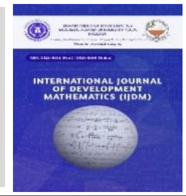
International Journal of Development Mathematics Vol 3 Issue 1 Page No. 236 - 00

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals



# Optimal Ordering Policy for Delayed Deteriorating Items under the Effects of Stock-Dependent Demand Rate, Price Discounting, and Trade Credit Policy

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**A R T I C L E I N F O**

***Article history:***
Received 10 January 2026
Received in revised form 20 March 2026
Accepted 25 March 2026

**Keywords:**
optimal ordering policy, delayed deteriorating items, stock-dependent demand rate, price discounting trade credit policy

**MSC 2020 Subject classification:**
**65K10**

**A B S T R A C T**

*Classical inventory models are developed under the restrictive assumptions of time-varying demand combined with a fixed unit selling price throughout the inventory cycle. These assumptions rarely hold in practical settings. Specifically, the quantity of items displayed or available in stock affects how much customers buy. Higher inventory levels may signal product popularity or availability, thereby stimulating demand. Products with limited shelf life or seasonal characteristics often prompt promotional or bulk sales to reduce wastage. Moreover, once deterioration begins, retailers frequently adopt price discount strategies to accelerate sales and reduce holding costs. These strategies also attract additional customers and mitigate losses arising from spoilage. Motivated by these practical considerations, this study develops an inventory model for delayed deteriorating items that simultaneously incorporates stock-dependent demand, price discounting, and trade credit policy. The demand rate is stock-dependent before deterioration and constant afterward; price discounts are applied only after deterioration begins. The proposed model determines the optimal cycle length and order quantity that maximize the average total profit of the inventory system. Rigorous analytical results are derived. The necessary and sufficient conditions for the existence and uniqueness of the optimal solution are established. Numerical examples are provided to demonstrate the applicability of the model. A comparative analysis with an existing benchmark model shows that the proposed approach yields a higher average total profit per unit time. Finally, a sensitivity analysis is conducted to examine the effects of key parameters on the optimal decision variables. Relevant managerial insights for profit maximization are also discussed.*

## 1. Introduction

Inventory systems in which demand depends explicitly on the on-hand stock level are commonly referred to as stock-dependent demand systems. They have received sustained attention due to their ability to capture key behavioural and managerial realities in retail and production environments. High inventory visibility can stimulate demand through display effects, perceived availability, and signalling mechanisms, whereas stockouts often induce demand loss or customer defection. Seminal work by Urban (1995) formalized these effects within a continuous-time deterministic framework. It demonstrated that demand behaviour differs fundamentally between in-stock and stockout periods, and such asymmetry significantly alters classical economic order quantity (EOQ) policies. Subsequent studies, including Ray (1997), Panda (2010), Yang (2014), Pal (2014), Tripathi (2018), and Pando (2021), extended stock-dependent demand models by incorporating shortages, imperfect quality, stock-dependent holding costs, trade credit, and profit- or return-based objective functions. Collectively, this literature establishes that stock-dependent demand generally increases optimal order quantities relative to classical EOQ models, although this effect

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https://doi.org/10.62054/ijdm/0301.19

may be moderated by rising holding costs or financing considerations.

Parallel to these developments, considerable research has focused on inventory systems for *deteriorating items*, particularly those characterized by rapid or instantaneous deterioration. Such items include fresh products, dairy products, live plants, radioactive materials, and certain chemicals that lose usability almost immediately upon arrival or production. This necessitates highly responsive replenishment policies. Studies such as Sivashankari (2016), Selvaraju and Ghuru (2018), Sivashankari and Panayappan (2014a, 2014b, 2015), Chaudhari et al. (2020), Khedlekar and Thumar (2022), Sharma *et al.* (2022), and Mallick*et al.* (2023) have proposed EOQ and EPQ frameworks for rapidly deteriorating goods under various holding cost structures, production regimes, and demand patterns. While analytically rich, these models implicitly assume that deterioration commences immediately, an assumption that is inappropriate for many real-world products.

In practice, a wide class of items exhibits non-instantaneous deterioration, whereby products remain fully usable for a finite period before deterioration begins. Examples include vaccines, pharmaceuticals, food items, and temperature-sensitive medical supplies that maintain their quality under controlled storage before degradation sets in. Ignoring this delay in deterioration can lead to suboptimal inventory decisions, including premature disposal, underproduction, or misaligned pricing strategies. To address this realism gap, several researchers have developed inventory models that explicitly incorporate delayed deterioration. Musa and Sani (2012) examined non-instantaneous decaying goods under permissible delay in payments. Choudhury et al. (2013) integrated stock-dependent demand, time-varying holding costs, and complete backlogging. Further extensions include Patoghi and Setak (2018), Babangida and Baraya (2019, 2020), Malumfashi*et al.* (2021a, 2021b), and Muazu*et al.* (2023), who considered multi-stage demand patterns, variable production rates, partial backlogging, and profit-oriented objectives. These studies collectively demonstrate that explicitly modelling deterioration delay greatly improves policy relevance and decision accuracy.

Another critical dimension in modern inventory systems is *trade credit financing*. Classical EOQ models assume immediate payment upon replenishment; however, in practice, suppliers frequently offer retailers a permissible delay in payment, allowing them to generate revenue and earn interest before settlement. Since the pioneering works of Haley and Higgins (1973) and Goyal (1985), extensive research has incorporated trade credit into inventory models for both non-deteriorating and deteriorating items (Aggarwal and Jaggi, 1995; Musa and Sani, 2012; Jaggi*et al.*, 2015; Shaikh*et al.*, 2018; Majumder and Kumar, 2019). These studies show that trade credit significantly influences optimal ordering policies by effectively reducing capital costs and altering cash-flow dynamics.

Despite these advances, many existing models for non-instantaneous deteriorating items assume a uniform selling price throughout the inventory cycle. This assumption is inconsistent with real-world retail practice, where price reductions are commonly implemented after deterioration begins to accelerate sales, reduce holding costs, and mitigate wastage. Addressing this gap, several studies have incorporated dynamic or multi-stage pricing strategies under trade credit (Tsaoand Sheen, 2008; Tsao, 2010; Wang *et al.*, 2015; Tsao*et al.*, 2017; Pang *et al.*, 2022; Ahmed *et al.*, 2025; Babangida and Baraya, 2021). However, even within this pricing-focused literature, demand is typically assumed to be time- or price dependent, with limited attention given to stock-dependent demand, particularly before deterioration begins.

In reality, customer purchasing behaviour is strongly influenced by the quantity of goods displayed or available. High stock levels often signal popularity, freshness, or promotional availability, thereby stimulating demand, especially for perishable or seasonal items. Conversely, once deterioration sets in, retailers frequently apply price discounts to encourage faster turnover. Yet, an integrated framework that simultaneously considers stock-dependent demand before deterioration, constant demand after deterioration, two-stage pricing, delayed deterioration, and trade credit financing remains largely unexplored.

Motivated by this gap, the present study develops an optimal ordering policy for non-instantaneous deteriorating items under stock-dependent demand, price discounting, and trade credit. Specifically, demand is assumed to depend on on-hand inventory prior to deterioration, while a constant demand rate applies thereafter. A price discount is introduced once deterioration begins, reflecting practical retail strategies. The objective is to maximize the average total profit of the inventory system by jointly determining the optimal order quantity and cycle length. The necessary and sufficient conditions for existence and uniqueness of the optimal solution are established analytically. Numerical examples are provided to illustrate the model's behavior, followed by a comparative analysis demonstrating superior profitability relative to existing models. Finally, sensitivity analyses are conducted to examine the impact of key parameters and to derive managerial insights for profit maximization.

Table 1: Comparison of some selected studies relevant to the proposed model

| Authors and year | Non-instant decaying goods | Two-stage consumption rates | Price discount | Trade credit | Closed form solution | Carrying charges | Stock dependent demand rates |
|---|---|---|---|---|---|---|---|
| Ouyang *et al.* (2006) | Yes | No | No | Yes | Yes | No | No |
| Tsao and Sheen (2008) | No | No | Yes | Yes | No | No | No |
| Chung (2009) | Yes | No | No | Yes | Yes | No | No |
| Tsao (2010) | No | No | Yes | Yes | No | No | No |
| Chen and Kang (2010) | No | No | Yes | Yes | No | No | No |
| Wang *et al.* (2015) | Yes | No | Yes | No | No | No | No |
| Tsao *et al.* (2017) | Yes | No | Yes | Yes | No | No | No |
| Pang *et al*. (2022) | No | No | Yes | No | No | No | No |
| Babangida and Baraya (2021) | Yes | Yes | Yes | Yes | Yes | No | No |
| Ahmed *et al.* (2025) | Yes | Yes | Yes | Yes | Yes | No | No |
| Proposed model | Yes | Yes | Yes | Yes | Yes | Yes | Yes |

# 2. Notation and Assumptions

## 2.1 Notation

$A$ The fixed ordering cost per order
$C$ The purchasing cost per unit
$S_1$ The initial selling price per unit time (\$/unit/ year) during the interval $[0,\ t_1]$.
$S_2$ The discounted selling price per unit time (\$/unit/ year) during the interval $[t_1,\ T](S_1 > S_2 > C)$.
$h$ The physical holding cost per unit per unit time (\$/unit/ year)
$i$ Interest on capital cost
$I_p$ The interest paid by the supplier per Dollar per year (\$/unit/year) $(I_p \geq I_g)$.
$I_e$ The interest gained by the retailer per Dollar per year (\$/unit/year).
M The trade credit period (in year) for settling accounts
$\omega$ The parameter of constant deterioration rate functions, where $0 < \omega < 1$ .
$t_1$ The length of time in which the product exhibits no deterioration
$T$ The length of the replenishment cycle time (time unit)
$Q$ The quantity of items received at the begging of the inventory cycle (units).
$T^*$ The optimal length of the replenishment cycle time
$EOQ^*$The optimal order quantity

## 2.2 Assumptions

1. The replenishment rate is infinite.
2. The lead time is zero.
3. During the fixed period, $t_1$, there is no deterioration and at the end of this period, the inventory item deteriorates at the constant rate $\omega$.
4. There is no replacement or repair for deteriorated items.
5. Demand rate before deterioration begins is a function of the on-hand inventory level $I_1$(t) and is defined as $R(I) = \alpha + I_1(t)\beta,\ \alpha > 0, 0 < \beta < 1, I_1(t)$ is the instantaneous stock level at time $t$, $\alpha$ is the initial demand rate, $\beta$ is the stock dependent parameter.
6. Demand rate after deterioration sets in is assumed to be constant and is given by $\gamma$.
7. price discount is offered as a result of deterioration or as a result of fear of items being deteriorated
8. The initial selling price during the interval $[0,\ t_1]\ is\ S_1$, while the discounted selling price when deterioration starts is given by $S_2$
9. During the trade credit period $M\ (0 < M < 1)$, the account is not settled; generated sales revenue is deposited in an interest-bearing account. At the end of the period, the retailer pays off all units bought, and starts to pay the capital opportunity cost for the items in stock.

10. The interest computed under simple interest and revenue accumulation assumptions
11. Shortages are not allowed.

**2.3 Formulation of the model**

The inventory system is developed as follows. $Q$ units of a single product from the supplier are received at the beginning of the cycle (i.e., at time $t = 0$). During the time interval $[0,\ t_1]$, the inventory leveldepletes gradually due to market demand only and the demand rate during this interval is assumed to be stock dependent. At time $t = t_1$, the inventory drops from $Q$ to $Q_1$. At time interval $[t_1, T]$, the inventory level depletes due to both demand from customers and deterioration and the demand rate during this interval is considered as constant. At time $t = T$, the inventory drops from $Q_1$ to 0

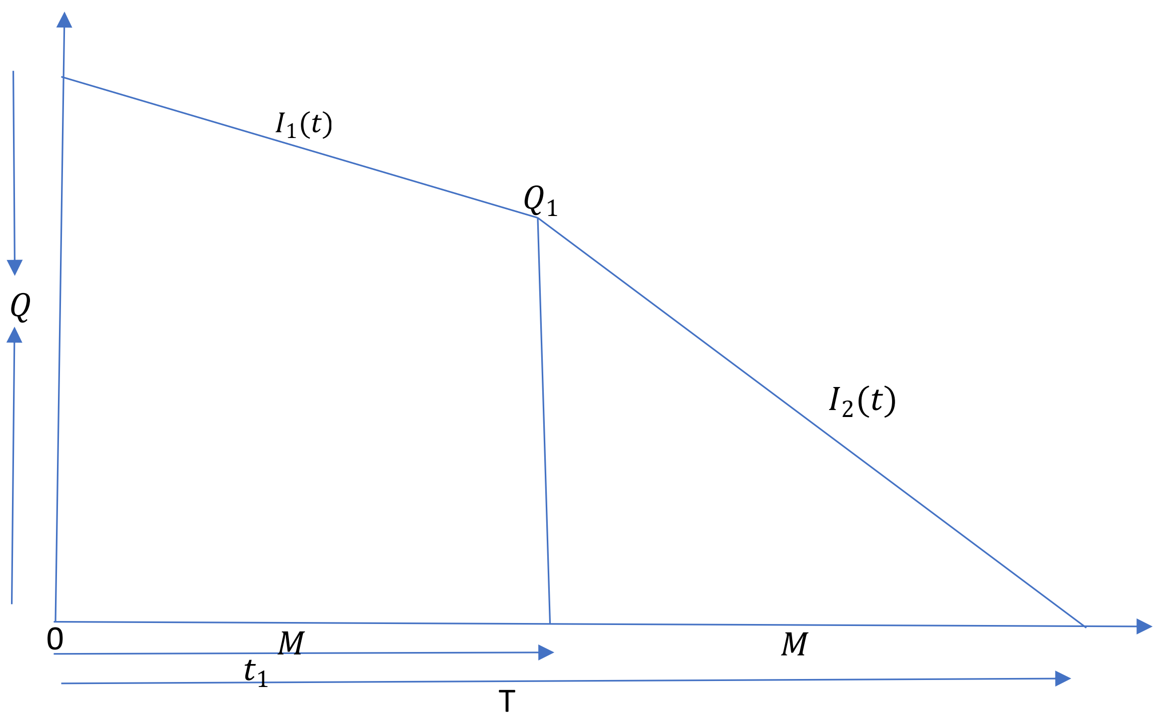


**Figure 1** The inventory system of delayed deteriorating items

Based on the above description during the time interval $[0,\ t_1]$, the change of inventory at any time $t$ is represented by the following differential equation:

$$\frac{dI_1(t)}{dt} = -\left(\alpha + I_1(t)\beta\right), \qquad 0 \le t \le t_1 \qquad (1)$$

With boundary conditions $I(t) = Q$ at $t = 0$ and $I(t) = Q_1$ at $t = t_1$.

During the second interval $[t_1,\ T]$, the inventory level decreases due to combined effects of demand and deterioration. Thus, the differential equation below represents the inventory status is

$$\frac{dI_2(t)}{dt} + \omega I_2(t) = -\gamma, \qquad t_1 \le t \le T \qquad (2)$$

With boundary conditions $I_2(t) = Q_1$at $t = t_1$and$I(t) = 0$ at $t = T$.

The solutions of equations (1) and (2) are

$$I_1(t) = \left(Q + \frac{\alpha}{\beta}\right)e^{-t\beta} - \frac{\alpha}{\beta} \qquad 0 \le t \le t_1 \qquad (3)$$

$$I_2(t) = \frac{\gamma}{\omega}\left(e^{\omega(T-t)} - 1\right), \qquad t_1 \le t \le T \qquad (4)$$

Also applying the conditions $I_1(t_1) = Q_1$ and $I_2(t_1) = Q_1$ at $t = t_1$ into equations (3) and (4), we obtain

$$Q_1 = \left(Q + \frac{\alpha}{\beta}\right) e^{-t_1\beta} - \frac{\alpha}{\beta}$$

$$Q_1 = \frac{\gamma}{\omega}\left(e^{\omega(T-t_1)} - 1\right)$$

Eliminating $Q_1$ from the above equations, we obtain

$$Q = \frac{\alpha}{\beta}\left(e^{t_1\beta} - 1\right) + \frac{\gamma}{\omega} e^{t_1\beta}\left(e^{\omega(T-t_1)} - 1\right) \tag{5}$$

Substituting equation (5) into equation (3) to obtain

$$I_1(t) = \frac{\alpha}{\beta}\left(e^{\beta(t_1-t)} - 1\right) + \frac{\gamma}{\omega} e^{\beta(t_1-t)}\left(e^{\omega(T-t_1)} - 1\right) \qquad 0 \le t \le t_1 \tag{6}$$

(i) The fixed ordering cost per order is given by $A$

(ii) The inventory holding cost during the period $[0,\ T]$ is given by

$$C_H = ih\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right] \tag{7}$$

Substituting (9) and (7) into (10), we obtain

$$C_H = ih\left[\int_0^{t_1}\left[\frac{\alpha}{\beta}\left(e^{\beta(t_1-t)} - 1\right) + \frac{\gamma}{\omega} e^{\beta(t_1-t)}\left(e^{\omega(T-t_1)} - 1\right)\right]dt + \int_{t_1}^{T}\left[\frac{\gamma}{\omega}\left(e^{\omega(T-t)} - 1\right)\right]dt\right]$$

$$= ih\left[\frac{\alpha}{\beta^2}\left(e^{\beta t_1} - \beta t_1 - 1\right) + \frac{\gamma}{\omega\beta}\left(e^{\beta t_1} - 1\right)\left(e^{\omega(T-t_1)} - 1\right) + \frac{\gamma}{\omega^2}\left(e^{\omega(T-t_1)} - 1 - \omega(T-t_1)\right)\right] \tag{8}$$

(iii) Total amount of the inventory that deteriorated during cycle is:

$$\int_{t_1}^{T} \omega I_2(t)dt = \int_{t_1}^{T} \omega\frac{\gamma}{\omega}\left(e^{\omega(T-t)} - 1\right)dt = \gamma\left[\left(-\frac{e^{\omega(T-t)}}{\omega} - t\right)\right]_{t_1}^{T} = \gamma\left(-1 + \frac{e^{\omega(T-t_1)}}{\omega} - (T-t_1)\right)$$

$$= \frac{\gamma}{\omega}\left(e^{\omega(T-t_1)} - 1\right) - \gamma(T-t_1) \tag{9}$$

(iii) Sale revenue (SR)

$$SR = S_1\left[\int_0^{t_1}(\alpha + I_1(t)\beta)dt\right] + S_2\left[\int_{t_1}^{T}\gamma dt\right]$$

$$= S_1\left[\int_0^{t_1}\left(\alpha + \left[\frac{\alpha}{\beta}\left(e^{\beta(t_1-t)} - 1\right) + \frac{\gamma}{\omega} e^{\beta(t_1-t)}\left(e^{\omega(T-t_1)} - 1\right)\right]\beta\right)dt\right] + S_2\left[\int_{t_1}^{T}\gamma dt\right]$$

$$= S_1\left(\alpha t_1 + \frac{\alpha}{\beta}\left(e^{\beta t_1} - 1 - \beta t_1\right) + \frac{\gamma}{\omega}\left(e^{\beta t_1} - 1\right)\left(e^{\omega(T-t_1)} - 1\right)\right) + S_2\gamma(T-t_1) \tag{10}$$

(iv) The average total profit per unit time for a replenishment cycle (denoted by $ATP(T)$ is given by

$$ATP(T) = \begin{cases} ATP_1(T) & 0 < M \le t_1 \\ ATP_2(T) & t_1 < M \le T \end{cases} \tag{11}$$

Where $ATP_1(T)$ and $ATP_2(T)$ are discussed as follows.

**Case 1:** $(0 < M \le t_1)$

**The Interest Payable**

This is the period before deterioration sets in, and payment for goods is settled with the capital opportunity cost rate $I_p$ for the items in stock.

Thus, the interest payable $= cI_p\left\{\int_M^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right\}$

$$= cI_p\left\{\int_M^{t_1}\left[\frac{\alpha}{\beta}\left(e^{\beta(t_1-t)} - 1\right) + \frac{\gamma}{\omega} e^{\beta(t_1-t)}\left(e^{\omega(T-t_1)} - 1\right)\right]dt + \int_{t_1}^{T}\left[\frac{\gamma}{\omega}\left(e^{\omega(T-t)} - 1\right)\right]dt\right\}$$

$$= cI_p\left[\frac{\alpha}{\beta^2}\left(e^{\beta(t_1-M)}-1-\beta(t_1-M)\right)+\frac{\gamma}{\beta\omega}\left(e^{\beta(t_1-M)}-1\right)\left(e^{\omega(T-t_1)}-1\right)+\frac{\gamma}{\omega^2}\left(e^{\omega(T-t_1)}-1-\omega(T-t_1)\right)\right] \quad (12)$$

**The interest Earn**

In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period $M$. Although, the retailer has to settle the accounts at period $M$, for that he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period $M$to $t_1$. The interest earn $= S_1I_g\left[\int_0^M(\alpha+I_1(t)\beta)tdt\right]$

$$= S_1I_g\left[\int_0^M\left(\alpha t+\left[\frac{\alpha t}{\beta}\left(e^{\beta(t_1-t)}-1\right)+\frac{\gamma t}{\omega}e^{\beta(t_1-t)}\left(e^{\omega(T-t_1)}-1\right)\right]\beta\right)dt\right]$$

$$= S_1I_g\left(\frac{\alpha}{\beta^2}e^{\beta t_1}-\frac{\alpha}{\beta^2}e^{\beta(t_1-M)}-\frac{\alpha M}{\beta}e^{\beta(t_1-M)}+\frac{\gamma}{\beta\boldsymbol{\omega}}e^{\beta t_1}e^{\omega(T-t_1)}-\frac{\gamma M}{\boldsymbol{\omega}}e^{\beta(t_1-M)}e^{\omega(T-t_1)}-\frac{\gamma}{\beta\boldsymbol{\omega}}e^{\beta(t_1-M)}e^{\omega(T-t_1)}+\frac{\gamma M}{\boldsymbol{\omega}}e^{\beta(t_1-M)}+\frac{\gamma}{\beta\boldsymbol{\omega}}e^{\beta(t_1-M)}-\frac{\gamma}{\beta\boldsymbol{\omega}}e^{\beta t_1}\right) \quad (13)$$

The average total profit per unit time for case 1 $(0 < M \leq t_1)$ is

$ATP_1(T) = \frac{1}{T}${Sale Revenue-Ordering cost - inventory holding cost - purchasing cost - interest payable during the permissible delay period + interest gained during the cycle}

$$= \frac{1}{T}\Bigg\{(S_1-C)\left(\frac{\alpha}{\beta}\left(e^{\beta t_1}-1\right)+\frac{\gamma}{\omega}e^{\beta t_1}\left(e^{\omega(T-t_1)}-1\right)\right)+S_1\left(\frac{\gamma}{\omega}\left(1-e^{\omega(T-t_1)}\right)\right)+S_2\gamma(T-t_1)-A$$
$$-\mathrm{i}h\left[\frac{\alpha}{\beta^2}\left(e^{\beta t_1}-\beta t_1-1\right)+\frac{\gamma}{\omega\beta}\left(e^{\beta t_1}-1\right)\left(e^{\omega(T-t_1)}-1\right)+\frac{\gamma}{\omega^2}\left(e^{\omega(T-t_1)}-1-\omega(T-t_1)\right)\right]$$
$$-cI_p\left[\frac{\alpha}{\beta^2}\left(e^{\beta(t_1-M)}-1-\beta(t_1-M)\right)+\frac{\gamma}{\beta\omega}\left(e^{\beta(t_1-M)}-1\right)\left(e^{\omega(T-t_1)}-1\right)+\frac{\gamma}{\omega^2}\left(e^{\omega(T-t_1)}-1-\omega(T-t_1)\right)\right]$$
$$+S_1I_g\left(\frac{\alpha}{\beta^2}e^{\beta t_1}-\frac{\alpha}{\beta^2}e^{\beta(t_1-M)}-\frac{\alpha M}{\beta}e^{\beta(t_1-M)}+\frac{\gamma}{\beta\boldsymbol{\omega}}e^{\beta t_1}e^{\omega(T-t_1)}-\frac{\gamma M}{\boldsymbol{\omega}}e^{\beta(t_1-M)}e^{\omega(T-t_1)}-\frac{\gamma}{\beta\boldsymbol{\omega}}e^{\beta(t_1-M)}e^{\omega(T-t_1)}+\frac{\gamma M}{\boldsymbol{\omega}}e^{\beta(t_1-M)}+\frac{\gamma}{\beta\boldsymbol{\omega}}e^{\beta(t_1-M)}-\frac{\gamma}{\beta\boldsymbol{\omega}}e^{\beta t_1}\right)\Bigg\} \quad (14)$$

**Case 2: $(\boldsymbol{t_1} < M \leq \boldsymbol{T})$.**

**The interest payable**

This is when the end point of credit period is greater than the period with no deterioration but shorter than or equal to the length of period with positive inventory stock of the items.

The interest payable$= cI_p\left[\int_M^T I_2(t)dt\right]$

$$= cI_p\left[\int_M^T\frac{\gamma}{\omega}\left(e^{\omega(T-t)}-1\right)dt\right]$$
$$= cI_p\left[\frac{\gamma}{\omega^2}\left(e^{\omega(T-M)}-1-\omega(T-M)\right)\right] \quad (15)$$

**The interest gained**

In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period $M$. Although, the retailer has to settle the accounts at period $M$, for that he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period $M$ to $T$. The interest earn $= S_1 I_g\left[\int_0^{t_1}(\alpha + I_1(t)\beta)tdt\right] + S_2 I_g\left[\int_{t_1}^{M}\gamma tdt\right]$

$$= S_1 I_g\left[\int_0^{t_1}\left(\alpha + \left[\frac{\alpha}{\beta}\left(e^{\beta(t_1-t)}-1\right)+\frac{\gamma}{\omega}e^{\beta(t_1-t)}\left(e^{\omega(T-t_1)}-1\right)\right]\beta\right)tdt\right] + S_2 I_g\left[\int_{t_1}^{M}\gamma tdt\right]$$

$$= S_1 I_g\left(\frac{\alpha}{\beta^2}\left(e^{\beta t_1}-1-\beta t_1\right)+\frac{\gamma}{\beta\boldsymbol{\omega}}\left(e^{\beta t_1}-1-\beta t_1\right)\left(e^{\omega(T-t_1)}-1\right)\right)+S_2 I_g\left(\frac{\gamma M^2}{2}-\frac{\gamma t_1^2}{2}\right) \quad (16)$$

The average total profit per unit time for case 2 ($t_1 < M \le T$) is

$ATP_2(T) = \frac{1}{T}${SaleRevenue-Ordering cost - inventory holding cost - purchasing cost - interest payable during the permissible delay period + interest gained during the cycle}

$$= \frac{1}{T}\left\{(S_1 - C)\left(\frac{\alpha}{\beta}\left(e^{\beta t_1}-1\right)+\frac{\gamma}{\omega}e^{\beta t_1}\left(e^{\omega(T-t_1)}-1\right)\right)+S_1\left(\frac{\gamma}{\omega}\left(1-e^{\omega(T-t_1)}\right)\right)+S_2\gamma(T-t_1)-A\right.$$
$$-ih\left[\frac{\alpha}{\beta^2}\left(e^{\beta t_1}-\beta t_1-1\right)+\frac{\gamma}{\omega\beta}\left(e^{\beta t_1}-1\right)\left(e^{\omega(T-t_1)}-1\right)\right.$$
$$\left.+\frac{\gamma}{\omega^2}\left(e^{\omega(T-t_1)}-1-\omega(T-t_1)\right)\right]-cI_p\left[\frac{\gamma}{\omega^2}\left(e^{\omega(T-M)}-1-\omega(T-M)\right)\right]$$
$$+S_1 I_g\left(\frac{\alpha}{\beta^2}\left(e^{\beta t_1}-1-\beta t_1\right)+\frac{\gamma}{\beta\boldsymbol{\omega}}\left(e^{\beta t_1}-1-\beta t_1\right)\left(e^{\omega(T-t_1)}-1\right)\right)$$
$$\left.+S_2 I_g\left(\frac{\gamma M^2}{2}-\frac{\gamma t_1^2}{2}\right)\right\} \quad (17)$$

Since $0 < \boldsymbol{\omega} < 1$and $0 < \beta < 1$ by utilizing a quadratic approximation for the exponential terms in equations (14) and (17) we have

$$ATP_1(T) = \frac{\gamma}{T}\left\{X_1\frac{T^2}{2}+X_2 T+X_3\right\} \quad (18)$$

Where

$$X_1 = \left[(S_1 - C)\left(1+\beta t_1+\frac{\beta^2 t_1^2}{2}\right)-S_1\omega-ih\left[\omega\left(t_1+\frac{\beta t_1^2}{2}\right)+1\right]-cI_p\left[\omega\left((t_1-M)+\frac{\beta(t_1-M)^2}{2}\right)+1\right]\right.$$
$$-S_1 I_g\left(\frac{t_1}{\beta}\left(1+\beta t_1+\frac{\beta^2 t_1^2}{2}\right)+M\left(1+\beta(t_1-M)+\frac{\beta^2(t_1-M)^2}{2}\right)\right.$$
$$\left.\left.+\frac{\omega}{\beta}\left(1+\beta(t_1-M)+\frac{\beta^2(t_1-M)^2}{2}\right)\right)\right]$$

$$X_2 = \left[(S_1 - C)\left(1+\beta t_1+\frac{\beta^2 t_1^2}{2}\right)(1-\omega t_1)+S_1(\omega t_1-1)+S_2-ih\left[\left(t_1+\frac{\beta t_1^2}{2}\right)(1-\omega t_1)-t_1\right]\right.$$
$$-cI_p\left[\left(\frac{\alpha(t_1-M)^2}{2\gamma}\right)+\left((t_1-M)+\frac{\beta(t_1-M)^2}{2}\right)(1-\omega t_1)-t_1\right]$$
$$\left.+S_1 I_g\left(\frac{1}{\beta}\left(1+\beta t_1+\frac{\beta^2 t_1^2}{2}\right)-\left(M+\frac{1}{\beta}\right)\left(1+\beta(t_1-M)+\frac{\beta^2(t_1-M)^2}{2}\right)\right)(1-\omega t_1)\right]$$

$$X_3 = \frac{1}{\gamma}\Bigg[S_1 I_g\left(\frac{\alpha t_1^2}{2} - \frac{\alpha(t_1-M)^2}{2} - \frac{\alpha M}{\beta}\left(\beta(t_1-M) + \frac{\beta^2(t_1-M)^2}{2}\right) + \frac{\gamma}{\beta}\left(1+\beta t_1 + \frac{\beta^2 t_1^2}{2}\right)\frac{\boldsymbol{\omega}}{2}t_1^2 + \left(\frac{\gamma M}{\boldsymbol{\omega}} + \frac{\gamma}{\beta\boldsymbol{\omega}}\right)\left(1+\beta(t_1-M) + \frac{\beta^2(t_1-M)^2}{2}\right)\left(\omega t_1 - \frac{\omega^2}{2}t_1^2\right)\right) + (S_1-C)\left(\alpha\left(t_1 + \frac{\beta t_1^2}{2}\right) + \frac{\gamma}{\omega}\left(1+\beta t_1 + \frac{\beta^2 t_1^2}{2}\right)\left(\frac{\omega^2}{2}t_1^2 - \omega t_1\right)\right) + S_1\gamma\left(t_1 - \frac{\omega}{2}t_1^2\right) - \Big\{S_2\gamma t_1 + A + ih\left[\frac{\alpha t_1^2}{2} + \frac{\gamma}{\omega}\left(t_1 + \frac{\beta t_1^2}{2}\right)\left(-\omega t_1 + \frac{\omega^2}{2}t_1^2\right) + \frac{\gamma}{2}t_1^2\right] + cI_p\left[\left(\frac{\alpha(t_1-M)^2}{2}\right) + \frac{\gamma}{\omega}\left((t_1-M) + \frac{\beta(t_1-M)^2}{2}\right)\left(-\omega t_1 + \frac{\omega^2}{2}t_1^2\right) + \frac{\gamma}{2}t_1^2\right]\Big\}\Bigg]$$

Similarly,

$$ATP_1(T) = \frac{\gamma}{T}\left\{Y_1\frac{T^2}{2} + Y_2 T + Y_3\right\} \tag{19}$$

where

$$Y_1 = \left[(S_1-C)\left(1+\beta t_1 + \frac{\beta^2 t_1^2}{2}\right) - S_1\omega + ih\left[\omega\left(t_1 + \frac{\beta t_1^2}{2}\right) + 1\right] - cI_p - S_1 I_g\left(\frac{\beta t_1^2\omega}{2}\right)\right]$$

$$Y_2 = \left[(S_1-C)\left(1+\beta t_1 + \frac{\beta^2 t_1^2}{2}\right)(1-\omega t_1) + S_1(\omega t_1 - 1) + S_2 - ih\left[\left(t_1 + \frac{\beta t_1^2}{2}\right)(1-\omega t_1) - t_1\right] + cI_p M + S_1 I_g\frac{\beta t_1^2}{2}(1-\omega t_1)\right]$$

$$Y_3 = \frac{1}{\gamma}\Bigg[S_1 I_g\left(\alpha\frac{t_1^2}{2} + \gamma\frac{\beta t_1^2}{2}\left(\frac{\omega}{2}t_1^2 - t_1\right)\right) + S_2 I_g\left(\frac{\gamma M^2}{2} - \frac{\gamma t_1^2}{2}\right) + (S_1-C)\left(\alpha\left(t_1 + \frac{\beta t_1^2}{2}\right) + \frac{\gamma}{\omega}\left(1+\beta t_1 + \frac{\beta^2 t_1^2}{2}\right)\left(\frac{\omega^2}{2}t_1^2 - \omega t_1\right)\right) + S_1\gamma\left(t_1 - \frac{\omega}{2}t_1^2\right) - \Big\{S_2\gamma t_1 + A + ih\left[\frac{\alpha t_1^2}{2} + \frac{\gamma}{\omega}\left(t_1 + \frac{\beta t_1^2}{2}\right)\left(-\omega t_1 + \frac{\omega^2}{2}t_1^2\right) + \frac{\gamma}{2}t_1^2\right] + cI_p\left[\gamma\frac{1}{2}M^2\right]\Big\}\Bigg]$$

## 3. Optimal Decision

In order to find the optimal ordering policy that maximizes the average total profit per unit time, we established the necessary and sufficient conditions. The necessary condition for the average total profit per unit time $ATP_i(T)$ to be maximum is $\frac{ATP_i(T)}{dT} = 0$ for $= 1, 2$ . The value of $T$ obtained from $\frac{ATP_i(T)}{dT} = 0$ and for which the sufficient condition $\frac{d^2 ATP_i(T)}{dT^2} < 0$ is satisfied gives a maximum for the average total profit per unit time $ATP_i(T)$.

**For $(0 < M \le t_d)$.**

The necessary and sufficient conditions to maximize $ATP_1(T)$ are respectively $\frac{dATP_1(T)}{dT} = 0$ and $\frac{d^2 ATP_1(T)}{dT^2} < 0$

The first derivative of the average total profit in (18) with respect to$T$is as follows.

$$\frac{dATP_1(T)}{dT} = \frac{\gamma}{T^2}\left\{X_1\frac{T^2}{2} - X_3\right\} \tag{20}$$

Therefore, the necessary condition that maximizes $ATP_1(T)$ is $\frac{ATP_1(T)(T)}{dT} = 0$ , which gives the following nonlinear equation in $T$

$$\frac{\gamma}{T^2}\left\{X_1\frac{T^2}{2} - X_3\right\} = 0 \tag{21}$$

which is equivalent to

$$2X_3 - T^2X_1 = 0 \tag{22}$$

Since $M \leq t_1$, it should be noted that

$$\left[(S_1 - C)\left(1 + \beta t_1 + \frac{\beta^2 t_1^2}{2}\right) - S_1\omega - ih\left[\omega\left(t_1 + \frac{\beta t_1^2}{2}\right) + 1\right] - cI_p\left[\omega\left((t_1 - M) + \frac{\beta(t_1 - M)^2}{2}\right) + 1\right]\right.$$
$$- S_1 I_g\left(\frac{t_1}{\beta}\left(1 + \beta t_1 + \frac{\beta^2 t_1^2}{2}\right) + M\left(1 + \beta(t_1 - M) + \frac{\beta^2(t_1 - M)^2}{2}\right)\right.$$
$$\left.\left. + \frac{\omega}{\beta}\left(1 + \beta(t_1 - M) + \frac{\beta^2(t_1 - M)^2}{2}\right)\right)\right] > 0$$

Let

$$\Delta_1 = 2X_3 - t_1^2X_1$$

**Lemma 1.** For $0 < M \leq t_1$, we have:

(i) If $\Delta_1 \geq 0$, then the solution of $T \in [t_1,\ \infty)$ (say $T_1^*$) which satisfies (22) not only exists but also is unique.
(ii) If $\Delta_1 < 0$, then the solution of $T \in [t_1,\ \infty)$ which satisfies (22) does not exist.

**Proof of part (i).** From (22), we define a new function $F_1(T)$ as follows:

$$F_1(T) = 2X_3 - T^2X_1,\ T \in [t_1, \infty). \tag{23}$$

Taking the first-order derivative of $F_1(T)$ with respect to $T \in [t_1, \infty)$, we have

$$\frac{F_1(T)}{dT} =- 2TX_1 < 0$$

We obtain that $F_1(T)$ is a decreasing of $T$in the interval $[t_1, \infty)$. Moreover, we have

$\lim_{T \to \infty} F_1(T) =- \infty$ and

$F_1(t_1) = 2X_3 - t_1^2X_1 = \Delta_1 \geq 0$

We have $F_1(t_1) \geq 0$. Therefore, by applying intermediate value theorem, there exists a unique $T_1^* \in [t_1,\ \infty)$ such that $F_1(T_1^*) = 0$. Hence $T_1^*$ isthe unique solution of (22). Thus, the value of $T$ (denoted by $T_1^*$) can be found from (22) and is given by

$$T_1^* = \sqrt{\frac{2X_3}{X_1}} \tag{24}$$

**Proof of part (ii).** If $\Delta_1 < 0$, then from (23), we have $F_1(T) < 0$. Since $F_1(T)$ is a decreasing function of $T \in [t_1,\ \infty)$, we have $F_1(T) < 0$ for all $T \in [t_1, \infty)$. Thus, we cannot find a value of $T \in [t_1,\ \infty)$ such that $F_1(T) = 0$. This completes the proof.

**Theorem 1.** When $0 < M \leq t_1$, we have

(i) If $\Delta_1 \geq 0$, then the average total profit$ATP_1(T)$ is concave and reaches its global maximum at the point $T_1^* \in [t_1, \infty)$, where $T_1^*$ is the point which satisfies (22).
(ii) If $\Delta_1 < 0$, then the average total profit$ATP_1(T)$ has a maximum value at the point $T_1^* = t_1$.

**Proof of part (i).** When $\Delta_1 \geq 0$, we see that $T_1^*$ is the unique solution of (22) from Lemma l(i). Taking the second derivative of $ATP_1(T)$ with respect to $T$ and then finding the value of the function at the point of $T_1^*$, we obtain

$$\left.\frac{d^2ATP_1(T)}{dT^2}\right|_{T_1^*} =- \frac{\gamma X_1}{T_1^*} < 0 \tag{25}$$

We thus conclude from (25) and Lemma 1 that $ATP_1(T_1^*)$ is concave and $T_1^*$ is the global maximum point of $ATP_1(T)$. Hence the value of $T$ in (24) is optimal.

**Proof of part (ii).** When $\Delta_1 < 0$, then we know that $F_1(T) < 0$ for all $T \in [t_1, \infty)$. Thus, $\frac{dATP_1(T)}{dT} = \frac{F_1(T)}{T^2} < 0$ for all $T \in [t_1, \infty)$ which implies $ATP_1(T)$ is a decreasing function of $T$. Thus $ATP_1(T)$ has a maximum value when $T$ is minimum. Therefore, $ATP_1(T)$ has a maximum value at the point $T = t_1$. This completes the proof.

**For ($t_d < M \leq T$).**

Applying the same procedure as in case 1, the value of the optimal cycle length, denoted by $T_2^*$, is given by

$$T_2^* = \sqrt{\frac{2Y_3}{Y_1}} \quad (26)$$

**Lemma 2:** Feasibility and Monotonicity $Q(T)$

$Q(T)$ is strictly increasing in $T$

**Proof:**

From (5), we define $Q(T)$ as follows:

$$Q(T) = \frac{\alpha}{\beta}\left(e^{\beta t_1} - 1\right) + \frac{\gamma}{\omega} e^{\beta t_1}\left(e^{\omega(T-t_1)} - 1\right),\ T \in [t_1, \infty). \quad (27)$$

Taking the first-order derivative of $Q(T)$ with respect to $T \in [t_1, \infty)$, we have

$$\frac{Q(T)}{dT} = \gamma e^{\beta t_1}\left(e^{\omega(T-t_1)}\right) > 0$$

We obtain that $Q(T)$ is a strictly increasing of $T$ in the interval $[t_1, \infty)$.

Thus, the EOQ corresponding to the optimal cycle length $T^*$will be computed as follows:

$EOQ^* = Total\ demand\ before\ deterioration\ starts + Total\ demand\ afterdeterioration\ satrts +$ Number of deteriorated items

$$= \int_0^{t_1} (\alpha + I_1(t)\beta)dt + \int_{t_1}^{T^*} \gamma dt + \frac{\gamma}{\omega}\left(e^{\omega(T-t_1)} - 1\right) - \gamma(T^* - t_1)$$

$$= \frac{\alpha}{\beta}\left(e^{\beta t_1} - 1\right) + \frac{\gamma}{\omega} e^{\beta t_1}\left(e^{\omega(T^*-t_1)} - 1\right) \quad (28)$$

# 4 Numerical Examples

**Example 4.1 (Case 1)**

The proposed model was validated numerically by adopting parameters from Babangida and Baraya (2021), with $\alpha$ and $\beta$ added in the proposed work, and the values of these added parameters' values were optimally estimated. The parameters and their values are summarized in Table 1.

Table 1 Parameters and their values

| Parameter | Value |
|---|---|
| $A$ | \$250/Order |
| $C$ | \$15/unit/year |
| $S_1$ | \$31/unit/year |
| $S_2$ | \$20/unit/year |
| $h$ | \$2/unit/year |
| $I$ | \$1/unit/year |
| $\omega$ | 0.01 units/year |
| $\alpha$ | 350 units |
| $\beta$ | 0.035 units |
| $\gamma$ | \$120units |
| $t_1$ | 0.1354 year (49 days) |
| $M$ | 0.0888year (32days) |
| $I_p$ | 0.1 |
| $I_g$ | 0.08 |

**4.1.1 Results 1**

We first check the condition $\Delta_1 =-175.9230 < 0$. Substituting the above values into $(24), (18)$ and $(28)$, we obtain as follows the values of the optimal cycle length, the optimal average total profit, and the economic order quantity respectively in Table 2.

Table 2 Decision Variables and their values

| Decision Variables | Values |
|---|---|
| $T_1^*$ | 0.8218 year (300 days) |
| $ATP_1(T_1^*)$ | \$793.8671 |
| $EOQ_1^*$ | 130.5507 Units |

**Example 4.2 (Case 2)**

The data are same as in Example 4.1 except that $M = 0.1523$ year (56 days).

**4.2.1 Results 2**

We first check the condition $\Delta_2 = 6.9437 > 0$. Substituting the above values into $(26), (19)$ and $(28)$, we obtain as follows the values of the optimal cycle length, the optimal average total profit and the economic order quantity respectively in Table 3.

Table 3 Decision Variables and their values

| Decision Variables | Values |
|---|---|
| $T_2^*$ | 0.7677 year (280 days) |
| $ATP_2(T_2^*)$ | \$1768.8265 |
| $EOQ_2^*$ | 123.9855 Units |

Therefore, $\boldsymbol{ATP}(\boldsymbol{T}^*) = \boldsymbol{max}\{ATP_1(T_1^*), ATP_2(T_2^*)\} = ATP_2(T_2^*) = \$1768.8265$ per year.

## 5. Numerical Validation

To assess the impact of the quadratic approximation, we compare the optimal cycle length obtained from approximated model $(T_{approx})$ and exact exponential model $(T_{exact})$. The relative error is defined as $Error = \left\{\frac{(T_{exact})-(T_{approx})}{(T_{exact})}\right\} x100$ as shown in the Table 4.

Table .4 Validation of Approximation Accuracy

| Model Type | Cycle length for case 1 | Cycle length for case 2 | Relative Error for case 1 | Relative Error for case 2 |
|---|---|---|---|---|
| Approximate model | 0.8218 | 0.7677 | − | − |
| Exact model | 0.8297 | 0.7704 | 0.008% | 0.35% |

Table 4.4 shows that the relative errors for both cases is less than 1%. This shows that the approximation has negligible impact on optimal decisions. Hence the managerial insights remain robust and reliable.

## 6. Comparison

Since both the proposed model and the model of Babangida and Baraya (2021) are designed to determine the optimal cycle length and order quantity that maximize the average total profit per unit, a direct comparison of their results is appropriate (see Table 5). The comparative analysis reveals that the proposed model yields substantially higher average total profit per unit in both scenarios, recording values of \$6.0809 and \$14.2664 for Case 1 and Case 2, respectively, compared with \$4.1341 and \$4.3176 reported by Babangida and Baraya (2021). These results demonstrate the superior performance and improved optimality of the proposed model relative to the benchmark model, underscoring its effectiveness in enhancing profit outcomes under the considered conditions.

Table 5 Comparison between the proposed and existing model

| Model | Average Total Profit Per Unit for Case 1 | Average Total Profit Per Unit For Case 2 |
|---|---|---|
| Babangida and Baraya (2021) | \$4.1341 | \$4.3176 |

| | | |
|---|---|---|
| Proposed Model | \$6.0809 | \$14.2664 |

## 7. Sensitivity Analysis

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from $-10\%, -5\%$ , $+5\%$ to $+10\%$ taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these changes on the decision variables are discussed.

Table 6: Effect of changes of some parameters on decision variables

| Parameter | % Change in Parameter | % Change in $EOQ^*$ | % Change in $ATP(T^*)$ |
|---|---|---|---|
| $A$ | $-20$ | 4.2138 | 3.5815 |
| | $-10$ | 2.1354 | 1.8151 |
| | $+10$ | -2.1974 | -1.8683 |
| | $+20$ | -4.4626 | -3.7946 |
| $\alpha$ | $-20$ | -22.2669 | -12.4246 |
| | $-10$ | -10.7439 | -5.8786 |
| | $+10$ | 10.1635 | 5.38137 |
| | $+20$ | 19.8730 | 10.3737 |
| $M$ | $-20$ | 0.0835 | -0.2290 |
| | $-10$ | 0.0198 | -0.0199 |
| | $+10$ | -0.1297 | 0.4172 |
| | $+20$ | -0.2155 | 0.6452 |
| $S_1$ | $-20$ | 14.9941 | -38.6644 |
| | $-10$ | 4.6261 | -19.1063 |
| | $+10$ | -2.6365 | 18.9664 |
| | $+20$ | -4.3427 | 37.8709 |
| $S_2$ | $-20$ | -5.4272 | -22.5242 |
| | $-10$ | -2.7605 | -11.2219 |
| | $+10$ | 2.8650 | 11.1324 |
| | $+20$ | 5.8475 | 22.1641 |
| $h$ | $-20$ | 1.1262 | 1.0979 |
| | $-10$ | 0.5700 | 0.5508 |
| | $+10$ | -0.5845 | -0.5545 |
| | $+20$ | -1.1840 | -1.1130 |
| $C$ | $-20$ | 1.6591 | 35.5819 |
| | $-10$ | 0.9238 | 17.7954 |
| | $+10$ | -1.1956 | -17.8079 |
| | $+20$ | -2.8045 | -35.6342 |

## 8. Discussion on sensitivity analysis

The sensitivity analysis, based on the computational results reported in Table 7.1, provides several economically meaningful insights into how key model parameters influence the optimal ordering policy and the resulting system profitability.

(i) The results indicate that variations in the ordering cost ($A$) significantly affect both the optimal order quantity($EOQ^*$) and the average total profit($TP(T^*)$). An increase in the ordering cost is associated with an expansion in the optimal lot size, reflecting the classical trade-off between fixed ordering costs and inventory-related costs. From a managerial perspective, this suggests that effective reductions in ordering or setup costs enable retailers to place orders more frequently, thereby improving overall profitability through greater operational flexibility.

(ii) Initial demand intensity $(\alpha)$ also plays a crucial role in shaping optimal decisions. Higher values of the initial demand parameter $(\alpha)$ lead to increases in both the optimal order quantity $(EOQ^*)$ and total profit $(TP(T^*))$, indicating that stronger market demand justifies larger replenishment quantities. In such settings, retailers benefit from stocking more units to exploit higher sales potential and achieve greater profit margins.

(iii) The length of the permissible delay in payment emerges as a key financial lever. The numerical results show that an extension of the trade credit period $(M)$ enhances profitability while influencing the retailer's replenishment behavior. Longer credit periods $(M)$ improve cash flow conditions, allowing the retailer to strategically adjust order quantities $(EOQ^*)$ and benefit from deferred payments. This highlights the importance of supplier financing policies in coordinating inventory and financial decisions.

(iv) Pricing decisions before and after the onset of deterioration exhibit distinct effects. An increase in the selling price prior to deterioration $(S_1)$ leads to a reduction in the optimal order quantity $(EOQ^*)$ while simultaneously improving total profit $(TP(T^*))$. This outcome is consistent with demand price sensitivity, where higher prices suppress demand and encourage more conservative ordering. Conversely, an increase in the selling price after deterioration $(S_2)$ results in higher order quantities $(EOQ^*)$ and greater profitability, particularly when the price gap between the two stages is narrow. In such cases, retailers are incentivized to extend the inventory cycle and place larger orders, as the revenue loss from deterioration-induced discounting is limited.

(v) Holding cost $(h)$ is found to have a uniformly adverse effect on both the optimal order quantity $(EOQ^*)$ and total profit $(TP(T^*))$. As storage-related expenses increase, retailers are motivated to reduce replenishment sizes to limit inventory exposure. This underscores the importance of efficient warehousing, improved handling practices, and cost-control measures aimed at mitigating holding costs in perishable inventory systems.

(vi) Finally, increases in the unit purchasing cost $(C)$ reduce both the optimal order quantity $(EOQ^*)$ and total profit $(TP(T^*))$. Higher acquisition costs compel retailers to adopt more conservative ordering strategies and shorten replenishment cycles, often to capitalize more frequently on trade credit benefits. This finding emphasizes the sensitivity of inventory decisions to procurement costs and reinforces the need for effective supplier negotiations.

Overall, the sensitivity analysis confirms the robustness of the proposed model and demonstrates how coordinated adjustments in ordering, pricing, and financing parameters can substantially enhance profitability in delayed-deterioration inventory systems.

## 9. Conclusion

This study develops a comprehensive inventory optimization framework for perishable products that integrates delayed deterioration, stock-dependent demand, price discounting, and trade credit within a unified analytical setting. By explicitly recognizing that demand may be influenced by inventory visibility prior to the onset of deterioration and stabilizes thereafter, the model captures a key behavioral dimension often overlooked in classical inventory formulations. The incorporation of a two-stage pricing policy, with a regular price before deterioration and a discounted price once quality degradation begins, reflects realistic retail practices aimed at stimulating demand, reducing waste, and mitigating holding losses. Departing from the predominantly cost-minimization orientation of traditional EOQ-type models, the proposed approach adopts an average profit-maximization objective and establishes the necessary and sufficient conditions for the existence and uniqueness of the optimal solution. The resulting inventory cycle structure is sufficiently general to encompass multiple operational phases, including inventory buildup, non-deteriorating consumption, deterioration-driven depletion, and fully backlogged shortages, thereby subsuming several well-known models as special cases. This analytical generality enhances both the theoretical contribution and the practical relevance of the framework.

Numerical illustrations and comparative analyses demonstrate that explicitly accounting for delayed deterioration and stock-dependent demand leads to superior profit performance relative to models that neglect these features. Sensitivity analyses further reveal the economic significance of key parameters, showing, for example, that optimal order quantities tend to decrease when ordering costs, holding costs, or purchasing prices increase, or when trade credit periods shorten. Conversely, pricing decisions before and after deterioration play a critical role in shaping demand and profitability, underscoring the importance of coordinated pricing and inventory policies for perishable goods.

Overall, the findings provide actionable managerial insights for retailers dealing with products such as fashion items, electronics, and other goods with limited shelf life, where visibility, pricing flexibility, and financing arrangements jointly influence demand and profitability. Future research may extend the proposed model by considering alternative assumptions, including partial or lost sales shortages, time-varying or stochastic deterioration rates, inflationary effects, advance payment schemes, or reliability considerations, nonlinear/power forms stock demand rates, gradual transition model. Such extensions would further enrich the applicability of the framework and deepen understanding of complex perishable inventory systems.

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