphical results in Table 3 and Figure 2.

Table 3. Results for Problem 5.1

TLev	Method	NS	MErr	TExec
10^{-2}	ode15s	37	$7.92983e-03$	0.009659
	IBP	26	$2.15015e-01$	0.000224
	RPA	20	$3.11030e-07$	0.000123
10^{-4}	ode15s	86	$1.16049e-04$	0.018109
	IBP	36	$8.24059e-03$	0.000344
	RPA	28	$1.04527e-09$	0.000197
10^{-6}	ode15s	162	$1.77877e-06$	0.028360
	IBP	58	$4.71831e-05$	0.000919
	RPA	40	$1.33176e-11$	0.000560
10^{-8}	ode15s	305	$3.83010e-08$	0.073445
	IBP	139	$1.45542e-09$	0.001936
	RPA	108	$6.88342e-13$	0.001056

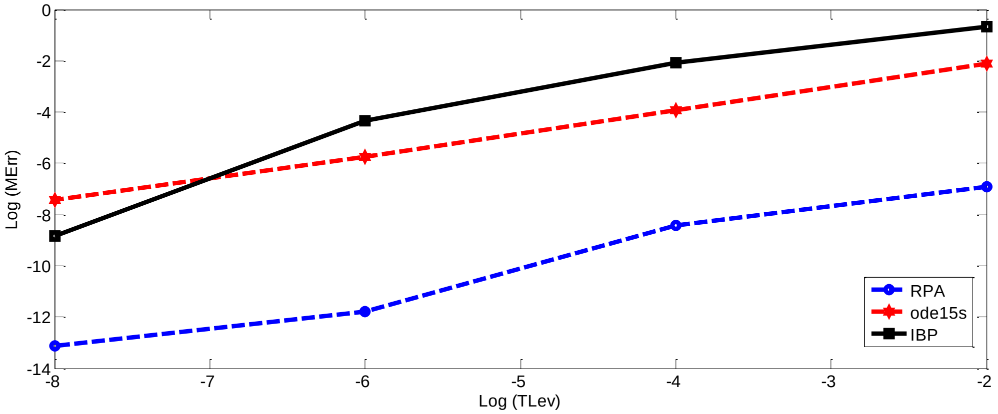


Figure 2. Accuracy curves for Problem 5.1

Problem 5.2

Consider the first-order differential system,

$$\left.\begin{aligned} y_1' &= -2y_1 + y_2 + 2\sin t, y_1(0) = 2, \\ y_2' &= 998y_1 - 999y_2 + 999(\cos t - \sin t),\ y_2(0) = 3, \end{aligned}\right\} \tag{41}$$

defined over $[0,5]$, with the exact solution,

$$\left.\begin{aligned} y_1(t) &= \sin t + 2e^{-t}, \\ y_2(t) &= \cos t + 2e^{-t}. \end{aligned}\right\} \tag{42}$$

On applying the RPA on Problem 5.2, we obtain the numerical and graphical results Table 4 and Figure 3.

Table 4. Results for Problem 5.2 at initial step-size $h_{init} = 10^{-1}$

TLev	Method	NS	NAS	NFS	MErr	TExec
10^{-3}	ode15s	29	28	1	$2.40e-03$	0.7938
	IPT	21	19	2	$1.2314e-09$	0.0021
	RPA	24	22	2	$3.5621e-11$	0.0025
10^{-4}	ode15s	38	38	0	$2.41e-04$	2.5669
	IPT	27	25	2	$7.0025e-10$	0.0033
	RPA	32	28	4	$1.9982e-12$	0.0038
10^{-5}	ode15s	55	54	1	$5.30e-05$	5.4463
	IPT	45	41	4	$4.5119e-12$	0.0078
	RPA	48	44	4	$2.6416e-13$	0.0084

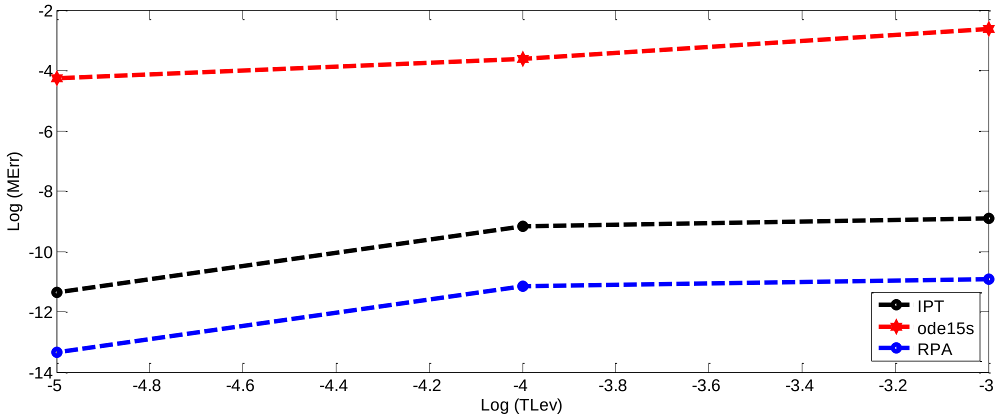


Figure 3. Accuracy curves for Problem 5.2

The RPA was tested two first-order differential systems and the numerical and graphical results obtained clearly showed that the algorithm is more accurate and efficient than the methods we compared our results with. In Table 3, for instance, it was observed that the RPA was more efficient and accurate than both ode15s and IBP developed by Mahayadin, Othman & Ibrahim (2017). This explains why the RPA had smaller maximum errors compared to those of the two methods. The RPA was also more efficient than the ode15s and IBP because its execution time was faster (or smaller) than the other two methods. The RPA was further proven to be more computationally reliable than the other two methods in view of the accuracy curves obtained in Figure 2. It was also observed in Table 3 that the RPA took fewer numbers of steps to achieve accuracy in comparison to ode15s. Similarly the numerical and graphical results obtained in Table 4 and Figure 3 respectively showed that the RPA is accurate and efficient than the ode15s and IPT developed by Sunday *et al.* (2025).

6. Concluding Remarks

In this research, a new algorithm called the RPA was formulated for computing the solutions of first-order differential systems. The algorithm composed of two sub-algorithms called the BHAA and BBDHA were formulated using integration and differentiation techniques respectively. Basic properties of the RPA were analysed and the results obtained showed that the algorithm is zero-stable, consistent and convergent. Numerical and graphical results obtained on the application of the RPA also showed that the algorithm is computationally reliable in computing the solutions to first-order differential systems. Going forward, the performance of the RPA may be explored on other forms of differential systems that exhibit complex dynamics like chaos, perturbation, etc. as discussed in Genesio & Tesi (1992), Ibrahim & Nasarudin (2020) and Sunday *et al.* (2024).

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