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An Effective Numerical Approach for Solving Second-Kind Fredholm Integral Equations

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ABSTRACT

In this paper, a numerical approach to solving second-kind Fredholm integral equations using shifted Chebyshev polynomials is presented. In order to approximate the solution and convert the integral equation into a system of algebraic equations, we use shifted Chebyshev polynomials as basis functions. The shifted Chebyshev polynomials provide superior approximation qualities that increase accuracy and convergence rates. Numerical experiments show that the suggested approach is successful in solving several classes of Fredholm integral equations, and its stability and efficiency are examined. The outcomes demonstrate the method's advantages over conventional numerical techniques.

1. Introduction

Integral equations are those in which an unknown function appears with an integral sign. It is applicable in a variety of domains, including engineering and science. There are various approaches for classifying integral equations. Some common classifications are linear and nonlinear, homogeneous and non-homogeneous, Fredholm and Volterra, first, second, and third order, and singular and regular integral equations. These discrepancies are usually based on some fundamental property, such as whether the equation is linear or homogeneous. Many mathematical models of physical phenomena incorporate integral equations, necessitating the use of numerical methods to solve them (Zarnan, 2018). Integral equations are typically difficult to solve analytically. As a result, obtaining an approximate answer is necessary. In recent years, numerical approaches for solving various forms of Fredholm integral equations have been intensively researched in several works, including Legendre multi-Galerkin methods for solving the Fredholm integral equation with a weakly singular kernel (Panigrahi, 2019). Bernstein method (Adhraa and Ayal, 2019), Adomian decompositions method (Khan and Bakoda, 2013), Collocation method by (Ajileye and Amoo, 2023; Agbolade and Anake, 2017), Hybrid linear multistep method (Mehdiyera, Ibrahim and Imanova, 2019), Chebyshev-Galerkin method (Issa and Saleh, 2017), Lagrange Interpolation (Shoukralla and Ahmed, 2020), Least-Squares Method (Al-Humedi and Shoushan, 2021), Chebyshev polynomials (Maadadi & Rahmoune, 2018), Optimal Auxiliary Function Method (OAFM) (Zada, Al-Hamami, Nawaz, Jehanzeb, Morsy, Abdei-Aty and Nisar, 2021) and many more.
This paper considered Fredholm integral equation of the second kind of the form

$$y(x) + \gamma \int_a^b G(x,t)\, y(t)dt = f(x), \quad a \le x \le b \tag{1}$$

where $G(x,t)$ is the Fredholm integral kernel, γ is the parameter given, $f(x)$ is a known function, and $y(x)$ is the unknown function to be determined.

2. Preliminaries

This section contains definitions and foundational concepts for the purpose of problem formulation.

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Definition 1 :(Agbolade & Anake, 2017) The desired collocation points within an interval are determined using this method. i.e., [a, b], and is provided by
$l_u = a + \frac{(b-a)u}{M}, u = 0, 1, 2 \dots \dots \text{M}, a \leq x \leq b.$

Definition 2: Shifted Chebyshev Polynomials (Adebisi *et al*., 2021**)**
This particular kind of polynomial is produced from the Chebyshev polynomial. Chebyshev has an interval of $[-1,1]$, but the shifted Chebyshev has an interval of $[0,1]$.
The shifted Chebyshev is generated using;
$T_n * (x) = T_n(2x - 1), n \geq 0$
Where $T_n^*(x)$ is the shifted Chebyshev term, while $T_n(x)$ is the Chebyshev term.
Few terms of the shifted Chebyshev terms are;
$T_0 * (x) = 1$
$T_1 * (x) = 2x - 1$
$T_2 * (x) = 8x^2 - 8x + 1$
$T_3 * (x) = 32x^3 - 48x^2 + 18x - 1$
$T_4 * (x) = 128x^4 - 256x^3 + 160x^2 + 32x + 1$

3. Materials and Methods

The approximation method for the numerical solution of Volterra integral equations is implemented in this section.

3.1 Method of Solution

Let the solution of equation (1) be approximated by

$$y(x) = \sum_{i=0}^{M} b_i L_i(x) = L(x)\boldsymbol{B} \quad (2)$$

where
$L_i(x) = \sum_{i=0}^{N} \left((-1)^i 2^{2k-2i} \frac{\Gamma(2k-i+2)}{\Gamma(i+1)\Gamma(2k-2i+2)} x^{k-i}\right), k = 0, \dots N$ and
$\boldsymbol{B} = [b_0 \ b_1 \cdots b_M]^T$

Substituting equation (2) into equation (1) gives

$$L(x)\boldsymbol{B} + \gamma \int_a^b G(x,t)L(t)\boldsymbol{B}dt = f(x) \quad (3)$$

Collocating equation (3) at $x_i$using standard collocation points equation
$x_i = \text{a} + \frac{(\text{b}-\text{a})i}{N}, \text{i} = 0, 1, 2, \dots \text{N}$

$$L(x_i)\boldsymbol{B} + \gamma \int_a^b G(x_i,t)L(t)\boldsymbol{B}dt = f(x_i) \quad (4)$$

Factorizing the value of **B** from equation (4) gives

$$\left[L(x_i) + \gamma \int_a^b G(x_i,t)L(t)dt\right]\boldsymbol{B} = f(x_i) \quad (5)$$

Equation (5) can be in the form

$$\beta(x_i)\boldsymbol{B} = f(x_i) \quad (6)$$

where

$$\beta(x_i) = L(x_i) + \gamma \int_a^b G(x_i,t)L(t)dt$$

$$\beta(x_i)=\begin{bmatrix}\beta_0(x_0) & \beta_1(x_0) & \beta_2(x_0) & \cdots & \beta_N(x_0)\\ \beta_0(x_1) & \beta_1(x_1) & \beta_2(x_1) & \cdots & \beta_N(x_1)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \beta_0(x_N) & \beta_1(x_N) & \beta_2(x_N) & \cdots & \beta_N(x_N)\end{bmatrix}, f(x_i)=\begin{bmatrix}f(x_0)\\ f(x_1)\\ \vdots\\ f(x_N)\end{bmatrix}$$

To find the unknown coefficient's value, we solved the system of equations (6) with the shifted Chebyshev polynomial as the base polynomial function and obtained the numerical result.

4. Numerical Examples

This section includes numerical examples to help evaluate the method's accuracy and usefulness. Let $y_n(x)$ and $y(x)$ denote the approximate and exact solutions, respectively. $Error_N = |y_n(x) - y(x)|$

Example 1: Consider the linear Fredholm integral equation (Djaidja *et al.*, 2024)

$$y(x) - \int_0^1 2e^{(x+t)}\, y(t)dt = e^x \quad (7)$$

Exact solution: $y(x) = \frac{e^x}{2-e^2}$.

Solution 1

The approximate solution of equation (7) at $N = 5$ gives

$$y_5(x) = 0.2570945043e - 2x^5 - 0.6469512545e - 2x^4 - 0.3162176587e - 1x^3 - 0.9260767882e - 1x^2 - 0.1855765216x - 0.1855611901$$

Table 1: Exact, approximate and absolute error values for Example 1

X	Exact	Our Method$_{N=5}$	Error$_5$	Error Djaidja *et al.*.N=10
0.2	-0.2266450257	-0.2266449496	7.61e-8	2.235202e-4
0.4	- 0.2768248596	- 0.2768247664	9.32e-8	2.730082e-4
0.6	- 0.3381146469	- 0.3381145344	125e-7	3.33453e-4
0.8	- 0.4129741623	- 0.4129740256	1.367e-7	4.072804e-4
1.0	- 0.5044077809	- 0.5044076140	1.669e-7	4.974534e-4

Example 2: Consider the linear Fredholm integral equation (Djaidja *et al.*, 2024)

$$y(x) - \int_{-1}^1 (xt + x^2t^2)\, y(t)dt = 1 \quad (8)$$

Exact solution: $y(x) = 1 + \frac{10}{9}x^2$

Solution 2

The approximate solution of equation (8) at N=5 gives

$y_5 = 9.094947018 \times 10^{-12}x^5 + 5.625366838 \times 10^{-10}x^4 - 1.640927394 \times 10^{-10}x^3 + 1.111111111x^2 - 4.185451985 \times 10^{-10}x + 1.000000000.$

Table 2: Exact, approximate and absolute error values for Example 2

x	Exact	Our Method$_{N=5}$	Error$_5$	Error Djaidja *et al.*.N=10
0.2	1.044444444	1.044444444	0.00	3.303384e-04
0.4	1.177777778	1.177777778	0.00	1.321354e-03
0.6	1.400000000	1.400000000	0.00	2.973046e-3
0.8	1.711111111	1.711111111	0.00	5.285414e-03
1.0	2.111111111	2.111111111	0.00	8.258460e-03

Example 3: Consider the linear Fredholm integral equation (Djaidja *et al*., 2024)

$$y(x) - \int_0^{\frac{\pi}{2}} \frac{4}{\pi} Cos(x-t)\, y(t)dt = -\frac{2}{\pi} Cos(x) \tag{9}$$

Exact solution: $y(x) = \text{Sin}(x)$

Solution 3

The approximate solution of equation (8) at N=5 gives

$$y_5 = 0.7255930396e - 2x^5 + 0.1572602669e - 2x^4 - 0.1676095739x^3 + 0.6745746086e - 3x^2 + 0.9999526268x - 0.8627603112e - 3$$

Table 3: Exact, approximate and absolute error values for Example 3

X	Exact	Our Method$_{N=5}$	Error$_5$	Error Djaidja *et al*..N=10
0.157	0.1563558123	0.1554994462	8.563661e-4	3.526219e-03
0.471	0.4537776271	0.4529971173	7.805098e-4	3.830231e-03
0.785	0.7068251811	0.7061969460	6.282351e-4	3.759312e-03
1.100	0.8912073601	0.8908012171	4.061430e-4	3.320406e-03
1.414	0.9877326198	0.9878636385	1.310187e-4	2.556476e-03

5. Results and Discussion

This section discusses the numerical findings obtained from the solved examples using the proposed numerical method.

According to Table 1, the approximate result for Example 1 when $N = 5$ yields $y_5 = 0.2570945043e - 2x^5 - 0.6469512545e - 2x^4 - 0.3162176587e - 1x^3 - 0.9260767882e - 1x^2 - 0.1855765216x - 0.1855611901$. The numerical results yielded an accurate solution, demonstrating that our method at N=5 beat the method proposed by Djaidja *et al*. (2024) at $N = 10$.

Table 2 shows the approximate solution for numerical Example 2 at $N = 5$: $y_5 = 9.094947018 \times 10^{-12}x^5 + 5.625366838 \times 10^{-10}x^4 - 1.640927394 \times 10^{-10}x^3 + 1.111111111x^2 - 4.185451985 \times 10^{-10}x + 1.000000000$. The numerical results converge to the exact solution, which gives a better result compared to the result obtained by Djaidja *et al*. (2024) at $N = 10$.

The approximate solution at $N = 5$ in numerical Example 3, as displayed in Table 3, yields $y_5 = 0.725593096e - 2x^5 + 0.157260266e - 2x^4 - 0.167609573x^3 + 0.67457461e - 3x^2 + 0.9999526268x - 0.8627603112e - 3$. In comparison to the result achieved by Djaidja *et al*. (2024) at $N = 10$, the numerical results obtained yield a better result at $N = 5$.

6. Conclusions

This paper investigates the numerical solution of Fredholm integral equations with the shifted Chebyshev as a basis function. This approach is simple to compute, reliable, and efficient. All of the computations in this work are done with Maple 18.

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