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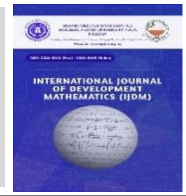

A Numerical Approach for Solving Second-Kind Linear and Nonlinear Volterra Integral Equations

Ojo O. Aduroja[a] and Ganiyu Ajileye[b*]

[a]Department of Mathematics, University of Ilesa, Ilesa, Osun State, Nigeria.
[b]Department of Mathematics, Federal University Wukari, Taraba State, Nigeria.

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ABSTRACT

This paper introduces a numerical method for solving second-kind linear and nonlinear Volterra integral equations using the collocation method. Standard collocation points are used to solve the modeled problem after it is converted into an algebraic system of equations. Numerical examples were utilized to evaluate the effectiveness of the method. The outcomes show that this approach performs better than others.

1. Introduction

Integral equations appear in a variety of scientific and engineering problems. Volterra or Fredholm integral equations can be used in a wide range of starting and boundary value problems. The potential theory made the most significant contribution to the creation of integral equations. The creation of integral equations was further aided by mathematical physics models of diffraction problems, astrophysics, quantum mechanics scattering, conformal mapping, and water waves (Abu-Ghuwaleh *et al*., 2022). Furthermore, the study of nonlinear Integro-Differential Equations (IDEs) has appeared in many domains of science, due to the large number of applications that may describe, such as chemical kinetics, queuing theory, and others (Mirzaee & Samadyar, 2018).
Numerous efforts have been made to construct and examine numerical techniques for solving Volterra integral equations of the second kind, including: Adomian decomposition method (Khan and Bakoda, 2013; Adomian, 2013), divided differences interpolation (Parandin and Gholamtabor, 2010), Bernstein method (Adhraa and Ayal, 2019), Collocation method (Ajileye *et al*., 2024; Agbolade and Anake, 2017), Hybrid linear multistep method (Mehdiyera *et al*., 2019), Chebyshev-Galerkin method (Issa and Saleh, 2017), Lagrange Interpolation (Shoukralla and Ahmed, 2020), Least-Squares Method (Al-Humedi and Shoushan, 2021;), Sinc Collocation Method (John *et al*., 2024), Chebyshev polynomials (Maadadi & Rahmoune, 2018), Optimal Auxiliary Function Method (OAFM) (Zada, Al-Hamami, Nawaz, Jehanzeb, Morsy, Abdei-Aty and Nisar, 2021), Modified Simpson's Rule (Djaidja and Khirani, 2024) and many other methods.
In this work, a novel technique for approximating the numerical solution of a second kind of linear and nonlinear Volterra integral equation of the form

$$y(x) = F(x) + \delta \int_0^x J(x,t)\,(y(t))^i dt \qquad 0 \le x \le 1, i \ge 1 \tag{1}$$

where $J(x,t)$ represents the Volterra integral kernel, δ is the supplied parameter, $F(x)$ is a known function, and $y(x)$ is the unknown function to be determined.

* Corresponding author. Tel.: +2348034906427
E-mail address: ajileye@fuwukari.edu.ng (Ajileye, G.)
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2. Preliminaries

In order to formulate the presented problem, we supply certain definitions and basic principles of integrals in this section.

Definition 1: (Ajileye *et al.*, 2024) Let (a_m) be a sequence of real integers and $m < 0$. The power series in t with coefficients a_m is an equation.

$u(x) = \sum_{m=0}^{N} a_m t^m = \theta(t)\boldsymbol{B}$ (2)

where $\theta(t) = [1 \ t \ t^2 \ \cdots \ t^N]$, $\boldsymbol{B} = [a_0 \ a_1 \cdots a_N]^T$

then $u(t,m) = t^m B, i = 0(1)N, m \in Z^+$

Definition 2: (Agbolade and Anake, 2017) Standard collocation point defined within the interval $[c,d]$ is given by

$x_i = c + \frac{(d-c)i}{M}, i = 0,1,2 \dots\dots M$ (3)

3. Methods and Materials

This section deals with the numerical solution of second-kind linear and nonlinear Volterra integral equations using an approximation approach.

3.1 Method of Solution

Let the solution of equation (1) be approximated by

$y(x) = \sum_{n=0}^{N} a_n x^n$ (4)

Substituting equation (4) into equation (1) gives

$\sum_{n=0}^{N} a_n x^n = F(x) + \delta \int_0^x J(x,t) \left(\sum_{n=0}^{N} a_n t^n\right)^i dt$ (5)

$\sum_{n=0}^{N} a_n \left(x^n - \delta \int_0^x J(x,t) (t^n)^i dt\right) = F(x)$ (6)

Equation (6) can be rewrite in the form

$\sum_{n=0}^{M} a_n \varphi(x) = F(x)$ (7)

where

$$\varphi(x) = x^n - \delta \int_0^x J(x,t)\,(t^n)^i dt$$

Collocate equation (7) at x_i using standard collocation points.

$x_i = a + \frac{(b-a)i}{M}, i = 0,1,2 \dots\dots M$

$\sum_{n=0}^{M} a_n \varphi(x_i) = F(x_i)$ (8)

where.

$$\varphi(x_i) = \begin{bmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \varphi_2(x_0) & \cdots & \varphi_N(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_N(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_0(x_N) & \varphi_1(x_N) & \varphi_2(x_N) & \cdots & \varphi_N(x_N) \end{bmatrix}, F(x_i) = \begin{bmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_N) \end{bmatrix}$$

To obtain the numerical result, we solve the system of equations (8) for the unknown variables and then incorporate the results into the approximate solution.

4. Numerical Examples

In this section, numerical examples are used to show the accuracy and applicability of the method. Let $y_n(x)$ and $y(x)$ represent the approximate and exact solutions, respectively. $Error_N = |y_n(x) - y(x)|$

Example 1: Consider the linear Volterra integral equation

$$y(x) = \int_0^x Sin(x-t)\, y(t)dt + x \quad (9)$$

Exact solution: $y(x) = x + \frac{x^3}{6}$.

Solution 1

The approximate solution to equation (9) at $N = 5$ yields

$$y_5 = 2.131628207 \times 10^{-14} + 0.999999999334250x + 8.265487850 \times 10^{-9}x^2 + 0.166666640376206x^3 + 3.096647561 \times 10^{-8}x^4 - 1.196167432 \times 10^{-8}x^5$$

Table 1: Exact, approximate and absolute error values for example 1

X	Exact	Our Method$_{N=5}$	Error$_5$	Error John *et al.* N=10
0.2	0.2013333333	0.2013333333	0.00	1.9857e-6
0.4	0.4106666667	0.4106666667	0.00	2.6239e-8
0.6	0.6360000000	0.6360000000	0.00	1.9848e-9
0.8	0.8853333333	0.8853333333	0.00	2.1996e-10
1.0	1.166666667	1.166666667	0.00	3.1784e-11

The result obtained for Example 1, as shown in Table 1 revealed that our method converges to exact solution at $N = 5$ than the method proposed in the literature at $N = 10$.

Example 2: Consider the linear Volterra integral equation

$$y(x) = Cos(x) - e^x Sin(x) + \int_0^x e^x\, y(t)dt \quad (10)$$

Exact solution: $y(x) = Cox(x)$

Solution 2

The approximate solution to equation (10) at $N = 5$ yields

$$y_5 = 1.00000000000002 - 0.531148325535469e - 4x - 0.499414632795379x^2 - 0.237877556355670e - 2x^3 + 0.461547014419921e - 1x^4 - 0.400828532292508e - 2x^5$$

Table 2: Exact, approximate and absolute error values for example 2

x	Exact	Our Method$_{N=5}$	Error$_8$	Error Shoukralla *et al.*=5
0.2	0.9800665778	0.9800663263	2.515e-07	3.336e-06
0.4	0.9210609940	0.9210606868	3.072e-07	4.968e-05
0.6	0.8253356149	0.8253350128	6.021e-07	5.405 e-04
0.8	0.6967067093	0.6967057408	1.85e-07	3.038 e-03
1.0	2.7182818280	2.7182820920	9.685e-07	1.173 e-02

In numerical Example 2, as shown in Table 2, the approximate solution at $N = 5$ yields a better result compared to the result obtained in the literature at $N = 5$.

Example 3: Consider the nonlinear Volterra integral equation

$$y(x) = 2x - \frac{x^4}{12} + \frac{1}{4}\int_0^x (x-t)\, y(t)^2 dt \quad (11)$$

Exact solution: $y(x) = 2x$.

Solution 3

The approximate solution to equation (11) at $N = 8$ yields

$$y_8 = 1.591615728 \times 10^{-12} + 2.00000132620335x - 0.272989273071289e - 4x^2 + 0.243186950683594e - 3x^3 - 0.427284240722656e - 1x^4 + 0.272750854492188e - 2x^5 - 0.407409667968750e - 2x^6 + 0.347900390625000e - 2x^7 - 0.137710571289062e - 2x^8$$

Table 3: Exact, approximate and absolute error values for example 3

x	Exact	Our Method$_{N=8}$	Error$_8$	Error $_{Saadeh.=20}$
0.2	0.4000000000	0.3999334063	6.65937e-05	1.332317e-4
0.4	0.8000000000	0.7989339186	1.06608e-03	2.1203322e-3
0.6	1.2000000000	1.194602162	5.39783e-03	1.05779450e-2
0.8	1.6000000000	1.582930850	1.70692e-03	3.28034650e-2
1.0	2.0000000000	1.958244100	4.17559e-02	1.166473750e-1

The approximate result in Example 3, as shown in Table 3, validated the reliability of our method of solution by producing a better result at $N = 8$ than the result found in the literature at $N = 20$.

Example 4: Consider the nonlinear Volterra integral equation

$$y(x) = x + \int_0^x (x - t)\, y(t)^2 dt \qquad (12)$$

Exact solution: $y(x) = tan(x)$

Solution 4

The approximate solution to equation (12) at $N = 20$ yields

$$y_{20} = 2.910383046 \times 10^{-10} + 1.00000194087625x - 0.352859497070312e - 4x^2 + 0.334045410156250x^3 - 0.585937500000000e - 2x^4 + 0.136718750000000x^5 + 0.937500000000000e - 1x^6 + 0.937500000000000x^8 - 1.56250000000000x^9 + 0.250000000000000x^{10} + 0.906250000000000x^{11} - 1.93750000000000x^{12} - 0.812500000000000x^{13} + 0.531250000000000x^{14} - 1.37500000000000x^{15} - 0.500000000000000x^{16} + 0.875000000000000x^{17} + 1.81250000000000x^{18} + 0.406250000000000x^{19} + 0.156250000000000x^{20}$$

Table 4: Exact, approximate and absolute error values for example 4

x	Exact	Our Method$_{N=20}$	Error$_{20}$	Error $_{Saadeh.=20}$
0.2	0.2027100355	0.2027133505	3.3150e-06	8.37612e-5
0.4	0.4227932187	0.4232391262	4.45908e-04	1.835594e-4
0.6	0.6841368083	0.6859003538	1.76355e-03	9.62880e-5
0.8	1.029638557	0.9333584287	2.96386e-03	3.3003770e-3
1.0	1.557407725	1.246121440	3.11286e-02	3.539545198e-2

The approximate result in Example 4, as shown in Table 4, revealed that our method of solution produced almost the same result found in the literature at the same value of $N = 20$.

5. Conclusions

This research examined the collocation method for solving linear and nonlinear Volterra integral equations using the power series collocation method. This approach is reliable, efficient, and simple to compute. All computations in this study were done with Maple 18. Some numerical examples were considered so as to show the effectiveness and reliability of the method.

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