International Journal of Development Mathematics Vol3 Issue 1Page No. 172 - 183

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals

Hybrid Block Scheme for the Solution of Fifth – Order Initial Value Problems in Ordinary Differential Equations

Fayose F. Ololade[a,*], Ogunrinde, R. Bosede[b], Akinbo, Y. Razaki[c] and Fayose, S. Taiwo[d,e]

[a]Department of Mathematics, The Federal Polytechnic Ado Ekiti, Nigeria.

[b]Department of Mathematics, Ekiti State University, Ado Ekiti, Nigeria

[c]Department of Mathematics and Statistics, Federal Polytechnic Ilaro, Ogun State, Nigeria

[d]Department of Statistics, The Federal Polytechnic Ado Ekiti, Nigeria;

[e]Department of Statistics, Federal University of Technology and Environmental Sciences, Iyin Ekiti, Nigeria

ARTICLE INFO

Article history:
Received 02 January 2026
Received in revised form 10 March 2026
Accepted 21 March 2026

Keywords:
Hybrid Block Method, Fifth Order ODEs, Initial Conditions, Linear Fifth Order Problem, Nonlinear Fifth Order Problem

MSC 2020 Subject classification:
65LO6

ABSTRACT

This study proposes a novel hybrid block scheme for the direct solution of fifth – order ordinary differential equations (ODEs) with initial conditions, thereby eliminating the computational burden associated with reduction to systems of first – order equations. Power series was used as the basis function for the development of the method. The approximate solution was obtained from the basis function and interpolated at some selected grid points. The fifth derivative of the approximate solution was collocated at all the grid points. This system was solved to determine the unknown parameters in the equations. The values of these parameters were substituted into the basis function to give a One – Step method with continuous coefficients. The discrete schemes obtained from the continuous method and its first, second, third and fourth derivatives were combined and implemented as a block method. The properties of the method were examined. The method was found to be zero stable, consistent and convergent. The method was tested with linear and nonlinear fifth – order problems to confirm its accuracy. Numerical results from this test revealed the method is efficient and it compares favourably with existing methods in the literature.

1. Introduction

In this paper, we considered the method of approximate solution of the fifth order ordinary differential equations of initial value problem of the form

$$y^{(v)} = f\left(x, y, y', y'', y''', y^{(iv)}\right), y^{(i)} = y_i, i = 0,1,2,3,4 \qquad (1)$$

Where x_n , is the initial point, y_n is the solution at x_n, f is continuous within the interval of integration. Equation (1) is of interest to researchers because of its wider relevance in engineering, control theory and other real life problem, hence the study of the methods of its solution.

The conventional method of solving (1) is to reduce it to a system of first order differential equation (Awoyemi, 2005; Awoyemi *et al.,* 2009; Adoghe and Omole, 2019). The reduction of (1) to a system of first order equations leads to

* Corresponding author. Tel.:+2348138220071
E-mail address: lolade.fayose12@gmail.com (Fayose O. F.)
https://doi.org/10.62054/ijdm/0301.14

serious computational burden as well as wastage of computer and human efforts. However, these setbacks have been addressed by some researchers (Bun and Vasil'yer, 1992), (Henrici, 1962), (Yap and Ismail, 2015), (Jain *et al.*, 2007), (Osa and Olaoluwa, 2019). It has been reported in literature that the direct method of solving the above equation is more efficient in terms of speed and accuracy than the method of reduction to a system of first order ODES (Ogunware and Omole, 2020), (Ibijola *et al.*, 2011), (Jena and Mohanty, 2019), (Olabode and Omole, 2015), (Lambert, 1973), (Fayose *et al.*, 2021). Implicit linear multistep methods have better stability properties than explicit methods and are solved using predictor and corrector method. Conversely, several authors have proposed multi – derivative multistep methods for the solution of (1). These methods were implemented in predictor – corrector mode (Awoyemi, 2005; Kayode *et al.*, 2018; Jator and Lee, 2014; Kayode, 2014). Although these methods yield good results but it has a major setback which includes computational burden and the reducing the order of accuracy of the predictors. To correct these set – backs scholars developed block methods (Ibijola *et al.*, 2011; Olabode and Omole, 2015; Osa and Olaoluwa, 2019; Jena and Mohanty, 2019; Ogunware and Omole, 2020). Block method is found to be cost effective in terms of execution and saves time.

While literature shows various numerical methods for first – through fourth – order ordinary differential equations (ODEs), direct solutions for **fifth – order ODEs** remain relatively scarce. This paper introduces a **hybrid block method** designed to solve fifth – order ODEs directly. By implementing the approach in **block mode**, we efficiently generate the desired results without the need for order reduction.

2. Method and Materials

The simple power series of the form

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j \tag{2}$$

is considered as the approximate solution of (1) where $a_j \in R, y \in C^m(a,b), k \in N$

The first, second, third, fourth and fifth derivatives of (2) are

$$y'(x) = \sum_{j=1}^{c+i-1} j a_j x^{j-1} \tag{3}$$

$$y''(x) = \sum_{j=2}^{c+i-1} j(j-1) a_j x^{j-2} \tag{4}$$

$$y'''(x) = \sum_{j=3}^{c+i-1} j(j-1)(j-2) a_j x^{j-3} \tag{5}$$

$$y^{iv}(x) = \sum_{j=4}^{c+i-1} j(j-1)(j-2)(j-3) a_j x^{j-4} \tag{6}$$

$$\boldsymbol{y^{v}(x) = \sum_{j=5}^{c+i-1} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5}} \qquad (\mathbf{7})$$

Substituting (1) in (7) yield

$$\boldsymbol{y^{v}(x) = \sum_{j=0}^{c+i-1} j(j-1)(j-2)(j-3)(j-4)\, a_j x^{j-5} = f(x, y, y^{i}, y^{ii}, y^{iii} y^{iv})} \quad (\mathbf{8})$$

Interpolating (2) at $\boldsymbol{x_{n+i}}$, i= $\frac{1}{7}$,$\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and collocating equation (8) at $\boldsymbol{x_{n+c}}$ c = 0($\frac{1}{7}$)1. This yields a system of non linear equation

$$\boldsymbol{XA = U} \qquad (9)$$

Where A= $\left[a_o,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ a_9,\ a_{10},\ a_{11},\ a_{12}\right]^T$

$$\boldsymbol{U = \left[y_{n+\frac{1}{7}}, y_{n+\frac{2}{7}}, y_{n+\frac{3}{7}}, y_{n+\frac{4}{7}}, y_{n+\frac{5}{7}}, f_n, f_{n+\frac{1}{7}}, f_{n+\frac{2}{7}}, f_{n+\frac{3}{7}}, f_{n+\frac{4}{7}}, f_{n+\frac{5}{7}}, f_{n+\frac{6}{7}}, f_{n+1}\right]^T}$$

$$X=\begin{bmatrix}
1 & x_{n+\frac{1}{7}} & x^2_{n+\frac{1}{7}} & x^3_{n+\frac{1}{7}} & x^4_{n+\frac{1}{7}} & x^5_{n+\frac{1}{7}} & x^6_{n+\frac{1}{7}} & x^7_{n+\frac{1}{7}} & \cdots & x^9_{n+\frac{1}{7}} & x^{10}_{n+\frac{1}{7}} & x^{11}_{n+\frac{1}{7}} & x^{12}_{n+\frac{1}{7}} \\
1 & x_{n+\frac{2}{7}} & x^2_{n+\frac{2}{7}} & x^3_{n+\frac{2}{7}} & x^4_{n+\frac{2}{7}} & x^5_{n+\frac{2}{7}} & x^6_{n+\frac{2}{7}} & x^7_{n+\frac{2}{7}} & \cdots & x^9_{n+\frac{2}{7}} & x^{10}_{n+\frac{2}{7}} & x^{11}_{n+\frac{2}{7}} & x^{12}_{n+\frac{2}{7}} \\
1 & x_{n+\frac{3}{7}} & x^2_{n+\frac{3}{7}} & x^3_{n+\frac{3}{7}} & x^4_{n+\frac{3}{7}} & x^5_{n+\frac{3}{7}} & x^6_{n+\frac{3}{7}} & x^7_{n+\frac{3}{7}} & \cdots & x^9_{n+\frac{3}{7}} & x^{10}_{n+\frac{3}{7}} & x^{11}_{n+\frac{3}{7}} & x^{12}_{n+\frac{3}{7}} \\
1 & x_{n+\frac{4}{7}} & x^2_{n+\frac{4}{7}} & x^3_{n+\frac{4}{7}} & x^4_{n+\frac{4}{7}} & x^5_{n+\frac{4}{7}} & x^6_{n+\frac{4}{7}} & x^7_{n+\frac{4}{7}} & \cdots & x^9_{n+\frac{4}{7}} & x^{10}_{n+\frac{4}{7}} & x^{11}_{n+\frac{4}{7}} & x^{12}_{n+\frac{4}{7}} \\
1 & x_{n+\frac{5}{7}} & x^2_{n+\frac{5}{7}} & x^3_{n+\frac{5}{7}} & x^4_{n+\frac{5}{7}} & x^5_{n+\frac{5}{7}} & x^6_{n+\frac{5}{7}} & x^7_{n+\frac{5}{7}} & \cdots & x^9_{n+\frac{5}{7}} & x^{10}_{n+\frac{5}{7}} & x^{11}_{n+\frac{5}{7}} & x^{12}_{n+\frac{5}{7}} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_n & 2520x^2_n & \cdots & 15120x^4_n & 30240x^5_n & 55440x^5_n & 954040x^7_n \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{1}{7}} & 2520x^2_{n+\frac{1}{7}} & \cdots & 15120x^4_{n+\frac{1}{7}} & 30240x^5_{n+\frac{1}{7}} & 55440x^6_{n+\frac{1}{7}} & 954040x^7_{n+\frac{1}{7}} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{2}{7}} & 2520x^2_{n+\frac{2}{7}} & \cdots & 15120x^4_{n+\frac{2}{7}} & 30240x^5_{n+\frac{2}{7}} & 55440x^6_{n+\frac{2}{7}} & 954040x^7_{n+\frac{2}{7}} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{3}{7}} & 2520x^2_{n+\frac{3}{7}} & \cdots & 15120x^4_{n+\frac{3}{7}} & 30240x^5_{n+\frac{3}{7}} & 55440x^6_{n+\frac{3}{7}} & 954040x^7_{n+\frac{3}{7}} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+4} & 2520x^2_{n+\frac{4}{7}} & \cdots & 15120x^4_{n+\frac{4}{7}} & 30240x^5_{n+\frac{4}{7}} & 55440x^6_{n+\frac{4}{7}} & 954040x^7_{n+\frac{4}{7}} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{5}{7}} & 2520x^2_{n+\frac{5}{7}} & \cdots & 15120x^4_{n+\frac{5}{7}} & 30240x^5_{n+\frac{5}{7}} & 55440x^6_{n+\frac{5}{7}} & 954040x^7_{n+\frac{5}{7}} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{6}{7}} & 2520x^2_{n+\frac{6}{7}} & \cdots & 15120x^4_{n+\frac{6}{7}} & 30240x^5_{n+\frac{6}{7}} & 55440x^6_{n+\frac{6}{7}} & 954040x^7_{n+\frac{6}{7}} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+1} & 2520x^2_{n+1} & \cdots & 15120x^4_{n+1} & 30240x^5_{n+1} & 55440x^6_{n+1} & 954040x^7_{n+1}
\end{bmatrix}$$

Solving equation (9) using Gaussian elimination approach for the unknown constants $a's$ and substituting into (2) gives a hybrid method with continuous coefficient of the form:

$$y(x) = \sum_i \alpha_i y_{n+1} + h^5 \sum_c \beta_c f_{n+c}, i = \tfrac{1}{7}\left(\tfrac{1}{7}\right)\tfrac{5}{7}, c = 0\left(\tfrac{1}{7}\right)1 \qquad (10)$$

where α_i $i = \frac{1}{7}(\frac{1}{7})\frac{5}{7}$ and β_c $c = 0(\frac{1}{7})1$ are continuous coefficients.

The first, second, third and fourth derivatives of (10) is given by

$$y^{(j)}(x) = \frac{1}{h^j}\left(\sum_i \alpha_i y_{n+i} + h^5 \sum_c \beta_c f_{n+c}\right), \text{i} = \frac{1}{7}\left(\frac{1}{7}\right)\frac{5}{7}, \text{c} = 0(\frac{1}{7})1 \qquad (11)$$

By evaluating (10) and (11) at t=0, $\frac{6}{7}$,1 and $t = 0(\frac{1}{7})1$. The output of (10) and (11) are combine together to form the hybrid block methods below which gives equation (12)

$$
\begin{bmatrix}
1&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
0&0&1&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&0&0&1&0&0\\
0&0&0&0&0&1&0\\
0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{7}}\\ y_{n+\frac{2}{7}}\\ y_{n+\frac{3}{7}}\\ y_{n+\frac{4}{7}}\\ y_{n+\frac{5}{7}}\\ y_{n+\frac{6}{7}}\\ y_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix}
y_{n-\frac{1}{7}}\\ y_{n-\frac{2}{7}}\\ y_{n-\frac{3}{7}}\\ y_{n-\frac{4}{7}}\\ y_{n-\frac{5}{7}}\\ y_{n-\frac{6}{7}}\\ y_{n}
\end{bmatrix}
$$

$$
+h\begin{bmatrix}
0&0&0&0&0&0&\frac{1}{7}\\
0&0&0&0&0&0&\frac{2}{7}\\
0&0&0&0&0&0&\frac{3}{7}\\
0&0&0&0&0&0&\frac{4}{7}\\
0&0&0&0&0&0&\frac{5}{7}\\
0&0&0&0&0&0&\frac{6}{7}\\
0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix}
y'_{n-\frac{1}{7}}\\ y'_{n-\frac{2}{7}}\\ y'_{n-\frac{3}{7}}\\ y'_{n-\frac{4}{7}}\\ y'_{n-\frac{5}{7}}\\ y'_{n-\frac{6}{7}}\\ y'_{n}
\end{bmatrix}
+h^2\begin{bmatrix}
0&0&0&0&0&0&\frac{1}{98}\\
0&0&0&0&0&0&\frac{2}{49}\\
0&0&0&0&0&0&\frac{9}{98}\\
0&0&0&0&0&0&\frac{8}{49}\\
0&0&0&0&0&0&\frac{25}{98}\\
0&0&0&0&0&0&\frac{18}{49}\\
0&0&0&0&0&0&\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
y''_{n-\frac{1}{7}}\\ y''_{n-\frac{2}{7}}\\ y''_{n-\frac{3}{7}}\\ y''_{n-\frac{4}{7}}\\ y''_{n-\frac{5}{7}}\\ y''_{n-\frac{6}{7}}\\ y''_{n}
\end{bmatrix}
$$

$$
+h^3\begin{bmatrix}
0&0&0&0&0&0&\frac{1}{2058}\\
0&0&0&0&0&0&\frac{4}{1029}\\
0&0&0&0&0&0&\frac{9}{686}\\
0&0&0&0&0&0&\frac{32}{1029}\\
0&0&0&0&0&0&\frac{125}{2058}\\
0&0&0&0&0&0&\frac{36}{343}\\
0&0&0&0&0&0&\frac{1}{6}
\end{bmatrix}
\begin{bmatrix}
y'''_{n-\frac{1}{7}}\\ y'''_{n-\frac{2}{7}}\\ y'''_{n-\frac{3}{7}}\\ y'''_{n-\frac{4}{7}}\\ y'''_{n-\frac{5}{7}}\\ y'''_{n-\frac{6}{7}}\\ y'''_{n}
\end{bmatrix}
+h^4\begin{bmatrix}
0&0&0&0&0&0&\frac{1}{57624}\\
0&0&0&0&0&0&\frac{2}{7203}\\
0&0&0&0&0&0&\frac{27}{19208}\\
0&0&0&0&0&0&\frac{32}{7203}\\
0&0&0&0&0&0&\frac{625}{57624}\\
0&0&0&0&0&0&\frac{54}{2401}\\
0&0&0&0&0&0&\frac{1}{24}
\end{bmatrix}
\begin{bmatrix}
y''''_{n-\frac{1}{7}}\\ y''''_{n-\frac{2}{7}}\\ y''''_{n-\frac{3}{7}}\\ y''''_{n-\frac{4}{7}}\\ y''''_{n-\frac{5}{7}}\\ y''''_{n-\frac{6}{7}}\\ y''''_{n}
\end{bmatrix}
$$

$$+h^5\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{9917}{29816962560} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9623}{1257903108} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{737073}{16564979200} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{35968}{238239225} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{61946875}{161011597824} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{21249}{25882780} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3541}{2280960} \end{bmatrix}\begin{bmatrix} f_{n-\frac{1}{7}} \\ f_{n-\frac{2}{7}} \\ f_{n-\frac{3}{7}} \\ f_{n-\frac{4}{7}} \\ f_{n-\frac{5}{7}} \\ f_{n-\frac{6}{7}} \\ f_n \end{bmatrix} \qquad (12)$$

$$+h^5\begin{bmatrix} \frac{425111}{1150082841600} & -\frac{126827}{268352663040} & \frac{269923}{536705326080} & \frac{297547}{805057989120} & \frac{94093}{536705326080} & -\frac{64837}{1341763315200} & \frac{1895}{322023195648} \\ \frac{4868}{314475777} & -\frac{8159}{499167900} & \frac{2438}{142943535} & -\frac{78119}{6289515540} & \frac{1024}{174708765} & -\frac{10127}{6289515540} & \frac{1538}{7861894425} \\ \frac{3961467}{33129958400} & -\frac{1690551}{16564979200} & \frac{20817}{189314048} & -\frac{265761}{3312995840}, & \frac{1254609}{33129958400} & -\frac{172341}{16564979200} & \frac{3807}{3011814400} \\ \frac{3735296}{7861894425} & -\frac{855104}{2620631475} & \frac{204032}{524126295} & -\frac{63488}{224625555} & \frac{349952}{2620631475} & -\frac{10688}{291181275} & \frac{35072}{7861894425} \\ \frac{430015625}{322023195648} & -\frac{13328125}{17890177536} & \frac{335703125}{322023195648} & -\frac{10703125}{14637417984}, & \frac{1773125}{5111479296} & \frac{-15359375}{161011597824} & \frac{3734375}{322023195648} \\ \frac{98496}{32353475} & -\frac{36207}{25882780} & \frac{15282}{6470695} & -\frac{39933}{25882780} & \frac{972}{1294139} & -\frac{3807}{18487700} & \frac{162}{6470695} \\ \frac{7511}{1244160} & -\frac{26383}{11404800} & \frac{7217}{1520640} & -\frac{19243}{6842880} & \frac{1337}{912384} & -\frac{889}{2280960} & \frac{3263}{68428800} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{7}} \\ f_{n+\frac{2}{7}} \\ f_{n+\frac{3}{7}} \\ f_{n+\frac{4}{7}} \\ f_{n+\frac{5}{7}} \\ f_{n+\frac{6}{7}} \\ f_n \end{bmatrix}$$

3. Analysis of the Hybrid Block Method

3.1 Order and error Constants of the Hybrid Block Method

According to (Waelah *et al.*, 2011), (Yap and Ismail, 2015), (Adoghe and Omole, 2019) and (Jena and Mohanty, 2019). The order of the new Hybrid Block Method (12) is obtained by using the Taylor series and it is found to be of uniform order eight.

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = C_9 = C_{10} = C_{11} = C_{12}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_{13} = \begin{bmatrix} \frac{132877}{15357487933477754880} \\ -\frac{62449}{2199640198786790760} \\ -\frac{6370407}{34759746351198668800} \\ -\frac{2671232}{4124325372725232675} \\ -\frac{189828125}{112621578177883686912} \\ -\frac{8991}{246873198570360} \\ -\frac{14243}{205128775627760} \end{bmatrix} \quad (13)$$

Error constant $$C_{p+5} = \begin{bmatrix} \frac{132877}{15357487933477754880} \\ -\frac{62449}{2199640198786790760} \\ -\frac{6370407}{34759746351198668800} \\ -\frac{2671232}{4124325372725232675} \\ -\frac{189828125}{112621578177883686912} \\ -\frac{8991}{246873198570360} \\ -\frac{14243}{205128775627760} \end{bmatrix} \quad (14)$$

3.2 Consistency

Definition 3.1:The Hybrid Block Method (12) is said to be consistent if it has an order more than or equal to one i.e. $P \geq 1$. Therefore, the method is consistent (Lambert, 1973; Jain *et al.*, 2007; Yap and Ismail, 2015; Olabode and Omole, 2015).

3.3 Zero Stability

Definition 3.2: The Hybrid Block Method (12) said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1 \quad and \quad if \quad |r_z| = 1$, then the multiplicity of r_z must not greater than six (Jator and Lee, 2014; Ogunware *et al.*, 2015). In order to find the zero – stability of Hybrid Block Method (12), we only considered the first characteristic polynomial of the method according to definition (3.2) as follows

$$\prod(r) = r\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = r^6 (r-1) \quad (15)$$

which implies $r = 0, 0, 0, 0, 0, 0, 1$. Hence the method is zero-stable since $|r_z| \leq 1$.

3.4 Convergence

Theorem (3.1): Consistency and zero stability are sufficient conditions for linear multistep method to be convergent. Since the method (12) is consistent and zero stable, it implies the method is convergent for all point (Lambert, 1973 and Fatunla, 2011).

3.5 Regions of Absolute Stability (RAS)

The absolute stability region of the new method is found according to (Lambert, 1973), (Fatunla, 1988), (Ibijola *et al.*, 2011), (Osa and Olaoluwa, 2019) and (Ogunware and Omole, 2020).

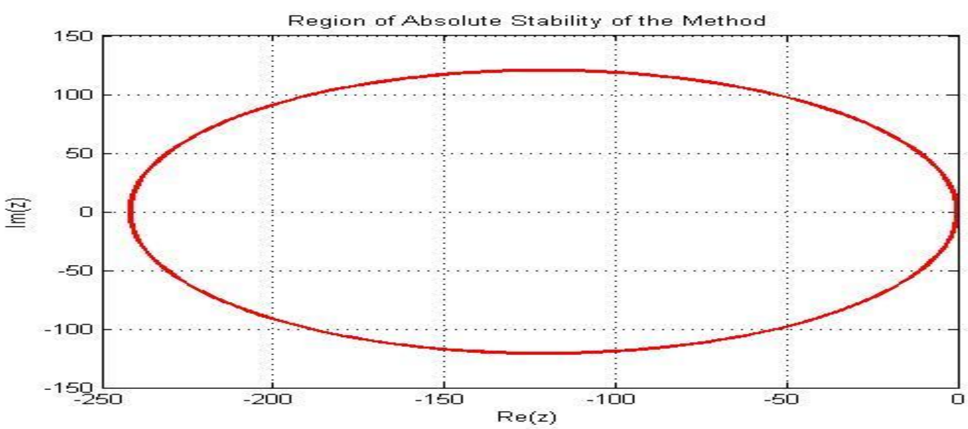

Figure 1: Graph of Region of Absolute Stability of the Hybrid Block Method.

4 Implementation of the Block Method

In this section, we implement our derived method with the aid of MATLAB coding to solve fifth order problems in order to show the level of accuracy and efficiency of the method.

4.1 Numerical Examples

The method is tested on linear and nonlinear fifth order problems to test the accuracy of the proposed methods and our results are compared with the results obtained using existing methods.

The following problems are taken as test problems:

Problem 1: Consider a Non – linear fifth order problem

$y^{v} = 6\left(2(y')^3 + 6yy'y''' + y^2y'''\right),$

$y(1) = 1, y'(1) = -1, y''(1) = 2, y'''(1) = -6, y^{iv}(1) = 24$

$y(x) = \frac{1}{x}$

$y^{v} = 5y''' - 4y',$

$y(0) = 3, y'(0) = -5, y''(0) = 11, y'''(0) = -23, y^{iv}(0) = 47, h = 0.1$

$y(x) = 1 - e^{x} + 3e^{-2x}$

Source: (Kayode, 2014)

Problem 2: Consider a Linear fifth order problem

Source: (Jena and Mohanty, 2019)

The following Notations were used in the tables

X – val	Value of the independent variables where numerical value is taken
Exact – solution	Exact solution at X – val
Computed-solution	Computed solution at X – val
Error	$\lvert Exact\ Solution - Computed\ Solution \rvert$

Table 1: Result of Problem 1 using the developed method with h = 0.1

x-values	Exact solution	Computed solution	Error in our Method
1.100000	0.909090909090907950	0.909090909090949810	4.185541e-014
1.200000	0.833333333333331370	0.833333333335711800	2.380429e-012
1.300000	0.769230769230766720	0.769230769254996340	2.422962e-011
1.400000	0.714285714285711410	0.714285714407756790	1.220454e-010
1.500000	0.666666666666663520	0.666666667086609040	4.199455e-010
1.600000	0.624999999999996670	0.625000001138331650	1.138335e-009
1.700000	0.588235294117643640	0.588235296739063380	2.621420e-009
1.800000	0.555555555555552030	0.555555560918773670	5.363222e-009
1.900000	0.526315789473680630	0.526315799505716720	1.003204e-008
2.000000	0.499999999999996450	0.500000017493390960	1.749339e-008

Table 2: Result of Problem 2 using the developed method with h = 0.1

x-value	Exact Solution	Computed Solution	Error in our Method
0.100000	2.551354841197985800	2.551354841197988500	2.664535e-015
0.200000	2.192229385028936500	2.192229385029126100	1.896261e-013
0.300000	1.905616687600363100	1.905616687602535600	2.172484e-012
0.400000	1.677666846316027400	1.677666846328145400	1.211808e-011
0.500000	1.497107663801695600	1.497107663847459900	4.576428e-011

0.600000	1.354770999642579600	1.354770999777845900	1.352662e-010
0.700000	1.243205588033408500	1.243205588371373100	3.379645e-010
0.800000	1.156360589866742400	1.156360590614063900	7.473215e-010
0.900000	1.089327004924157900	1.089327006430507400	1.506349e-009
1.000000	1.038126408538393000	1.038126411362454900	2.824062e-009

Table 3: Comparison of errors in the developed method and (Kayode, 2014) using Problem 1 with h = 0.1

x-values	Error in our Method P=8 k=1	Error in (Kayode, 2014) P=7 k=6
1.100000	4.185541e-014	6.147755e-010
1.200000	2.380429e-012	1.711526e-008
1.300000	2.422962e-011	1.186268e-007
1.400000	1.220454e-010	4.597295e-007
1.500000	4.199455e-010	1.299344e-006
1.600000	1.138335e-009	3.013204e-006
1.700000	2.621420e-009	6.103066e-006
1.800000	5.363222e-009	1.120406e-005
1.900000	1.003204e-008	1.909085e-005
2.000000	1.749339e-008	3.068296e-005

Table 4: Comparison of errors in the developed method and (Jena and Mohanty, 2019) using Problem 2 with h = 0.1

x-values	Error in our Method P=8, k=1	Error in (Jena and Mohanty, 2019), p=8, k=6
0.100000	2.664535e-015	3.108624e-015
0.200000	1.896261e-013	2.398081e-014
0.300000	2.172484e-012	6.894929e-012
0.400000	1.211808e-011	1.453641e-010
0.500000	4.576428e-011	1.681372e-009
0.600000	1.352662e-010	1.226419e-008
0.700000	3.379645e-010	6.574432e-008
0.800000	7.473215e-010	2.808989e-007
0.900000	1.506349e-009	1.009538e-006
1.000000	2.824062e-009	3.165451e-006

5. Discussion of Results

The developed method was tested on linear and nonlinear initial value problems of fifth – order ordinary differential equations. The exact and computed solution with the absolute errors of Problems 1 and Problem 2 were presented in Tables 1 and Table 2 using h = 0.1. The comparison of the absolute errors of the proposed method with other authors in literature was made in Table 3 and Table 4. The proposed hybrid block method presented in this work has the order of accuracy to be eight and compares favorably to the method of (Kayode, 2014) and (Jena and Mohanty, 2019). The performance of the developed method can be traced to the introduction of multiple hybrid points.

6. Conclusion

We have proposed a new hybrid block method for the numerical solution of fifth order initial value problems in ordinary differential equations in this paper. The method is consistent, convergent and zero stable. The method derived efficiently solved linear and nonlinear fifth order IVPS as can be seen in the low error constant and gives better approximation than the existing methods in literature. The exact solution, computed solution and the absolute error of problem 1 and 2 are displays in Table 1 and Table 2. Presentations of the comparison of our method are shown in Table 3 and 4. It is very clear that our method has a minimal error when compared with other existing methods in the literature. On the other hand, more analysis might broaden the approach to directly solving sixth and seventh order ordinary differential equations or sixth and seventh boundary value problems in view of the advantages of this method. Other basis functions in place of the power series polynomial used in this work are also suggested for further research in this direction. Introduction of more hybrid points for direct solution of higher order ordinary differential equations is also recommended

Acknowledgements

The authors acknowledged the contribution of lecturers in the Department of Mathematics, Federal University of Technology and Environmental Sciences, Iyin Ekiti, Landmark University and Ekiti State University respectively.

Author Contributions

O.F.F.: Original Draft Preparation, Conceptualization, Methodology, Results Discussion and Review and Editing. All authors have read and agreed to the published version of the manuscript.

R.B.O.: Conceptualization, Methodology and Original Draft Preparation.

T.S.F.: Results Discussion, Review and Editing.

Y.R.A: Methodology and Result Discussion.

Declaration of Competing Interest

The authors declared that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this journal. Finally, the authors agreed that all information supplied here are real and original.

Funding

No financial assistance was received for the study.

REFERENCES

Adoghe, L. O. & Omole, E. O. (2019). A Two – Step Hybrid Block Method for the Numerical Integration of Higher Order Initial Value Problems of Ordinary Differential Equations. *World Scientific News*, *118*, 236 – 250.

Awoyemi, D. O. (2005). An Algorithm Collocation Methods for Direct Solution of Special and General Fourth Order Initial Value Problems of Ordinary Differential Equation. *Journal of the Nigeria Association of Mathematical Physics*, *6*, 211 – 238.

Awoyemi, D. O., Adesanya, A. O. & Ogunyebi, S. N. (2009). Construction of Self – Starting Numerov Method for the Solutions of Initial Value Problem of General Order Differential Equations. *Journal of Numerical Mathematics*, *4*(2), 267 – 278.

Bun, R. A. & Vasil'yer, Y. D. (1992). A Numerical Method for Solving Differential Equations of Any Order. *Computational Mathematics and Mathematical Physics*, *32*, 317 – 330.

Fatunla, S. O. (1988). *Numerical Methods for Initial Value Problems in Ordinary Differential Equations*. Academic Press.

Fatunla, S. O. (2011). Block Method for Second Order IVPs. *International Journal of Computer Mathematics*, *41*(1 – 2), 55 – 63.

Fayose, O. F., Ogunrinde, R. B. & Fayose, T. S. (2021). Optimization of One – Step Hybrid Method for Direct Solution of Fifth Order Ordinary Differential Equations of Initial Value Problems. *World Journal of Advanced Research and Reviews*, *9*(1), 239 – 249.

Henrici, P. (1962). *Discrete Variable Method in Ordinary Differential Equations*. John Wiley and Sons.

Ibijola, E. A., Skwame, Y. & Kumleng, G. (2011). Formulation of Hybrid Block of Higher Step Sizes, through the Continuous Multi – Step Collation. *American Journal of Scientific and Industrial Research*, *2*(2), 161 – 173.

Jain, M. K., Iyengar, S. R. K. & Jain, R. K. (2007). *Numerical Methods for Scientific and Engineering Computation* (5th edition). New Age International.

Jator, S. N. & Lee, L. (2014). Implementing a Seventh – Order Linear Multistep Method in a Predictor – Corrector Mode or Block Mode: Which is More Efficient for the General Second Order Initial Value Problem. *SpringerPlus*, *3*(1), 1 – 8.

Jena, S. R. & Mohanty, M. (2019). Numerical Treatment for ODE (Fifth Order). *International Journal on Emerging Technologies*, *10*(4), 191 – 196.

Kayode, S. J. (2014). Symmetric Implicit Multi – derivative Numerical Integrators for Direct Solution of Fifth – Order Differential Equations. *Thammasat International Journal of Science and Technology*, *19*(2), 1 – 8.

Kayode, S. J., Ige, S. O., Obarhua, F. O. & Omole, E. O. (2018). An Order Six Stormer – Cowell – Type Method for Solving Directly Higher Order Ordinary Differential Equations. *Asian Research Journal of Mathematics, 11*(3), 1 – 12.

Lambert, J. D. (1973). *Computational Methods in Ordinary Differential Equations*. John Wiley and Sons.

Olabode, B. T. & Omole, E. O. (2015). Implicit Hybrid Block Numerov – Type Methods for the Direct Solution of Fourth Order Ordinary Differential Equations. *American Journal of Computational and Applied Mathematics*, *5*(5), 129 – 139.

Ogunware, B. G. & Omole, E. O. (2020). A Class of Irrational Linear Multistep Block Method for the Direct Numerical Solution of Third Order Ordinary Differential Equations. *Turkish Journal of Analysis and Number Theory*, *8*(2), 21 – 27.

Ogunware, B. G., Omole, E. & Olayemi, O. O. (2015). Hybrid and Non – Hybrid Schemes for Solving Third Order ODEs using Block Methods as Predictors. *Journal of Mathematical Theory and Modeling*, *5*(3), 10 – 23.

Osa, A. L. & Olaoluwa, O. E. (2019). A Fifth – Fourth Continuous Block Implicit Hybrid Method for the Solution of Third Order Initial Value Problems in Ordinary Differential Equations. *Journal of Applied and Computational Mathematics*, *8*(3), 50 – 57.

Waeleh, N., Majid, Z. A. & Ismail, F. (2011). A New Algorithm for Solving Higher Order IVPs of ODEs. *Journal of Applied Mathematical Sciences*, *5*(26), 2795 – 2805.

Yap, L. K. & Ismail, F. (2015). Block Hybrid Collocation Method with Application to Fourth Order Differential Equations. *Mathematical Problems in Engineering*, Article 561489.