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Stability and Coexistence in an Improved Three-Species Predator–Prey System with Logistic Growth and Nonlinear Interactions

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ABSTRACT

Understanding the mechanisms that sustain coexistence among multiple interacting species remains a fundamental problem in mathematical ecology. In this study, we develop an enhanced three-trophic-level predator–prey model representing a food chain consisting of lions (top predators), leopards (intermediate predators/preys), and hares (preys). The model incorporates logistic growth functions to account for environmental carrying capacities and nonlinear predation terms with energetic conversion efficiencies to reflect realistic trophic interactions. Analytical results establish the positivity, boundedness, and biological feasibility of the model solutions. The equilibrium points and their local stability properties are examined through Jacobian linearization and Routh–Hurwitz conditions, while global stability is investigated via a suitable Lyapunov function. Numerical simulations illustrate diverse dynamic behaviors, including coexistence equilibria, extinction scenarios, and oscillatory coexistence depending on the predation and conversion parameters. The proposed framework provides a more biologically consistent and mathematically robust tool for studying multispecies interactions and can be extended to incorporate stochastic perturbations, time delays, or spatial heterogeneity.

1. Introduction

The study of predator–prey dynamics is one of the central pillars of mathematical ecology, providing fundamental insights into species coexistence, population stability, and the emergence of ecological complexity (Hastings & Powell, 1991; Kot, 2001; Murray, 2007; Bacaër, 2011; Chakraborty et al., 2012; Wang & Jiang, 2012; Diz-Pita & Otero-Espinar, 2021). At its core, this line of inquiry seeks to understand how nonlinear biological interactions translate into observable population-level patterns across time and space.

The mathematical study of predator–prey interactions originated with the pioneering deterministic models of Lotka and Volterra, which demonstrated that simple nonlinear coupling between predator and prey populations can generate sustained oscillatory dynamics consistent with certain empirical observations (Lotka, 1925; Volterra, 1926). Despite their historical importance, these classical models rely on biologically restrictive assumptions, including exponential prey growth in unbounded environments and perfectly efficient energy transfer between trophic levels. Such assumptions, while analytically convenient, limit the ecological realism of the resulting predictions.

A major conceptual advance in the development of population models was the incorporation of density dependence through logistic growth, introduced by Verhulst (1838). The logistic term captures the influence of finite environmental resources by imposing an upper bound, or carrying capacity, on population growth. Parallel developments focused on refining the predator functional response, which describes the rate of prey consumption as a function of prey density (Holling, 1959). Moving beyond the simplest linear (Type I) response, Holling Type II and Type III functional forms

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were introduced to account for predator handling time, satiation, and switching behavior. These refinements substantially altered the qualitative dynamics of predator–prey systems, leading to richer stability and bifurcation structures (Lu & Xia, 2002; Liu & Yuan, 2004; Huang et al., 2006; Jiang et al., 2007; Upadhyay & Naji, 2009).

As the field matured, it became increasingly evident that two-species models are insufficient to capture the hierarchical organization of real ecosystems. This realization motivated a shift toward multi-trophic food-chain models, typically consisting of a basal prey species, an intermediate predator, and a top predator. Such three-dimensional systems can exhibit a far broader range of dynamical behaviors, including chaos, multi-stability, and complex bifurcation phenomena that cannot arise in planar systems (Liu et al., 2005; Wang, 2006; Lv & Zhao, 2008; Meng et al., 2011; Panja & Mondal, 2015; Sen et al., 2018). Although the analysis of these models is considerably more challenging due to the proliferation of parameters and equilibrium points, they provide a more faithful representation of trophic structure and energy flow in ecological networks.

In the pursuit of increasing biological fidelity, contemporary predator–prey models have incorporated a variety of additional mechanisms. These include explicit time delays to represent gestation or maturation periods, spatial heterogeneity and dispersal processes, and non-lethal effects of predation risk, commonly referred to as the "fear effect" (Kuang, 1993; Okubo & Levin, 2001; Cantrell & Cosner, 2003; Creel et al., 2007; Cressman & Garay, 2011; Zanette et al., 2011; Clinchy et al., 2013; Wang et al., 2016; Wang & Zou, 2017; Pingping et al., 2021). These mechanisms have been shown to profoundly influence population persistence, stability, and oscillatory behavior, even in the absence of direct predation mortality.

Other sophisticated extensions further enhance ecological realism by incorporating stage structure, Allee effects, toxin production, and the provision of supplementary or alternative food sources (Arancibia-Ibarra, 2018; Arancibia-Ibarra et al., 2019; Arancibia-Ibarra & Flores, 2020; Terry, 2015; Rihan et al., 2020; Santra, 2023; Reshma & Ankit, 2024). While these developments have significantly advanced the modeling framework, a careful review of the literature reveals a persistent and often overlooked limitation: the explicit treatment of biomass conversion efficiency.

The transfer of energy from prey to predator is inherently inefficient, as only a fraction of consumed biomass is converted into new predator growth due to metabolic losses and assimilation constraints (Borges et al., 2024). Nevertheless, many three-species food-chain models either assume perfect conversion efficiency or embed this efficiency implicitly within other parameters, thereby obscuring its distinct ecological role. Such simplifications can lead to misleading conclusions regarding system stability, resilience, and long-term coexistence (Kwaghkora et al., 2024).

It is within this context that the present study is situated. We address this gap by developing and analyzing a refined three-trophic-level predator–prey model that explicitly incorporates biomass conversion efficiency while maintaining mathematical rigor and biological consistency. The model is grounded in a representative food chain consisting of hares (basal prey), leopards (intermediate predator), and lions (apex predator). This ecological configuration captures essential features of natural systems, including high prey turnover, intraguild predation pressure on mesopredators, and top-down regulation by an apex predator.

2. Methodology

In order to create a mathematical model which describes the relationship between predators and the preys, few assumptions were made as presented below.

2.1 assumptions of the model

1. Each species grows logistically in the absence of predation, with intrinsic growth rate $r_i > 0$ and carrying capacity $K_i > 0$ $(i = 1,2,3)$.
2. Each species experiences a natural per-capita mortality rate $d_i > 0$, representing deaths due to aging, disease, or other non-predation factors.
3. The interactions are hierarchical and unidirectional

4. The functional response of each predator is assumed to be bilinear
5. Only a fraction of the prey biomass consumed is converted into predator growth.
6. The populations are assumed to be uniformly mixed in a homogeneous environment with no migration, spatial heterogeneity, or external recruitment.

2.2 Improved Lion-Leopard-Hare interaction model

The system formulated below represents the dynamics of a three-trophic-level food chain comprising hares (x), leopards(y), and lions (z). Each equation consists of four main components: intrinsic growth, environmental limitation, predation loss, and natural mortality, as well as predator gain terms governed by energetic efficiencies**.**
The nonlinear system is given by

$$\frac{dx}{dt} = r_1 x\left(1-\frac{x}{K_1}\right) - a_1xy - a_2xz - d_1x, \tag{1}$$

$$\frac{dy}{dt} = r_2 y\left(1-\frac{y}{K_2}\right) + e_1a_1xy - a_3yz - d_2y \tag{2}$$

$$\frac{dz}{dt} = r_3 z\left(1-\frac{z}{K_3}\right) + e_2a_2xz + e_3a_3yz - d_3z \tag{3}$$

with initial conditions

$$x(0) = x_0 > 0, y(0) = y_0 > 0, z(0) = z_0 > 0.$$

Assume all parameters are nonnegative and $K_i > 0, r_i \geq 0$. (Biologically meaningful parameter regime.)

We prove: for all $t \geq 0$

1. $x(t), y(t), z(t) \geq 0$, (positivity), and

2. x, y, z are uniformly bounded (there exist finite bounds independent of t)

r_i: Intrinsic growth rate, K_i: carrying capacity, d_i: natural death rate.

a_i: Predation coefficients.

e_i: Conversion efficiency (how much predator growth is supported by prey consumed).

2.3 Model analysis

2.3.1 Positivity of the solution

Each right-hand side is C^1 and has the species factor x, y, z multiplying every term that could change sign. More precisely:

$x(t_1) = 0$, for some t_1 while $(t_1), z(t_1) \geq 0$, then from the equation

$$x^{\cdot}(t_1) = r_1 \cdot 0 \cdot (1-\frac{0}{K_1}) - a_1 \cdot 0 \cdot y(t_1) - a_2 \cdot 0 \cdot z(t_1) - d_1 \cdot 0 = 0.$$

Thus x cannot instantaneously decrease below zero. By uniqueness of solutions to ODEs (Lipschitz RHS) and continuity, once x reaches zero it cannot cross to negative values. The same argument applies to y and z.

Since $x(0), y(0), z(0) > 0$, there is no time $t > 0$ at which any variable first becomes negative. Therefore, for all ≥ 0 , $x(t) \geq 0, y(t) \geq 0, z(t) \geq 0$.

This is the standard invariance argument: the nonnegative orthant is positively invariant.

2.3.2 Boundednes (comparison theorem)

We will bound each species from equation (1)-(3) above by comparing with scalar logistic-type equations.

Step I— boundedness for $x(t)$

From the equation (1),

$$\dot{x} = r_1 x\left(1-\frac{x}{K_1}\right) - a_1 xy - a_2 xz - d_1 x \leq r_1 x\left(1-\frac{x}{K_1}\right),$$

Because the subtracted terms a_1xy, a_2xz, d_1x are nonnegative. Define the scalar logistic equation

$$\dot{u} = r_1 u\left(1-\frac{u}{K_1}\right), u(0) = x_0$$

By the scalar comparison theorem (since the RHS of x is ≤ RHS of u when replacing x by u), we get

$x(t) \leq u(t)$ for all $t \geq 0$.

The logistic solution $u(t)$ is bounded and satisfies $\limsup_{t\to\infty} u(t) \leq K_1$. Hence,

$0 \leq x(t) \leq max\{x_0, K_1\}$ for all $t \geq 0$.

Thus x is uniformly bounded by $K_1^* := max\{x_0, K_1\}$.

Step II— boundedness for $y(t)$

From the equation (2), $\dot{y} = r_2 y\left(1-\frac{y}{K_2}\right) + e_1 a_1 xy.$

Drop the nonpositive terms $-a_3yz$ and $-d_2y$ to get the upper bound.

$$\dot{y} \leq r_2 y\left(1-\frac{y}{K_2}\right) + e_1 a_1 xy.$$

Apply the bound for x from Step I: $x(t) \leq K_1^*$. Therefore

Define constants

$$A := r_2 + e_1 a_1 K_1^*,\ B := \frac{r_2}{K_2},$$

so, the inequality becomes

$$\dot{y} \leq Ay - By^2 =: f(y).$$

Consider the scalar logistic-type equation

$$\dot{v} = Av - Bv^2, v(0) = y0.$$

By comparison, $y(t) \leq v(t)$ for all t. The scalar solution $v(t)$ is bounded and it carrying capacity is

$v_\infty = A / B = \dfrac{r_2(r_2 + e_1 a_1 K_1^*)}{K_2}$. Hence,

$$y(t) \le max\{y0, v_\infty\} =: K_2^* < \infty.$$

Step III — boundedness for $z(t)$

From the equation (3),

$$\dot{z} = r_3 z(1 - \frac{z}{K_3}) + e_2 a_2 xz + e_3 a_3 yz - d_3 z.$$

Drop $-d_3 z$ (nonpositive) to get

$$\dot{z} \le r_3 z(1 - \frac{z}{K_3}) + e_2 a_2 xz + e_3 a_3 yz.$$

Use the bounds from Steps I and II: $x(t) \le K_1^*,\ y(t) \le K_2^*$. Thus

$$\dot{z} \le r_3 z(1 - \frac{z}{K_3}) + (e_2 a_2 K_1^* + e_3 a_3 K_2^*)z.$$

Set

$$C := r_3 + e_2 a_2 K_1^* + e_3 a_3 K_2^*,\ D := \frac{r_3}{K_3},$$

So

$$\dot{z} \le Cz - Dz^{2.}$$

Again, by scalar comparison with the logistic-type equation $\dot{w} = Cw - Dw^2$ with $w(0) = z_0$. We obtain

$$z(t) \le w(t) \le max\{z0, C / D\} =: K_3^* < \infty.$$

Therefore, by invariance/uniqueness arguments, the nonnegative orthant is positively invariant:

$x(t), y(t), z(t) \ge 0$ for all $t \ge 0$.

By stepwise scalar comparison, each species is bounded above by an explicit logistic solution. Concretely

$0 \le x(t) \le K_1^*,\ 0 \le y(t) \le K_2^*,\ 0 \le z(t) \le K_3^*,$ were

$K_1^* = \max\{x_0, K_1\}$, $K_2^* = \max\{y_0, \dfrac{(r_2 + e_1 a_1 K_1^*)K_2}{r_2}\}$, and $K_3^* = \max\{z_0, \dfrac{CK_3}{r_3}\}$ with C as above

Thus, all solutions remain nonnegative and uniformly bounded for $t \ge 0$

2.3.3 Local Stability of the Equilibrium Points

Let f_1, f_2, f_3 denote the right-hand sides of equation (1), (2) and (3) respectively. The Jacobian $J(x, y, z) = \left(\partial f_i / \partial u_j\right)$ is

$$J(x,y,z)=\begin{pmatrix} r_1(1-\frac{2x}{K_1})-a_1y-a_2z-d_1 & -a_1x & -a_2x \\ e_1a_1y & r_2(1-\frac{2y}{K_2})-a_1x-a_3z-d_2 & -a_3y \\ e_2a_2z & e_3a_3z & r_3(1-\frac{2z}{K_3})-a_1x-a_2y-d_3 \end{pmatrix}$$

Denote entries by a_{ij} when evaluating at a specific equilibrium: $a_{ij}=\frac{\partial f_i}{\partial x_j}$

i. for the trivial solutions

Evaluating the Jacobean at E_0:

$$J\left(E_0\right)=diag\left(r_1-d_1, r_2-d_2, r_3-d_3\right)$$

Thus, the linearized eigenvalues are

$$\lambda_1=r_1-d_1,\ \lambda_2=r_2-d_2,\ \lambda_3=r_3-d_3$$

E_0 is locally asymptotically stable, iff $r_i-d_i<0$ for all i, i.e $r_1<d_1,\ r_2<d_2,\ r_3<d_3$

If any $r_i-d_i>0$ then E_0 is unstable.

ii. Single-species boundary equilibria

Solve scalar nonzero steady states for species i (ignoring others): for species x the nonzero equilibrium satisfies

$$r_1\left(1-\frac{x*}{K_1}\right)=d_1 \Rightarrow x*=K_1\left(1-\frac{d_1}{r_1}\right),$$

which is biologically feasible (positive), iff $r_1>d_1$. Similarly

$$y*=K_2\left(1-\frac{d_2}{r_2}\right),\ z*=K_3\left(1-\frac{d_3}{r_3}\right), \text{ when } r_1>d_1.$$

We examine the Jacobian at $E_x=\left(x^*,0,0\right)$. Evaluating entries:

$$J\left(E_x\right)=\begin{pmatrix} r_1(1-\frac{2x^*}{K_1})-d_1 & -a_1x^* & -a_2x^* \\ 0 & r_2-a_1x^*-d_2 & 0 \\ 0 & 0 & r_3-a_1x^*-d_3 \end{pmatrix}$$

Eigenvalues are the diagonal entries:

$$\lambda_1^{(x)}=-(r_1-d_1),\ \lambda_2^{(x)}=r_2-d_2+e_1a_1x*,\ \lambda_3^{(x)}=r_3-d_3+e_2a_2x*.$$

The eigenvalue $\lambda_1^{(x)}<0$, whenever $r_1>d_1$

E_x are locally asymptotically stable iff all three eigenvalues being negative, i.e.

$$r_1>d_1,\ r_2-d_2+e_1a_1x*<0,\ r_3-d_3+e_2a_2x*<0.$$

Replacing $x* = K_1\left(1-\frac{d_1}{r_1}\right)$ one obtains explicit conditions:

$$r_2 - d_2 + e_1 a_1 K_1\left(1-\frac{d_1}{r_1}\right) < 0, \quad r_3 - d_3 + e_2 a_2 K_1\left(1-\frac{d_1}{r_1}\right) < 0.$$

Biologically these inequalities mean that the resident x-population prevents invasion of y and z.

Analogous statements hold for the equilibria $E_y = \left(0, y^*, 0\right)$ and $E_z = \left(0, 0, z^*\right)$

iii. coexistence equilibrium $E^* = (x^*, y^*, z^*)$

Assume a biologically feasible interior equilibrium $E^* > 0$ exists. Denote the Jacobian evaluated at E^* with $J^* = \left(\alpha_{ij}\right)$ with entries

$$\alpha_{11} = r_1(1-\frac{2x^*}{K_1}) - a_1 y^* - a_2 z^* - d_1$$

$$\alpha_{12} = -a_1 x \ \alpha_{13} = -a_1 x$$

$$\alpha_{21} = e_1 a_1 y \ , \quad \alpha_{22} = r_2(1-\frac{2y}{K_2}) - a_1 x - a_3 z - d_2 \ , \quad \alpha_{23} = - a_3 y$$

$$\alpha_{31} = e_2 a_2 z^* \ , \quad \alpha_{31} = e_2 a_2 z^* \ ,$$

$$\alpha_{33} = r_3(1-\frac{2z^*}{K_3}) - a_1 x^* - a_2 y^* - d_3$$

The characteristic polynomial of $J^* = \left(\alpha_{ij}\right)$ is $\chi(\lambda) = det\left(\lambda I - J^*\right) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$

where the coefficients are $a_1 = -tr\left(J^*\right) = -(\alpha_{11} + \alpha_{22} + \alpha_{33}),$

$$a_2 = \alpha_{11}\alpha_{22} + \alpha_{11}\alpha_{33} + \alpha_{22}\alpha_{33} - \alpha_{12}\alpha_{21} - \alpha_{13}\alpha_{31} - \alpha_{23}\alpha_{32} \ , \quad a_3 = -\det\left(J^*\right)$$

3rd order Routh–Hurwitz stability criterion.

All roots of $\chi(\lambda)$ have negative real parts (i.e E^* is locally asymptotically stable) iff the following three Routh–Hurwitz conditions hold:

$$a_1 > 0, \ a_2 > 0, \ a_1 a_2 > a_3$$

2.3.4 Sensitivity Analysis

To evaluate the influence of key parameters on the system dynamics, a local sensitivity analysis was conducted. Two biologically important parameters were selected: the predation rate of leopards on hares(a_1) and the conversion efficiency from hares to leopards (e_1). Each parameter was varied within a biologically plausible interval while all other parameters were fixed at their baseline values. The system of differential equations was then numerically integrated using the `solve_ivp` routine in Python. The resulting final population densities were recorded and compared to assess how variations in these parameters affect species persistence and coexistence.

2.3.5 Numerical Simulations

To illustrate the analytical results and explore the dynamical behavior of the proposed three–species predator–prey system, numerical simulations were performed. The system of nonlinear ordinary differential equations defined in equations (1) – (3) was solved numerically using the Runge–Kutta method implemented through the **`solve_ivp`** routine in Python's SciPy library**.**

Baseline parameter values were selected within biologically plausible ranges commonly used in ecological predator–prey modeling studies. The parameter set used for the simulations is given in **Table 1.**

Table 1: Baseline Parameter Values and Sources

Parameter	Description	Value	Units	Source
(r_1)	Intrinsic growth rate of hares	1.8	year^{-1}	Typical small mammal growth rates reported in Mathematical Biology I: An Introduction
(K_1)	Carrying capacity of hares	200	individuals	Ecological population models in Dynamic Models in Biology
(d_1)	Natural mortality rate of hares	0.20	year^{-1}	Predator–prey demographic studies in Elements of Mathematical Ecology
(r_2)	Intrinsic growth rate of leopards	0.15	year^{-1}	Large carnivore demographic studies
(K_2)	Carrying capacity of leopards	50	individuals	Predator population ecology
(d_2)	Natural mortality rate of leopards	0.05	year^{-1}	Carnivore survival estimates
(r_3)	Intrinsic growth rate of lions	0.05	year^{-1}	Large predator reproduction rates
(K_3)	Carrying capacity of lions	30	individuals	Wildlife population ecology models
(d_3)	Natural mortality rate of lions	0.02	year^{-1}	Lion demographic data
(a_1)	Predation rate of leopards on hares	0.02	Individual^{-1} year^{-1}	Predator–prey interaction models
(a_2)	Predation rate of lions on hares	0.015	Individual^{-1} year^{-1}	Three-species food chain models
(a_3)	Predation rate of lions on leopards	0.01	Individual^{-1} year^{-1}	Trophic chain interaction studies
(e_1)	Conversion efficiency (hare → leopard)	0.10	dimensionless	Empirical trophic efficiency (~10%) reported in Fundamentals of Ecology
(e_2)	Conversion efficiency (hare → lion)	0.08	dimensionless	Ecological energy transfer literature
(e_3)	Conversion efficiency (leopard → lion)	0.12	dimensionless	Predator energetic efficiency studies

The baseline parameter values used for the numerical simulations are summarized in **Table 1**. These parameters were selected from ranges commonly reported in ecological and predator–prey modeling literature. In particular, intrinsic growth rates and mortality rates follow typical demographic estimates for small mammals and large carnivores, while conversion efficiencies are assumed to lie within the commonly observed ecological energy transfer range of 5–15% (Odum, 1971; Murray, 2002).

3. Results

In this section, we Simulate the refined three-trophic-level predator-prey model, produce time-series and phase-plane plots, and perform a basic sensitivity analysis for two parameters (predation rate a_1 and conversion efficiency e_1).

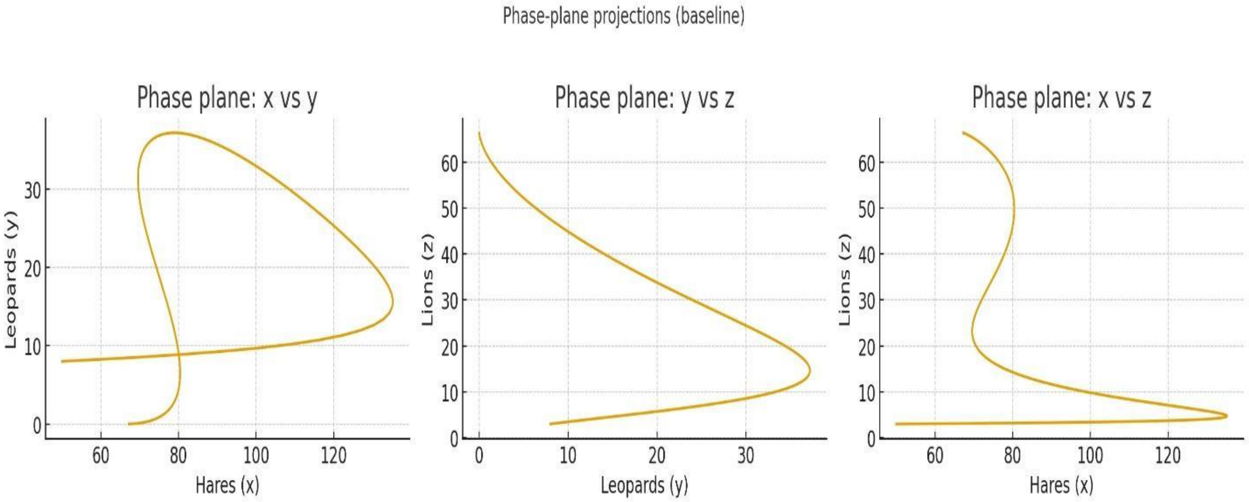

Figure 1. Phase-plane projections: $x\ vs\ y, y\ vs\ z, x\ vs\ z$.

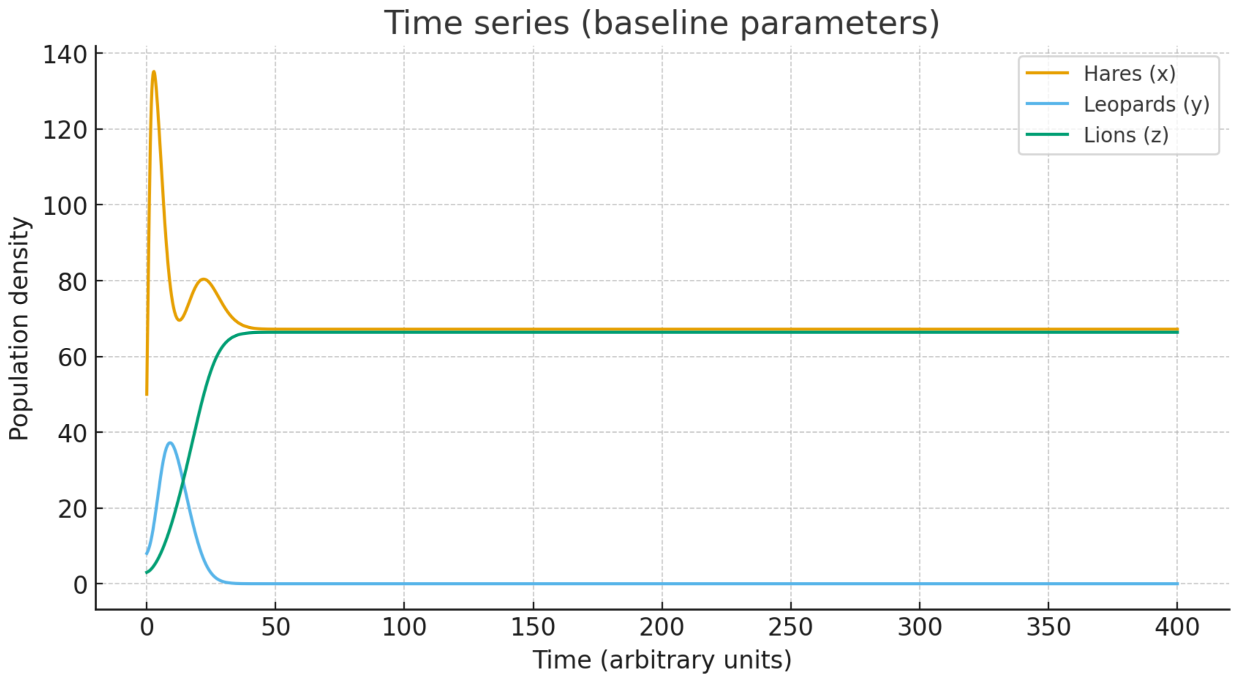

Figure 2. Time-series plot of $x(t)$ (hares), $y(t)$ (leopards), $z(t)$ (lions) — baseline parameters

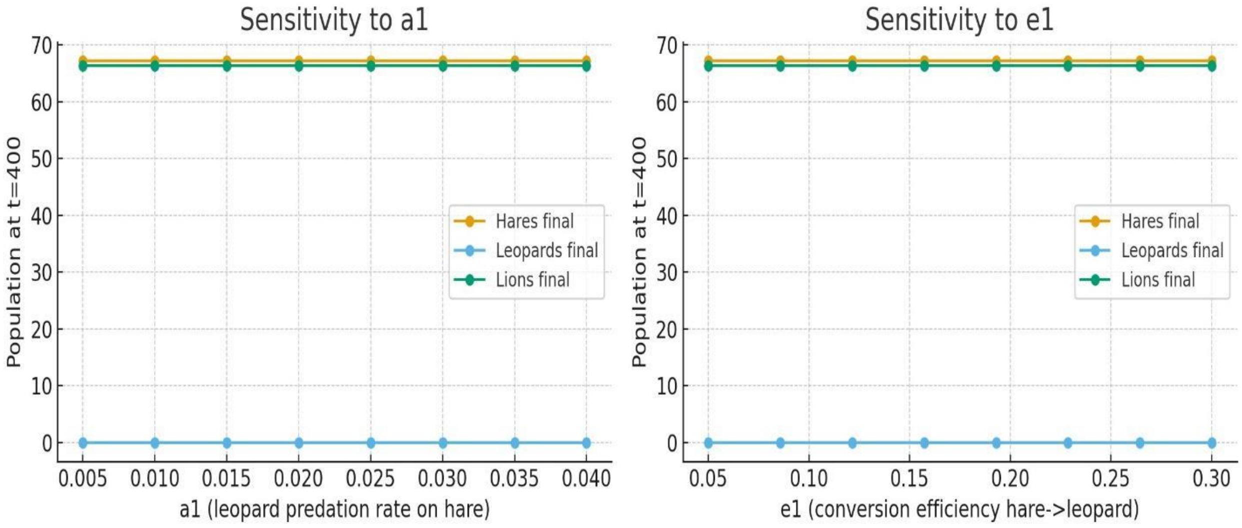

Figure 3. Sensitivity analysis: end-of-simulation populations (at $t = 400$) while varying: a_1 from 0.005 to 0.04, and e_1 from 0.05 to 0.30.

4. Discussion of Results

This work established a refined three-trophic-level predator-prey model that sheds new light on the interactions between lions (top predator), hares (prey), and leopards (intermediate predator/prey). The system is still mathematically constrained, ecologically realistic, and biologically possible by adding logistic growth and conversion efficiency. The combination of numerical and analytical results reveals important ecological processes that control species population stability and coexistence.

The time-series simulations with baseline parameters that show constrained oscillatory coexistence among the three species are shown in Figure 2. Predation pressure reduces hare abundance as the leopard population grows, however at first the prey (hares) grows quickly because of their high intrinsic rate $(r_1 = 1.8)$. Due to its reliance on both prey species and low intrinsic rate $(r_3 = 0.05)$., the lion population reacts more slowly. The system eventually settles around a quasi-steady oscillatory equilibrium, which is characteristic of many natural food chains and in which all populations live with sporadic oscillations.

The coexistence equilibrium is dynamically stable, as seen in Figure 1, where the phase-plane trajectories$(x - y, y - z\ )$ and $(x - z)$ form closed or spiral orbits approaching an attractor. These trends are in line with ecological findings indicating, in the presence of self-limiting effects, predator-prey systems typically display damped oscillations toward equilibrium as opposed to prolonged limit cycles.

Sensitivity analysis was used to examine the effects of changes in the leopard's conversion efficiency and rate of predation on hares on long-term population densities (Figure 3). The findings show that:

There is an ideal threshold beyond which excessive predation destabilizes both predator and prey populations. Beyond that threshold, increasing predation pressure on hares' results in a dramatic fall in hare population and a moderate increase in leopard density. Because of the food chain's top-down dependence, lions also undergo a secondary decrease when hares become too scarce.

Since lions eat both hares and leopards, increasing the energy transfer efficiency from hares to leopards improves the growth and survival of leopards, which in turn lowers the quantity of hares but indirectly supports a greater lion population. Nevertheless, the system exhibits rhythmic amplification at high e1 values (>0.25), indicating the beginning of a possible Hopf-type bifurcation. Overall, the results of the sensitivity analysis show that the efficiency of energy transfer and the intensity of predation are important control elements that decide whether the ecosystem is stable or not.

5. Conclusion

In order to describe the ecological interactions between hares (prey), leopards (intermediate predator/prey), and lions (top predator), this study created and examined an improved three-trophic-level predator–prey model. Logistic growth, energetic conversion efficiency, and natural mortality variables were added to the traditional Lotka–Volterra framework to modify it and create a mathematically sound and physiologically realistic model.

The comparison theorem was used in analytical studies to prove the positivity and boundedness of all state variables, guaranteeing that population trajectories are nonnegative and limited over time. Clear circumstances under which the system achieves asymptotic stability were found by the local stability analysis using the Routh–Hurwitz criterion and

Jacobian matrix, especially when self-regulating processes take precedence over interspecific interactions.

All three species can coexist in a stable biological regime when predation and conversion efficiencies are moderate, as shown by numerical simulations that validated the theoretical predictions. The sensitivity analysis showed that the predation rate and conversion efficiency are important factors influencing system stability, while the phase-plane trajectories showed the existence of damped oscillations that were getting close to equilibrium. The system may become unstable due to too high predation or energy conversion, which could result in population collapses or long-lasting oscillations.

From an ecological perspective, the improved model emphasizes the need for balance between environmental carrying capacity, predation pressure, and reproductive growth, and it mirrors the natural energy flow in a trophic chain. These results are in line with actual ecological evidence indicating density-dependent restrictions that prevent overexploitation of prey populations and moderate energy transfer efficiencies (usually 5–15%) govern healthy ecosystems.

6. Recommendations

Future studies should extend this model to include climate-dependent parameters, such as temperature-driven growth rates or rainfall-influenced carrying capacities. Such extensions can help understand the impact of climate change on food-chain stability, particularly for temperature-sensitive species.

Real-world ecological systems experience random fluctuations and memory effects. Introducing stochastic perturbations or fractional derivatives can provide a deeper understanding of population resilience and long-term persistence under uncertainty.

The results suggest that sensitivity coefficients of key parameters (e.g.,a_1, e_1, r_1) can serve as early-warning indicators for ecological imbalance. Future work should quantify these thresholds for practical wildlife management applications.

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