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Bi-Basis Function Second Derivative Block Method for the Solution of Optimal Control Problems

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ABSTRACT

This paper determines the state trajectory of a dynamic system over a period to optimize a given performance index. A one-step second derivative hybrid method based on the combination of Lucas and first Boubaker polynomials as basis functions is formulated for the solution of optimal control problems via the forward-backward sweep method. The technique is used to generate a set of hybrid schemes at selected grid and off-grid points with implementation in block form. The stability and convergence of the new method is discussed and the accuracy of the method is tested on some numerical examples. The results obtained correspond to a large extent with the results obtained by the exact solution.

1. Introduction

Mathematical models are usually found in the field of natural sciences, medical sciences, biological sciences, and engineering which result to ordinary differential equations. Sometimes, solutions to these ordinary differential equations cannot be obtained analytically, hence the need for approximate solution by the application of numerical methods (Adamu, 2023).

Optimal Control (OC) is the process of determining control and state trajectories for a dynamic system, over a period of time, in order to optimize a given performance index (Rodrigues *et at.*, 2014). The goal of this theory is to select a stream of values over time for the control variable that will optimize the functional subject to constraint set on the state variable (Adamu, 2023). Optimal control problem may be to find the way to drive the car so as to minimize its fuel consumption, given that it must complete a given course in a time not exceeding some amount. Or it may be to minimize the total monetary cost of completing the trip, assuming monetary prices for time and fuel (Garrett, 2015).

Forward Backward Sweep (FBS) is an iterative method named base on how the algorithm solve the problem state and adjoint. Given an approximation of the control function, FBS solve the state `forward' in time, then solve the control `backward' in time. Finding the optimal control by solving the necessary conditions is known as the indirect approach (Garrett, 2015; Adamu *et al.*, 2023).

Pontrygin's Principle is a powerful method for the computation of optimal controls, which has the crucial advantage that it does not require prior evaluation of the infimal cost function. In the 1950′s, a Russian mathematician by the name of Lev Pontryagin and his co-workers in Moscow derived necessary conditions that is needed to solve the basic optimal problems. Pontryagin introduced the adjoint function to affix to the differential equation to the objective functional (Garrett, 2015).

This paper considers the numerical solution to Optimal Control Problems (OCPs) of the form

$$J\left[x(\cdot),u(\cdot)\right]=\int_{t_0}^{t_f} f\left(t,x(t),u(t)\right)dt, \tag{1}$$

subject to the dynamic system

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$$x' = g(t,\ x(t),\ u(t)), \tag{2}$$

$$x(t_0) = x_0, \tag{3}$$

where $x(t)$ and $u(t)$ are real valued functions, $f(t,x,u)$ and $g(t,x,u)$ are continuously differentiable functions. The control(s) is piecewise continuous, the associated state is piecewise differentiable.

Forward-backward sweep method is extensively discussed by (Lenhart and Workman, 2007) for the solution of optimal control problem, using classical order Runge-Kutta via Pontryagin's principle. They are able to establish the necessary and sufficient conditions for the solution of optimal control problem as follows:

Considering Equation (1) and Equation (2), the Hamiltonian of the problem is

$$H = f\left(t, x(t), u(t)\right) + \lambda g(t, x(t), u(t)), \tag{4}$$

the necessary and sufficient conditions are:

$$x'(t) = \frac{\partial H}{\partial \lambda}, \quad x(t_0) = x_0$$

$$\frac{\partial H}{\partial u} = 0, \quad \text{(optimality condition)}$$

$$\lambda'(t) = -\frac{\partial H}{\partial x}, \ \text{(adjoint condition)}$$

$$\lambda(t_f) = 0, \quad \text{(Transversality condition)}$$

and subsequently applied the forward-backward sweep on classical Runge-Kutta method given as:

$$x_{n+1} = x_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \tag{5}$$

where

$$k_1 = f(t_n, x_n), \ k_2 = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_1\right),$$

$$k_3 = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_2\right), \ k_4 = f(t_n + h, x_n + hk_3).$$

Forward Backward Sweep (FBS) is adopted by (Garrett, 2015) to solve OCP using classical Runge-Kutta method via pontrygin's principle. Hence classical Runge-Kutta method became the most popular choice method for solving OCP using FBS.

Block methods are formulated in terms of Linear Multistep Method (LMM), which generates simultaneous solutions at all grid points, it preserves the traditional advantage of one step methods, of being self-starting and permitting easy change of step length (Alkali *et al.*, 2023). Akinnukawe and Odekunle (2023) presents the derivation of block scheme based on the combination of Hermite and shifted Chebyshev polynomials as basis functions of the collocation technique, for the solution of linear and non-linear initial value problems of ordinary differential equations.

2. Method

2.1 Development of the block method

We approximate the exact solution $y(x)$ of a differential equation $y'(x) = f(x, y(x))$ on the partition $\pi[a,b] = [a = x_0 < x_1 < \ldots < x_n < x_{n+1} < \ldots < x_N = b]$ of the integration interval $[a,b]$ using first Boubaker polynomials of the form

$$y(x) = \sum_{n=0}^{3} \alpha_n B_n(x) + \sum_{n=0}^{3} \alpha_n L_n(x) \tag{6}$$

where $x \in [a,b]$, $a_n \in \mathrm{R}$ are unknown parameters to be determined. To generate the collocation points, closed Newton Cote's collocation point is considered as

$$x_i = a + hi, \quad i = 0,1,2,\ldots,N \tag{7}$$

where $h=\dfrac{(b-a)}{N}$.

first Boubaker polynomials terms are

$$B_0(x)=1, B_1(x)=x, B_2(x)=x^2+2,$$
$$B_{n+1}(x)=xB_n(x)-B_{n-1}(x),\ \text{for } n\geq 2$$

Lucas polynomial terms are

$$l_0(x)=2, l_1(x)=x, l_2(x)=x^2+2,$$
$$L_{n+1}(x)=xL_{n-1}(x)+L_n(x),\ \text{for } n\geq 1$$

Equation (6) yield

$$y(x)=11a_0+8xa_1+12x^2a_2+6x^3a_3+6x^4a_4+2x^5a_5+2x^6a_6$$
$$y'(x)=8a_1+24a_2x+18a_3x^2+24a_4x^3+10a_5x^4+12a_6x^5$$
$$y''(x)=24a_2+36a_3x+72a_4x^2+40a_5x^3+60a_6x^4$$

Interpolating and collocating (6) using the points

$$y(x_{n+j})=y_{n+j},\quad j=0;\ y'(x_{n+j})=f_{n+j},\quad j=0,\frac{1}{5},\frac{3}{5},1;\ y''(x_{n+j})=g_{n+j},\quad j=\frac{3}{5},1$$

gives a system of equations

$$XA=U \tag{8}$$

$$X=\begin{bmatrix} 11 & 8x_n & 12x_n^2 & 6x_n^3 & 6x_n^4 & 2x_n^5 & 2x_n^6 \\ 0 & 8 & 24x_n & 18x_n^2 & 24x_n^3 & 10x_n^4 & 12x_n^5 \\ 0 & 8 & 24x_{n+\frac{1}{5}} & 18x_{n+\frac{1}{5}}^2 & 24x_{n+\frac{1}{5}}^3 & 10x_{n+\frac{1}{5}}^4 & 12x_{n+\frac{1}{5}}^5 \\ 0 & 8 & 24x_{n+\frac{3}{5}} & 18x_{n+\frac{3}{5}}^2 & 24x_{n+\frac{3}{5}}^3 & 10x_{n+\frac{3}{5}}^4 & 12x_{n+\frac{3}{5}}^5 \\ 0 & 8 & 24x_{n+1} & 18x_{n+1}^2 & 24x_{n+1}^3 & 10x_{n+1}^4 & 12x_{n+1}^5 \\ 0 & 0 & 24 & 36x_{n+\frac{3}{5}} & 72x_{n+\frac{3}{5}}^2 & 40x_{n+\frac{3}{5}}^3 & 60x_{n+\frac{3}{5}}^4 \\ 0 & 0 & 24 & 36x_{n+1} & 72x_{n+1}^2 & 40x_{n+1}^3 & 60x_{n+1}^4 \end{bmatrix},$$

$$A=[a_0,a_1,a_2,a_3,a_4,a_5,a_6]^T,\ U=\left[y_n,\ f_n,\ f_{n+\frac{1}{5}},\ f_{n+\frac{3}{5}},\ f_{n+1},\ g_{n+\frac{3}{5}},\ g_{n+1}\right]^T$$

Solving Equation (8) using Crammer's rule for the unknown parameters and substituting the result into Equation (6) gives the continuous linear multistep method

$$y_{n+t}=\alpha_0(t)y_n+\beta_0(t)f_n+\beta_{\frac{1}{5}}(t)f_{n+\frac{1}{5}}+\beta_{\frac{3}{5}}(t)f_{n+\frac{3}{5}}+\beta_1(t)f_{n+1}+\gamma_{\frac{3}{5}}(t)g_{n+\frac{3}{5}}+\gamma_1(t)g_{n+1} \tag{9}$$

where

$$\alpha_0(t)=y_n$$
$$\beta_0(t)=-\tfrac{1}{54}htf_n\left(125t^5-510t^4+825t^3-668t^2+279t-54\right)$$
$$\beta_{\frac{1}{5}}(t)=\tfrac{1}{3456}f_{n+\frac{1}{5}}\left(28125ht^6-108\,000ht^5+158\,625ht^4-108\,000ht^3+30\,375ht^2\right)$$
$$\beta_{\frac{3}{5}}(t)=\tfrac{1}{3456}f_{n+\frac{3}{5}}\left(12\,500ht^6-24\,000ht^5+1500ht^4+16\,000ht^3-4500ht^2\right)$$
$$\beta_1(t)=-\tfrac{1}{3456}f_{n+1}\left(32\,625ht^6-99\,360ht^5+107\,325ht^4-49\,248ht^3+8019ht^2\right)$$
$$\gamma_{\frac{3}{5}}(t)=\tfrac{25}{144}h^2t^2g_{n+\frac{3}{5}}(25t-9)(t-1)^3$$
$$\gamma_1(t)=\tfrac{1}{96}h^2t^2g_{n+1}\left(125t^4-360t^3+375t^2-168t+27\right)$$

Evaluating Equation (9) at to get the discrete linear multistep methods

$$y_{n+\frac{1}{5}} = y_n + \frac{h}{2160\,000}\begin{pmatrix} 152\,768 f_n + 357\,525 f_{n+\frac{1}{5}} - 35\,300 f_{n+\frac{3}{5}} \\ -42\,993 f_{n+1} + 30\,720 hg_{n+\frac{3}{5}} + 5148 hg_{n+1} \end{pmatrix} \tag{10}$$

$$y_{n+\frac{3}{5}} = y_n - \frac{h}{80\,000}\begin{pmatrix} -4672 f_n - 24\,975 f_{n+\frac{1}{5}} - 17\,300 f_{n+\frac{3}{5}} \\ -1053 f_{n+1} + 1920 hg_{n+\frac{3}{5}} + 108 hg_{n+1} \end{pmatrix} \tag{11}$$

$$y_{n+1} = y_n - \frac{h}{1152}\left(-64 f_n - 375 f_{n+\frac{1}{5}} - 500 f_{n+\frac{3}{5}} - 213 f_{n+1} + 12 hg_{n+1}\right) \tag{12}$$

Writing Equations (10), (11) and (12) in block to get

$$A^{(1)}Y_{m+1} = A^{(0)}Y_m + hB^{(0)}F_m + hB^{(1)}F_{m+1} + h^2\gamma^{(1)}G_{m+1} \tag{13}$$

where

$$Y_{m+1} = \left[y_{n+\frac{1}{5}} \;\; y_{n+\frac{3}{5}} \;\; y_{n+1}\right]^T, Y_m = \left[y_{n-1} \;\; y_{n-2} \;\; y_n\right]^T,$$

$$F_m = \left[f_{n-1} \;\; f_{n-2} \;\; f_n\right]^T,\; F_{m+1} = \left[f_{n+\frac{1}{5}} \;\; f_{n+\frac{3}{5}} \;\; f_{n+1}\right]^T \; G_{m+1} = \left[g_{n-1} \;\; g_{n+\frac{3}{5}} \;\; g_{n+1}\right]^T,$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\; A^{(0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} 0 & 0 & \frac{2387}{33750} \\ 0 & 0 & \frac{73}{1250} \\ 0 & 0 & \frac{1}{18} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} \frac{1589}{9600} & -\frac{353}{21600} & -\frac{4777}{240000} \\ \frac{999}{3200} & \frac{173}{800} & \frac{1053}{80000} \\ \frac{125}{384} & \frac{125}{288} & \frac{71}{384} \end{bmatrix} \text{ and } \gamma^{(1)} = \begin{bmatrix} 0 & \frac{16}{1125} & \frac{143}{60000} \\ 0 & -\frac{3}{125} & -\frac{27}{20000} \\ 0 & 0 & -\frac{1}{96} \end{bmatrix}$$

2.2 Stability Analysis

2.2.1 Order of the Bi-Basis Function Block Method (BBFBM)

Definition 1: *Order and Error constant* (Adamu *et al.*, 2019)

The operator L is associated with the linear method defined by

$$L[y(x):\; h] = y_{n+t} - \sum_{j=0}^{k}\alpha_j(t)y_{n+j} - \sum_{j=0}^{k}\gamma_j(t)f_{n+j} \tag{14}$$

where $y(x)$ is an arbitrary function, continuously differentiable on interval of integration $[a,b]$. The LMM can be written in Taylor expansion about the point x to obtain

$$L[y(x):\; h] = c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) + \ldots + c_p h^p y^p(x) \tag{15}$$

where

$$c_p = \frac{1}{p!}\left[\sum_{j=1}^{r} j^p\theta_j - \frac{1}{(p-1)!}\sum_{j=1}^{r} j^{p-1}\gamma_j - \frac{1}{(q-2)!}\sum_{j=1}^{r} j^{q-2}g_j\right] \tag{16}$$

The method is of order p if

$$L[y(x):\; h] = 0\left(h^{p+2}\right),\;\; c_0 = c_1 = \ldots = c_p = c_{p+1} = 0,\;\; c_{p+2} \neq 0. \tag{17}$$

Hence c_{p+2} is called the error constant and $c_{p+2}h^{p+2}y^{(p+2)}(x)$ is called the local truncation error (LTE).

Considering Definition 1, and evaluating equations (10), (11) and (12) in a Taylor series about x_n gives

$$L[y(x); h] = y_{n+t} - \alpha_0(t)y_n - \beta_0(t)f_n - \beta_{\frac{1}{5}}(t)f_{n+\frac{1}{5}} - \beta_{\frac{3}{5}}(t)f_{n+\frac{3}{5}} - \beta_1(t)f_{n+1} - \gamma_{\frac{3}{5}}(t)g_{n+\frac{3}{5}} - \gamma_1(t)g_{n+1} = 0$$

$$h^{p+n} \neq 0,\;\; p+n = 7$$

where n is the order of the differential equation. Therefore, the order of the block method is $p = [6,6,6]^T$ with error constants

$$\text{Error Constant} = \left[-\frac{911}{2362500000}h^7, \frac{3}{87500000}h^7, \frac{1}{3780000}h^7 \right]^T.$$

2.2.2 Zero Stability of the Bi-Basis Function Block Method (BBFBM)

The block method (13) is zero stable since the roots $z_s, s = 1,2,3,...n$ of the first characteristics polynomial $\rho(z)$ is defined by

$$\rho(\lambda) = \det\left[A^{(1)}\lambda - A^{(0)}\right] = 0$$

$$\lambda^3 - \lambda^2 = \lambda^2(\lambda - 1) = 0$$

Solving for λ we have $\lambda = [0,0,1]$. Hence the block method (13) is zero stable.

2.2.3 Consistency of the BBFBM

A block method is said to be consistent if it has order $p \geq 1$ (Adamu *et al.*, 2019). Therefore, since the block method has order $p = 6 > 1$, therefore, the method is consistent.

2.2.4 Convergence of the BBFBM

The block method (3.4) is said to be convergent if and only if the method is consistent and satisfies the root condition (Jain *et al.*, 2012). Therefore, the block method is convergent since it is consistent and zero-stable.

2.2.5 Region of absolute stability of BBFBM

The region of absolute stability of a LMM is the set of all points λh in the complex plane for which the method is absolutely stable (Endre and David, 2003).

The region of absolute stability of the block method is shown in Figure 1.

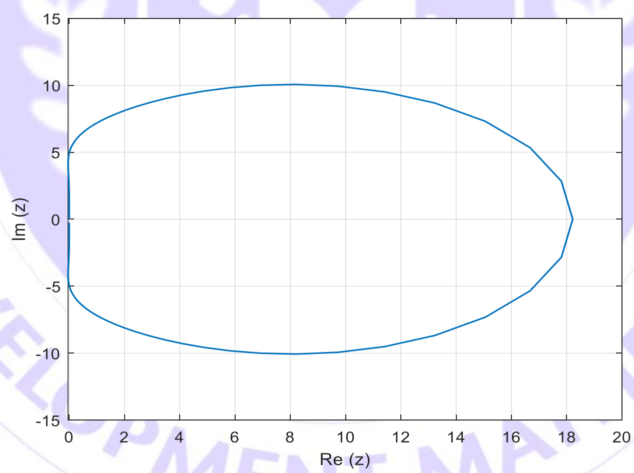

Figure 1: Region of Absolute Stability of the method

3. Results

Example 1: *Consider the optimal control problem*

$$\min_u \frac{1}{2}\int_0^1 x(t)^2 + u(t)^2\,dt$$

$$\text{subject to } \begin{cases} x'(t) = -x(t) + u(t) \\ x(0) = 1, \quad x(1) \text{ free} \end{cases}$$

Source: (Garrett, 2015)

Solution: The Hamiltonian is constructed as

$$H = \frac{1}{2}x^2(t) + \frac{1}{2}u^2(t) + \lambda\left(-x(t) + u(t)\right)$$

The optimality and adjoint conditions are:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u = -\lambda; \quad \lambda' = -\frac{\partial H}{\partial x} = \lambda - x.$$

This problem is solved using the optimality system with $N = 100$ and the results are shown in Figures.

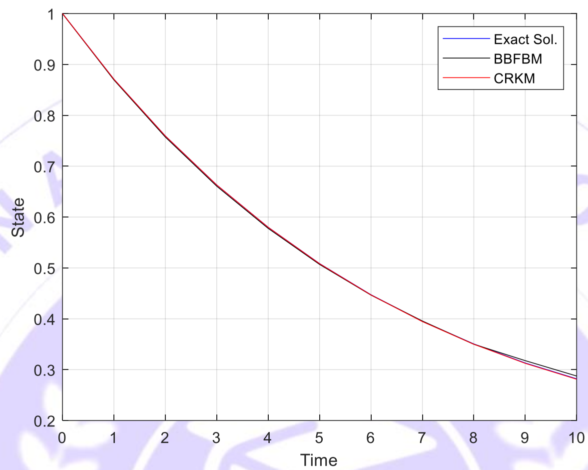

Figure 2: State Result for Example 1

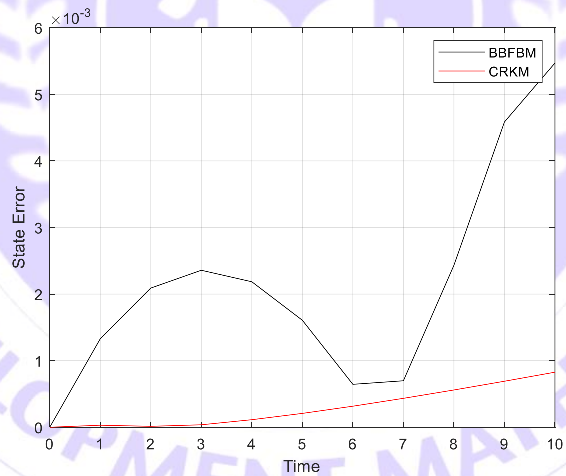

Figure 3: State Error for Example 1

The results show that BBFBM matches the result of Classical Runge Kutta method used by (Garrett, 2015) but with better accuracy.

Example 2: *Consider the optimal control problem*

$$\min_u \int_0^1 u(t)^2 \, dt$$

$$\text{subject to} \begin{cases} x'(t) = x(t) + u(t), \\ x(0) = 1, \; x(1) \text{ free} \end{cases}$$

Source: (Lenhart and Workman, 2007).

Solution: The Hamiltonian is constructed as

$$H = u^2(t) + \lambda\big(x(t) + u(t)\big)$$

The optimality and adjoint conditions are:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u = -\frac{1}{2}\lambda;\ \lambda' = -\frac{\partial H}{\partial x} = -\lambda$$

This problem is solved using the optimality system with $N = 100$ and the results are shown in Figures.

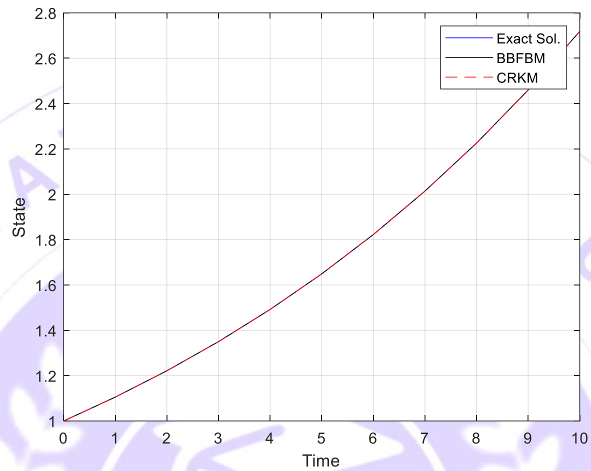

Figure 4: State Result for Example 2

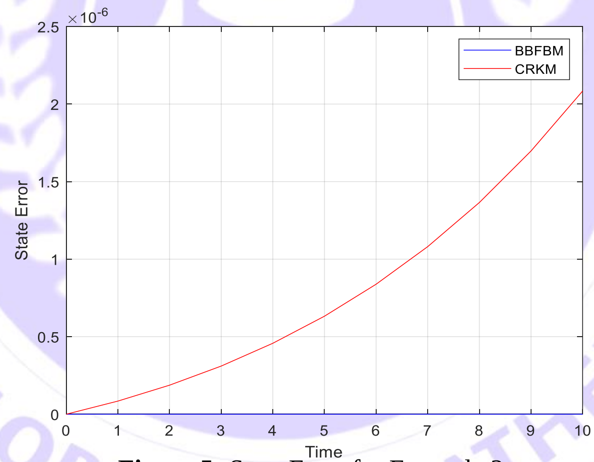

Figure 5: State Error for Example 2

The results show that BBFBM matches the result of Classical Runge Kutta method used by (Lenhart and Workman, 2007).

Example 3: *Consider the optimal control problem*

$$\min\ J = \int_0^1 \big[x(t) + u(t)\big]dt$$

$$\text{subject to} \begin{cases} x'(t) = 1 - u(t), \\ x(0) = 1 \end{cases}$$

Source: (Lenhart and Workman, 2007)

Solution: The Hamiltonian is constructed as:

$$H = x(t) + u(t) + \lambda(1 - u(t))$$

The optimality and adjoint conditions are:

$$\lambda' = -\frac{\partial H}{\partial x} = 1; \ \frac{\partial H}{\partial u} = 1 - \lambda \Rightarrow u = \lambda;$$

This problem is solved using the optimality system with $N = 100$ and the results are shown in Figures.

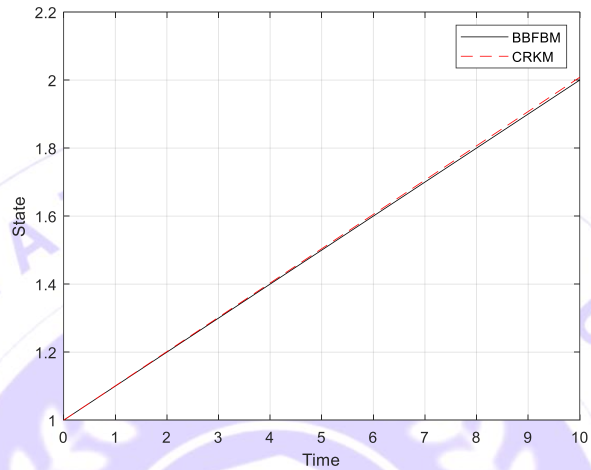

Figure 6: State Result for Example 3

The results show that BBFBM matches the result of Classical Runge Kutta method used by (Lenhart and Workman, 2007).

4. Discussion of Results

Three examples are solved in this paper. In Problem 1, the Figure 2 show that, the state is minimized to the optimal level. It starts from 1, and as the time increase, the state decrease to about 0.28 at time 10 for both the exact solution, the new method and the method of (Garrett, 2015) which is the CRKM. Figure 3 show the error in BBFBM and CRKM.

In Problem 2, Figure 4 show that, the state increased from 1, and as the time increase, the state increase to about 2.7 at time 10 for both the exact solution, the new method and the method of (Lenhart and Workman, 2007) which is the CRKM. Figure 5 show the error in BBFBM and CRKM. The error in BBFBM is almost zero.

In Problem 3, Figure 6 show that, the state increased from 1, and as the time increase, the state increase up to 2 at time 10 for both the exact solution, the new method and the method of (Lenhart and Workman, 2007). There is no error table for this problem because the exact solution for the problem does not exist.

Therefore, the results show that, the BBFBM approximate better than the result produced by classical Runge-Kutta method used by authors in (Garrett, 2015; Lenhart and Workman, 2007). The new method has fewer number of iteration when compared with the classical Runge Kutta method. In term of development of the methods, the new method is easier to develop than the Runge Kutta method especially when higher order methods are required.

5. Conclusion

B-basis block method is developed for the solution of optimal control problems using forward backward sweep method via pontrygin's principle. The new method developed is of order six and verified to be stable and convergent. The accuracy of the method is tested on some numerical examples and compared the result produced by the new method with that of classical Runge Kutta method. It can be seen from Figure 2 to Figure 6 that; the computed solution match the solution produced by classical Runge-Kutta method with better accuracy.

This paper therefore, show that, numerical methods based on bi-polynomial basis function, can effectively handle optimal control problems using forward-backward sweep method.

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