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Rank, Subdegrees, and Primitivity in Product-Action Wreath Products Acting on the Cartesian Power of a Base Set

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ABSTRACT

We study the product action of the wreath product $G \wr S_m$ on the Cartesian power X^m, where $G \leq S_n$ is a finite transitive permutation group of $rank\ r$. A complete combinatorial description of the suborbit structure is obtained by showing that suborbits correspond bijectively to weak compositions of m determined by the G_1 −orbit decomposition of X. This yields the explicit rank formula. $rank(G \wr S_m) = \binom{m+r-1}{r-1}$ together with closed subdegrees expressions given by multinomial coefficients weighted by stabilizer orbit sizes. Consequently, orbit enumeration for product action wreath products reduces to classical stars-and-bars counting and multinomial expansion. A unified primitivity criterion is established. $G \wr S_m$ acting in product action is primitive if and only if G is primitive and non-regular (with $m \geq 2$). Specializations to symmetric, alternating, cyclic, and dihedral groups produce explicit rank polynomials, concrete subdegrees formulas, and a complete block classification in each case. These results provide a unified enumerative and structural framework for wreath product actions, connecting permutation group theory, multinomial combinatorics, and algebraic graph theory.

1. Introduction

The study of finite permutation groups and it associated combinatorial structures is a central theme in algebraic combinatorics and finite group theory. A classical approach to understanding a transitive action $G \curvearrowright \Omega$ is through the analysis of its orbitals, i.e., the orbits of G on ordered pairs $\Omega \times \Omega$. The orbits of a point stabilizer G_α on Ω are called $suborbits$, and the graphs induce is $suborbital\ graphs$ which encode deep structural properties of the underlying action, including rank, connectivity, and regularity Cameron (1999).

Permutation groups frequently arise as wreath products acting in product action on Cartesian powers of finite sets. Given two groups $G < Sym(\Gamma)$ and $H < Sym(\Delta)$, the wreath product $G \wr H$ yields a natural permutation group on Γ^Δ, incorporating both individual group action on components and permutation of the coordinates by H Praeger (2011). This product action is a cornerstone of modern permutation group theory and has been instrumental in the description and classification of primitive and imprimitive groups Praeger (2011), as well as in the study of codes and graph symmetry Grech (2019).

Furthermore, wreath products of cyclic and symmetric groups appear in the enumeration of combinatorial structures and algebraic properties of symmetric functions Tout (2021).

Recent research on wreath product actions has focused on statistical properties of group elements, such as the prevalence of derangements in power actions, as well as combinatorial and structural aspects of suborbital configurations Arumugam (2024). For example, the suborbital graphs of direct products of the symmetric group acting on Cartesian products have been constructed and analysed, demonstrating explicit suborbit lengths, girth, connectivity, and regularity Muriuki (2024). More general structural results on transitivity and orbit decomposition for wreath products with symmetric groups have

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been developed, including connections to character theory and combinatorial design Klawuhn (2024).
In the context of algebraic graph theory, the automorphism groups of composed or product graphs often contain wreath products, and the interplay between graph structure and group symmetry has led to new insights and open problems Grech (2019). These developments motivate a unified analysis of suborbital graphs arising from wreath product actions of classical permutation groups.
In this study, we investigate the suborbital graphs associated with wreath product actions of the symmetric group S_n, the Alternating group A_n, the cyclic group C_n, and the dihedral group D_{2n} on the Cartesian power n^m with $n, m \geq 2$.
The wreath product $G \wr S_m$ acts on n^m via:

$$(g_1, \cdots, g_m: \sigma): (\omega_1, \cdots, \omega_m) \to \left(\omega_{\sigma^{-1}(1)}^{g_{\sigma^{-1}(1)}}, \cdots, \omega_{\sigma^{-1}(m)}^{g_{\sigma^{-1}(m)}}\right)$$

where each $g_i \in G$ and $\sigma \in S_m$. This action blends coordinate permutations with component wise application of G, producing a highly structured family of suborbits.
The combinatorial properties of these suborbital graphs such as the number of suborbits (rank), the lengths of suborbits (subdegrees), the self-pairing of orbitals, and structural attributes of the associated graphs (e.g., connectivity, regularity, arc-transitivity) can be traced back to both group-theoretic and combinatorial principles. Moreover, explicit computations in special cases (e.g., direct product of symmetric groups) show that suborbital ranks are bounded and the associated graphs admit significant regularity properties Muriuki (2024). The interplay between base group structure (abelian and non-abelian, cyclic and dihedral) and the wreath products coordinate permutation also gives rise to a rich spectrum of suborbital behaviours.

2. Preliminary

Let S_n denote the symmetric group on $\{1, \ldots, n\}$. For $n \geq 2$, S_n acts transitively and primitively on $\{1, \ldots, n\}$ set.
Definition 2.1. The wreath product of $S_n \wr S_m$ is defined as $S_n \wr S_m = S_n^m \rtimes S_m$
Where; S_n^m is the base group consisting of m independent copies of S_n, and S_m permutes the m factors.

Definition 2.2. Let $G \leq S_n$ act transitively on a finite set X with $|X| = n$. Fix $\alpha \in X$. The *rank* of G is the number of orbits of the stabilizer G_α acting on X. Since G is transitive, the stabilizer G_α partitions X into r disjoint orbits:

$$X = \Delta_1 \cup \cdots \cup \Delta_r, \qquad (1)$$

where $\Delta_1 = \{\alpha\}$ and $|\Delta_i| = \delta_i$. The integers δ_i are the subdegrees of G.

3. Main Results

Let $G \leq S_n$ act transitively on $X = \{1, \ldots, n\}$ and

$$W = G \wr S_m = G^m \rtimes S_m$$

act in product action on

$$\Omega = X^m.$$

Definition 3.1. Let G act $transitively$ on a finite set Ω. Then, we define the rank of the action as the number of orbits of a point stabilizer G_α on Ω.
These orbits of G_α are called *suborbits*, and its sizes are the *subdegrees*.
Fix the base point

$$\alpha = (1,1, \ldots, 1) \in \Omega.$$

Then an element

$$(\sigma_1, \ldots, \sigma_m; \tau) \in G$$

stabilizes α if and only if:

i. $\tau \in S_m$ is arbitrary,
ii. each σ_i fixes the point $1 \in \{1, \ldots, n\}$.

Thus,

$$G_\alpha \cong (S_{n-1})^m \wr S_m$$

For $x = (x_1, \ldots, x_m) \in \Omega$, define

$$t(x) = |\{i : x_i \neq 1\}|.$$

Let G have rank r on X. Thus G_1 has exactly r orbits on X. Denote these by

$$\Delta_1 = \{1\}, \Delta_2 \cdots \Delta_r.$$

Lemma 3.1 (Transitivity of Product Action)**.** Let $G \leq S_n$ be transitive on $X = \{1, \ldots, n\}$. Then $G \wr S_m$ acts transitively on $\Omega = X^m$.

Proof. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be in Ω.
Since G is transitive on X, for each i there exists $g_i \in G$ such that

$$g_i(x_i) = y_i$$

Then

$$(g_1, \ldots, g_m; 1). x = y$$

Hence the action is transitive.

□

Lemma 3.2 (Stabilizer Structure)**.**
Let $\alpha = (1, \ldots, 1)$.
Then,

$$W_\alpha = (G_i)^m \wr S_m$$

Proof. Let

$$(g_1, \ldots, g_m; \tau) \in W$$

Using product action,

$$(g_1, \ldots, g_m; \tau) \cdot (1, \ldots, 1) = (g_1(1), \ldots, g_m(1)).$$

Thus, stabilization requires $g_i(1) = 1$ for all *i*, so $g_i \in G_1$. No condition is imposed on τ. □

Lemma 3.3 (Orbit Characterization)**.** Let G have rank r on X. Let $\Delta_1, \cdots, \Delta_r$ be the $G_1 - orbits$. Two points $x, y \in \Omega$ lie in the same $W_\alpha - orbit$ if and only if

$$|\{i: x_i \in \Delta_j\}| = |\{i: y_i \in \Delta_j\}| \ \ for\ all\ j.$$

Proof. (⇒) The stabilizer preserves coordinate counts since coordinate permutations preserve multiplicities and G_1 preserves membership in each Δ_j.

(⇐) If the multiplicities coincide, first permute coordinates to align indices. Then apply suitable elements of G_1 coordinate wise to match entries. □

Theorem 3.1 (General Rank Formula)**.** *If* G has $rank\ r\ on\ X$, then

$$rank(G \wr S_m) = \binom{m + r - 1}{r - 1}$$

Proof. By lemma 3.3, suborbits correspond bijectively to *r*-tuples $(a_1 \cdots a_r)$ satisfying

$$a_1 + \cdots + a_r = m, \qquad a_j \geq 0.$$

Such tuples are weak compositions of *m* into *r* parts.
By stars-and-bars, the number of weak compositions equals

$$\binom{m + r - 1}{r - 1}$$

Since rank equals the number of suborbits, the result follows. □

Theorem 3.2 (Explicit Subdegrees)**.** Let δ_j=|Δ_j|. The subdegrees corresponding to $(a_1 \cdots\ a_r)$ equals

$$\binom{m}{a_1 \cdots\ a_r} \delta_1^{a_1} \cdots \delta_r^{a_r}.$$

$Proof.$ Fix a composition $(a_1 \cdots\ a_r)$.
Step 1: Choose which coordinates belong to each orbit. The number of ways equals the multinomial coefficient

$$\binom{m}{a_1 \cdots\ a_r}.$$

Step 2: For each coordinate assigned to Δ_j, choose one of its δ_j elements. Total possibilities:

$$\binom{m}{a_1 \cdots\ a_r} \delta_1^{a_1} \cdots \delta_r^{a_r}$$

Since each such point lies in the same suborbit, this equals the subdegrees. □

Theorem 3.3 (Primitivity Criterion)**.** Let G be transitive on X. Then $G \wr S_m$ in product action on X^m is primitive if and only if:

1. G is primitive,
2. G is non-regular,
3. $m \geq 2$.

Proof. (⇒)
Assume $W = G \wr S_m$ is primitive.
1. If G were imprimitive, let B be a nontrivial block system in X. Then,

$$\mathcal{B}^m = \{\mathcal{B}_1 \times \cdots \times \mathcal{B}_m\}$$

forms a nontrivial $W - invariant$ partition of X^m, contradicting primitivity.
Hence G is primitive.
2. If G were regular, then G_1 is trivial. Thus

$$W_\alpha \cong S_m$$

Then the partition

$$\{x \in \Omega : x_1 = c\}$$

is $W - invariant$, yielding imprimitivity. Thus, G must be non-regular.
3. If $m = 1$, then $W \cong G$. Primitivity reduces to G.

(⇐) Assume G primitive and non-regular, $m \geq 2$. Let C be a nontrivial block of W. Project onto coordinates. Using transitivity of S_m, any coordinate plays identical role. Projection argument shows each coordinate projection must be either singleton or entire X. Non-regularity ensures stabilizer is nontrivial, preventing coordinate partitions. Hence only trivial blocks exist.
Therefore, W is primitive. □

3.1 Structure of $S_n \wr S_m$

Theorem 3.4 (Structure of $S_n \wr S_m$)**.**
Let $G = S_n \wr S_m = S_n^m \rtimes S_m$ act on $\Omega = \{1, \ldots, n\}^m$. For $n \geq 3,\ m \geq 2$. Then,

1. The action is Primitive,
2. Has rank m + 1,
3. Has subdegrees $\binom{m}{t}(n-1)^t$

$Proof$. We have that, S_n has rank 2 since

$$(S_n)_1 \text{ has orbits } \{1\}, \{2, \ldots, n\}.$$

Apply general rank formula with $r = 2$:

$$\binom{m+1}{1} = m + 1.$$

Subdegrees follow from multinomial formula with $\delta_1 = 1$, $\delta_2 = n - 1$.
Primitivity holds because S_n is primitive and non-regular for $n \geq 3$, so apply primitivity criterion.

□

Theorem 3.5 Two points $x, y \in \Omega$ lie in the same $G_\alpha - orbit$ if and only if

$$t(x) = t(y).$$

Proof. The group S_m permutes coordinates transitively. Each copy of S_{n-1} acts transitively on $\{2, \dots, n\}$. Hence only the number of non-1 coordinate is invariant. □

3.2 Structures of $A_n \wr S_m$

Lemma 3.4 (Transitivity of $(A_n)_1$)**.** For $n \geq 4$, the stabilizer $(A_n)_1$ acts transitively on $\{2, \dots, n\}$.
Proof. Let $i, j \in \{2, \dots, n\}$. Since A_n is 2-transitive for $n \geq 4$, there exists $\sigma \in A_n$ such that $\sigma(1) = 1$ and $\sigma(i) = j$.
Thus $(A_n)_1$ is transitive on $\{2, \dots, n\}$. □

Lemma 3.5 (Rank of A_n)**.** For $n \geq 4$, A_n has rank 2. The $A_n - stabilizer$ of 1 has exactly two orbits:

$$\{1\} \text{ and } \{2, \dots, n\},$$

by the lemma 3.4. □

Theorem 3.6 (Structure of $A_n \wr S_m$)**.** For n ≥ 4, the product action of $A_n \wr S_m$ satisfies:

1. Degree n^m,
2. Rank m + 1,
3. Subdegrees
$$d_t = \binom{m}{t} (n-1)^t,$$
4. Primitive action.

Proof. By Lemma 3.5, A_n has rank 2. Apply the general rank formula:

$$rank = \binom{m+2-1}{2-1} = m+1$$

Subdegrees formula follows from the general multinomial formula with $\delta_1 = 1$ and $\delta_2 = n - 1$.
Primitivity holds since A_n is primitive and non-regular. □

3.3 Structure of $C_n \wr S_m$

Lemma 3.6 (Regularity)**.** Assume C_n acts regularly on X. The cyclic group C_n has trivial point stabilizers.

Proof. If $g \in C_n$ fixes a point, then by regularity it fixes all points, so $g = 1$. □

Lemma 3.7 (Stabilizer in the Wreath Product)**.** *Let* $\alpha = (1, \dots, 1) \in \Omega$. Then

$$W_\alpha \cong S_m$$

Proof. Since C_n has trivial stabilizers, each coordinate stabilizer is trivial. Thus, only the top group S_m remains. □

Theorem 3.7 (Complete Block Classification for $C_n \wr S_m$)**.** Let $W = C_n \wr S_m$ in product action on X^m. Then every nontrivial block system arises from a coordinate partition. More precisely, all nontrivial blocks are unions of sets of the form

$$B_{I,c} = \{x \in \Omega : x_i = c_i \text{ for all } i \in I\}.$$

where $I \subseteq \{1, \dots, m\}$ is nonempty.

Proof. Since $W_\alpha \cong S_m$, suborbits correspond to orbits of S_m on coordinate positions.

Thus, any block must be invariant under coordinate permutations.
Let B be a nontrivial block system. Projection onto coordinates shows that if a coordinate value is fixed inside a block, then this condition must hold for all coordinates in the same $S_m - orbit$.
Hence blocks are determined solely by fixing coordinate positions. No further refinement is possible because the base group acts regularly.
Thus, blocks correspond exactly to coordinate constraints, and W is imprimitive. □

Corollary 3.1. *The action of* $C_n \wr S_m$ is never primitive for $m \geq 2$.

Example 3.7 ($C_3 \wr S_2$). *Let* $\Omega = \{1,2,3\}^2$. *A nontrivial block system is:*

$$\{(1,1),(1,2),(1,3)\},\{(2,1),(2,2),(2,3)\},\{(3,1),(3,2),(3,3)\}.$$

These are coordinate fibres determined by fixing the first coordinate. All blocks arise from such coordinate conditions.

Lemma 3.8 (Regularity of C_n)**.** Assume C_n acts regularly on $\{1,\ldots,n\}$. Then, the stabilizer of any point in C_n is trivial.

Proof. If $g \in C_n$ fixes a point, then by regularity it must be the identity. □

Lemma 3.9 (Rank of C_n)**.** The cyclic group C_n has rank n.

Proof. Since stabilizers are trivial, each element forms its own orbit under G_1.
Thus, the number of G_1-orbits equals n. □

Theorem 3.8 (Structure of $C_n \wr S_m$)**.** *The product action of* $C_n \wr S_m$ *has:*

1. Rank
$$\binom{m+n-1}{n-1}$$
2. *Subdegrees*
$$\frac{m!}{a_1!\cdots a_n!}$$
3. *Imprimitive action.*

Proof. Apply general rank formula with $r = n$.
Subdegrees follow from multinomial expansion since $\delta_i = 1$ for all j.
For imprimitivity: Define partition

$$B_k = \{x \in \Omega: x_1 = k\}$$

This partition is preserved, so the action is imprimitive. □

Theorem 3.9. *Let* $W = C_n \wr S_m$ act in product action on X^m. Then every nontrivial block system arises from coordinate constraints. In particular, W is imprimitive for $m \geq 2$.

$Proof.$ Since C_n acts regularly on X, its point stabilizers are trivial.
$Fix\ \alpha = (1,\ldots,1) \in \Omega.$
Then

$$W_\alpha \cong S_m$$

Thus, the only stabilizing action comes from coordinate permutations.
Step 1: Orbit structure of W_α.
Because $W_\alpha = S_m$, its orbits on Ω consist of points having the same multiset of coordinates.
Thus, suborbits correspond to weak compositions

$$(a_1 \cdots a_n), \qquad \sum a_i = m$$

Step 2: Construction of nontrivial blocks.

For any nonempty proper subset $I \subseteq \{1, \ldots, m\}$ and any fixed values $c_i \in X$, define

$$B_{I,c} = \{x \in \Omega : x_i = c_i \text{ for all } i \in I\}.$$

We show this is a block.

Let $W = (g_1, \ldots g_m; \tau) \in W$. Then

$$wB_{I,}c = B_{\tau(I), g(c)}$$

where $g(c)$ denotes coordinatewise images.

Hence either τ(I) = I and $g(c) = c$, so $wB = B$, or the image block is disjoint. Thus $B_{I,}c$ is a block. Since $1 \leq |I| \leq m - 1$, these blocks are nontrivial.

Step 3: Exhaustion of all blocks.

Let B be any nontrivial block containing α.

Because the base group acts regularly, only coordinate permutations can stabilize α.

Projection to each coordinate:

If $x \in B$ and $x_i \neq 1$ for some coordinate i, then applying a base element that changes only coordinate i moves x to any other value at coordinate i.

Hence B cannot restrict values within a coordinate unless that coordinate is fixed entirely.

Therefore, blocks must be defined by fixing entire coordinate positions, not partial value sets.

Thus, all blocks arise from coordinate constraints. Hence W is imprimitive.

□

3.4 Structure of $D_{2n} \wr S_m$

Lemma 3.10 (Stabilizer Structure in D_{2n})**.** Let D_{2n} act naturally on vertices of a regular *n*-gon. Fix vertex 1. The stabilizer $(D_{2n})_1$ consists of the identity and the reflection fixing 1.

Proof. The dihedral group consists of rotations and reflections. Only the identity rotation fixes 1. Exactly one reflection fixes vertex 1. Thus, the stabilizer has order 2. □

Lemma 3.11 (Primitivity of D_{2n})**.** Let D_{2n} act naturally on vertices of a regular $n-gon$, then D_{2n} *is* primitive on X *if and only if* n is *prime*.

Proof. If n is composite, the rotation subgroup preserves cosets of a proper subgroup, yielding nontrivial blocks. If n is prime, the rotation subgroup is cyclic of prime order, hence acts primitively. □

Lemma 3.12 (Rank of D_{2n})**.**

1. If n is odd,

$$r = \frac{n+1}{2}$$

1. If n is even,

$$r = \frac{n}{2} + 1$$

Proof. Under $(D_{2n})_1$, vertices pair symmetrically about axis through 1.

If n odd:

$$1 + \frac{n-1}{2} = \frac{n+1}{2}$$

If n even: one opposite vertex fixed separately, yielding

$$1 + \frac{n-2}{2} + 1 = \frac{n}{2} + 1$$

□

Theorem 3.10 (Structure of $D_{2n} \wr S_m$)**.** Let r be as above. Then:

1. *degree* n^m,
2. *Rank*

$$\binom{m+r-1}{r-1}$$

3. *Subdegrees given by multinomial formula,*
4. *Primitive* iff D_{2n} *is primitive.*

Proof. Apply general rank formula using computed r. Subdegrees follow from general lemma.
Primitivity: D_{2n} is primitive iff n is prime. Apply general primitivity criterion. □

Theorem 3.11. Let $W = D_{2n} \wr S_m$ act in product action on X^m.

1. If n is composite, then W is imprimitive and all blocks arise from lifting base-group blocks coordinate wise.
2. If n is prime, then W is primitive for $m \geq 2$.

Proof.

Case 1: n composite.

Then D_{2n} is imprimitive on X.
Let C be a nontrivial block system of D_{2n}. Define

$$C^m = \{C_1 \times \cdots \times C_m : C_i \in C\}.$$

We verify this is $W - invariant$.
Let $w = (g_1, \ldots g_m; \tau) \in W$. Then

$$w(C_1 \times \cdots \times C_m) = g_1(C_{\tau-1}(1)) \times \ldots \times g_m(C_{\tau-1}(m))$$

Since each C_i is a block of D_{2n}, $g_i(C_{\tau-1}(i))$ is again a block.
Thus, the partition C^m is preserved.
Hence W is imprimitive.

Exhaustion.

Suppose B is any nontrivial block of W. Project onto coordinate j.
Because S_m acts transitively on coordinates, all projections must be identical.
Each projection must be a union of D_{2n}-blocks. Thus, every block arises from coordinate wise lifting.

Case 2: n *prime*.

Then D_{2n} is primitive and non-regular.
We apply the general wreath primitivity criterion:
If G is primitive and non-regular, then $G \wr S_m$ in product action is primitive.
We give a direct argument.
Assume B is a nontrivial block containing α.
Projection argument:
Because S_m acts transitively on coordinates, if B restricts coordinate i, it must restrict all coordinates symmetrically.
If projection to coordinate i is a proper subset of X, then that subset is a block of D_{2n}, contradicting primitivity.
Thus, projection must be either a singleton or all of X.
If singleton for one coordinate, then base stabilizer action (nontrivial since group nonregular) forces collapse to trivial block.
Hence no nontrivial blocks exist.
Therefore, W is primitive. □

Proposition 3.1. The stabilizer of α in W is $W_\alpha \cong (G_1)^m \wr S_m$
where G_1 denotes the stabilizer of 1 in G.

$Proof$. An element of W has the form

$$(g_1 \cdots g_m; \tau).$$

It stabilizes α if and only if

$$(g_1 \cdots g_m; \tau) \cdot (1,\ldots,1) = (1,\ldots,1).$$

By definition of product action,

$$(g_1 \cdots g_m; \tau) \cdot (1,\ldots,1) = (g_1(1) \cdots g_m(1)).$$

Thus, we must have $g_i(1) = 1$ for all i. No restriction is imposed on τ.
Hence each $g_1(1) \in G_1$ and $\tau \in S_m$, giving

$$W_\alpha \cong (G_1)^m \wr S_m$$

□

Theorem 3.12 (General Rank Formula)**.** If G has *rank r on X,* then,

$$rank(G \wr S_m) = \binom{m+r-1}{r-1}$$

Proof. Let $x = (x_1, \ldots x_m) \in \Omega$.

Each coordinate x_i lies in exactly one $G_1 - orbit\ \Delta_j$. Define

$$a_i = |\{i: x_i \in \Delta_j\}|, \qquad j = 1,\ldots,r.$$

Then

$$a_1 + \cdots + a_r = m.$$

The stabilizer W_α acts by:

1. Permuting coordinates via S_m,
2. Acting independently on each coordinate via G_1.

Thus, W_α preserves the numbers a_j. Conversely, if two points have identical $(a_1, \ldots a_m)$, coordinate permutations align positions, and elements of G_1 move entries inside each Δ_j. Hence suborbits correspond exactly to weak compositions

$$a_1 + \cdots + a_r = m.$$

The number of such compositions equals

$$\binom{m+r-1}{r-1}.$$

□

Theorem 3.13 (Subdegrees)**.** *Let* $\delta_j = |\Delta_j|$. *The subdegree corresponding to a composition* $(a_1, \ldots a_m)$ equals

$$\binom{m}{a_1 \cdots a_r} \delta_1^{a_1} \cdots \delta_r^{a_r}.$$

Proof. To construct a point in the suborbit:

1. Choose which a_j coordinates lie in Δ_j:

$$\binom{m}{a_1 \cdots a_r}.$$

2. For each coordinate in Δ_j, choose one of its δ_j possible values.
   Multiplying yields the formula. □

Theorem 3.14 (Full Classification for Product-Action Wreath Products)**.** Let $G \leq S_n$ be transitive and $m \geq 2$. Then $G \wr S_m$ in product action is primitive if and only if:

1. G is primitive,
2. G is non-regular.

In particular:

$C_n \wr S_m$ is always imprimitive,

$D_{2n} \wr S_m$ is primitive $iff\ n$ is prime.

Proof. Combining theorem 3.12 and 3.13 with the general primitivity criterion established.

□

Theorem 3.15 (Computational Verification Theorem)**.** *Let* G be a finite permutation group of $rank\ r$ acting on X. Let

$$W = G \wr S_m = G^m \rtimes S_m$$

act on $\Omega = X^m$.

in product action. Then the following hold:

1. The rank of W equals
$$\binom{m+r-1}{r-1},$$
2. Subdegrees are given by
$$\frac{m!}{a_1! \cdots a_r!} \prod_{i=1}^{r} \delta_i^{a_i}$$
where $(a_1 \cdots a_r)$ is a composition of m.
3. The sum of all subdegrees equals
$$|X|^m.$$
4. The action of W is primitive if and only if G is primitive and non-regular.

Proof. The rank and subdegrees formulas follow from the orbit classification theorem established. The degree identity follows from the multinomial theorem:

$$\sum_{a_1+\cdots+a_r=m} \frac{m!}{a_1! \cdots a_r!} \prod_{i=1}^{r} \delta_i^{a_i} = \left(\sum_{i=1}^{r} \delta_i\right)^m = |X|^m$$

Primitivity follows from the wreath product structure theorem.

□

4. Conclusions

We determined the suborbit structure, rank, and subdegrees of $G \wr S_m$ acting in product action on X^m in terms of the rank and $G_1 - orbit$ structure of the base group G. If G has rank r, then

$$rank(G \wr S_m) = \binom{m+r-1}{r-1}$$

and subdegrees are given explicitly by multinomial coefficients weighted by orbit sizes. Thus, suborbits correspond naturally to weak compositions of m, providing a direct combinatorial interpretation of wreath product rank.

We further established that $G \wr S_m$ is primitive in product action if and only if G is primitive and non-regular with $m \geq 2$. Applications to classical groups yield explicit rank formulas. These results give a unified and explicit description of product-action wreath structures.

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