



INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: <https://ijdm.org.ng/index.php/Journals>



Adams-Type Block Hybrid Method for Direct Solutions of Second-Order Differential Equations

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ARTICLE INFO

Article history:

Received 03 February 2026

Received in revised form 15 March 2026

Accepted 20 March 2026

Keywords:

Accuracy, Adams-type method, block hybrid method, second-order

MSC 2020 Subject classification:

65L05, 65L06

ABSTRACT

In this study, an eighth-order Adams-Type Block Hybrid Method (ATBHM) is derived for the solutions of second-order ordinary differential equations. The two-step method was developed using collocation and interpolation techniques, using five off-step points within the interval of integration to enhance accuracy. In contrast to most existing methods that reduce higher-order differential equations to first-order systems, often resulting in loss of inherent characteristics, the proposed ATBHM maintains essential properties of the equations by solving them directly. Basic stability and convergence properties of the ATBHM were also analysed to check its computational reliability in handling second-order ordinary differential equations. The results of the analysis showed that the method is consistent, zero-stable and convergent. Additionally, the method was tested on some problems and the results obtained show that it performed better than existing methods with which we compared our results.

1. Introduction

Second-order differential equations are very important in modelling real-life phenomenon across diverse fields. This explains why derivations of numerical methods for the solution of such problems are very important. Over the years, a lot of methods have been formulated to solve such differential equations. It has however been observed that some of such methods have some setbacks such as low rate of convergence, smaller (or finite) stability regions, inefficiency in terms of central processing unit time, to mention a few. It is in view of these challenges that an ATBHM is proposed in this study for the solution of second-order differential equations of the form,

$$y'' = f(t, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_0' \quad (1)$$

on the interval $x \in [x_0, x_N]$. It is hoped that the proposed method will address some of these setbacks. Suffice to state that second-order differential equations are applied in various areas of human endeavours. These include vibration analysis, population dynamics, mechanical vibrations, control systems, aerodynamic drag, planetary motion and quantum mechanics. Others are weak signal detection and phase lags, (Adeyeye & Zurmi, 2016; Anemee & Latif, 2022; Onyekonwu, Sunday & Ndam, 2025).

Derivation of numerical methods and algorithms for the direct solution of second-order differential equations is paramount. A lot of researchers have proposed various methods to solve such problems. Previous works have demonstrated the effectiveness of hybrid methods, often implemented in a block structure, for directly solving second-order differential equations without converting them into first-order systems. Skwame et al. (2018) proposed a two-step implicit second derivative block method for the solution of initial value problems of general second-order ordinary differential equations. The method, which is A-stable, was found to be consistent, convergent and zero-stable. The Jator et al. (2023) formulated hybrid block method of order fourteen for the solution of second-order initial value problems. The method, implemented in variable-step mode was found to be computationally reliable. Kwari, Sunday & Ndam (2024) derived a four-point integrator for the solutions of second-order oscillatory problems. The integrator

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<https://doi.org/10.62054/ijdm/0301.04>

was proven to be computationally reliable

Sunday *et al.* (2025) developed block hybrid integrators for the solution of second order problems of the form (1). The integrators were consistent, convergent and stable. Other authors that proposed numerical methods for the solution of second-order differential equations are Nazreen & Zanariah (2017), Areo, Adeyanju & Kayode (2020), Obarhua & Kayode (2020), among others. For more details on the solution approach for other types of hybrid block methods, see James, Adesanya & Sunday (2012), Sunday, James, Odekunle & Adesanya (2015), Yakusak & Adeniyi (2022), Kamoh, Dang & Sunday (2024), Sunday, Kolawole, Ibijola & Ogunrinde (2025), Omole *et al.* (2025) and Sunday *et al.* (2026). The remaining part of the paper is composed of derivation of the ATBHM, analyses of basic properties of the method, results and discussion and conclusion.

2. Methodology

2.1 Derivation of the Adams-Type Block Hybrid Method with Five Off-grid Points

Consider the power series of the form,

$$y(x) = \sum_{j=0}^{r+s-1} \tau_j x^j = \tau_0 + \tau_1 x + \tau_2 x^2 + \dots + \tau_{r+s-1} x^{r+s-1} \quad (2)$$

Suppose equation (2) is the approximate solution $y(x)$ to the second-order differential equation of the form (1) for $x \in [x_n, x_{n+2}]$, where r is the number of collocation points, s is the number of interpolation points and $h = x_{n+1} - x_n$ is the step-size. Equation (2) is differentiated twice,

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1)\tau_j x^{j-2} \quad (3)$$

Substituting (3) into (1) gives

$$f(x, y, y') = \sum_{j=2}^{r+s-1} j(j-1)\tau_j x^{j-2} \quad (4)$$

To derive ATBHM, five off-grid points were inserted within the two-step interval of integration $[x_n, x_{n+2}]$, that is

$x_{n+1/4}, x_{n+1/2}, x_{n+3/4}, x_{n+4/3}$ and $x_{n+5/3}$. Interpolating (2) at points $x_{n+l}, l = \frac{4}{3}, \frac{5}{3}$ and the collocating (3) at points

$x_{n+m}, m = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{4}{3}, \frac{5}{3}, 2$ give a system of $(r+s)$ equations of the form,

$$\sum_{j=0}^{r+s-1} \tau_j x^j = y_{n+l}, l = \frac{4}{3}, \frac{5}{3} \quad (5)$$

$$\sum_{j=2}^{r+s-1} j(j-1)\tau_j x^{j-2} = f_{n+m}, m = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{4}{3}, \frac{5}{3}, 2 \quad (6)$$

The system of $(r+s)$ equations in (5) and (6) is then written compactly in matrix form as

$$XT = U \quad (7)$$

where

$$T = [\tau_0 \ \tau_1 \ \tau_2 \ \tau_3 \ \tau_4 \ \tau_5 \ \tau_6 \ \tau_7 \ \tau_8 \ \tau_9]^T$$

$$U = \left[y_{n+\frac{4}{3}} \ y_{n+\frac{5}{3}} \ f_n \ f_{n+\frac{1}{4}} \ f_{n+\frac{1}{2}} \ f_{n+\frac{3}{4}} \ f_{n+1} \ f_{n+\frac{4}{3}} \ f_{n+\frac{5}{3}} \ f_{n+2} \right]^T$$

and

$$X = \begin{bmatrix} 1 & x_{n+\frac{3}{4}} & x_{n+\frac{3}{4}}^2 & x_{n+\frac{3}{4}}^3 & x_{n+\frac{3}{4}}^4 & x_{n+\frac{3}{4}}^5 & x_{n+\frac{3}{4}}^6 & x_{n+\frac{3}{4}}^7 & x_{n+\frac{3}{4}}^8 & x_{n+\frac{3}{4}}^9 \\ 1 & x_{n+\frac{5}{3}} & x_{n+\frac{5}{3}}^2 & x_{n+\frac{5}{3}}^3 & x_{n+\frac{5}{3}}^4 & x_{n+\frac{5}{3}}^5 & x_{n+\frac{5}{3}}^6 & x_{n+\frac{5}{3}}^7 & x_{n+\frac{5}{3}}^8 & x_{n+\frac{5}{3}}^9 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12x_{n+\frac{1}{4}}^2 & 20x_{n+\frac{1}{4}}^3 & 30x_{n+\frac{1}{4}}^4 & 42x_{n+\frac{1}{4}}^5 & 56x_{n+\frac{1}{4}}^6 & 72x_{n+\frac{1}{4}}^7 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 & 30x_{n+\frac{1}{2}}^4 & 42x_{n+\frac{1}{2}}^5 & 56x_{n+\frac{1}{2}}^6 & 72x_{n+\frac{1}{2}}^7 \\ 0 & 0 & 2 & 6x_{n+\frac{3}{4}} & 12x_{n+\frac{3}{4}}^2 & 20x_{n+\frac{3}{4}}^3 & 30x_{n+\frac{3}{4}}^4 & 42x_{n+\frac{3}{4}}^5 & 56x_{n+\frac{3}{4}}^6 & 72x_{n+\frac{3}{4}}^7 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 \\ 0 & 0 & 2 & 6x_{n+\frac{4}{3}} & 12x_{n+\frac{4}{3}}^2 & 20x_{n+\frac{4}{3}}^3 & 30x_{n+\frac{4}{3}}^4 & 42x_{n+\frac{4}{3}}^5 & 56x_{n+\frac{4}{3}}^6 & 72x_{n+\frac{4}{3}}^7 \\ 0 & 0 & 2 & 6x_{n+\frac{5}{3}} & 12x_{n+\frac{5}{3}}^2 & 20x_{n+\frac{5}{3}}^3 & 30x_{n+\frac{5}{3}}^4 & 42x_{n+\frac{5}{3}}^5 & 56x_{n+\frac{5}{3}}^6 & 72x_{n+\frac{5}{3}}^7 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 & 72x_{n+2}^7 \end{bmatrix}$$

Solving equation (7) for τ_j 's and substituting into the approximate solution (2) gives a continuous ATBHM with five off-grid points of the form,

$$y(t) = \alpha_{\frac{4}{3}}(t)y_{n+\frac{4}{3}} + \alpha_{\frac{5}{3}}(t)y_{n+\frac{5}{3}} + h^2 \left(\sum_{j=0}^2 \beta_j(t)f_{n+j} + \beta_{\frac{1}{4}}(t)f_{n+\frac{1}{4}} + \beta_{\frac{1}{2}}(t)f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}}(t)f_{n+\frac{3}{4}} + \beta_{\frac{4}{3}}(t)f_{n+\frac{4}{3}} + \beta_{\frac{5}{3}}(t)f_{n+\frac{5}{3}} \right), \quad (8)$$

where

$$\begin{aligned}
\alpha_{\frac{4}{3}}(t) &= 5 - 3t \\
\alpha_{\frac{5}{3}}(t) &= 3t - 4 \\
\beta_0(t) &= -\frac{1}{110224800} \begin{pmatrix} 3674160t^9 - 35429400t^8 + 144298260t^7 - 323785350t^6 + 437776164t^5 \\ -366267825t^4 + 187075980t^3 - 55112400t^2 + 8175961t \\ -378060 \end{pmatrix} \\
\beta_{\frac{1}{4}}(t) &= \frac{256}{1065736035} \begin{pmatrix} 918540t^9 - 8562105t^8 + 33220530t^7 - 69319152t^6 + 83449359t^5 \\ -56796390t^4 + 18370800t^3 - 1570628t + 296880 \end{pmatrix} \\
\beta_{\frac{1}{2}}(t) &= -\frac{8}{24111675} \begin{pmatrix} 18379800t^9 - 16533720t^8 + 61126650t^7 - 119104020t^6 + 129560067t^5 \\ -75167190t^4 + 18370800t^3 + 246893t - 318780 \end{pmatrix} \\
\beta_{\frac{3}{4}}(t) &= \frac{256}{265228425} \begin{pmatrix} 918540t^9 - 7971615t^8 + 28102950t^7 - 51438240t^6 + 51576021t^5 \\ -27096930t^4 + 6123600t^3 - 417296t + 212160 \end{pmatrix} \\
\beta_1(t) &= -\frac{1}{5511240} \begin{pmatrix} 3674160t^9 - 30705480t^8 + 103357620t^7 - 179084628t^6 + 169149141t^5 \\ -84352590t^4 + 18370800t^3 + 1103635t - 1532100 \end{pmatrix} \\
\beta_{\frac{4}{3}}(t) &= \frac{1}{13759200} \begin{pmatrix} 3674160t^9 - 29130840t^8 + 92510100t^7 - 151099830t^6 + 135576504t^5 \\ -64986705t^4 + 13778100t^3 - 4702739t + 5663940 \end{pmatrix} \\
\beta_{\frac{5}{3}}(t) &= -\frac{1}{49480200} \begin{pmatrix} 3674160t^9 - 27556200t^8 + 83062260t^7 - 129973410t^6 + 112842639t^5 \\ -52816050t^4 + 11022480t^3 + 1087141t - 1840860 \end{pmatrix} \\
\beta_2(t) &= \frac{1}{385786800} \begin{pmatrix} 3674160t^9 - 25981560t^8 + 75014100t^7 - 113745870t^6 + 96538554t^5 \\ -44472645t^4 + 9185400t^3 + 61651t - 395460 \end{pmatrix}
\end{aligned} \tag{9}$$

and $t = (x - x_n)/h$. Solving (8) for the independent solution gives the continuous ATBHM of the form,

$$y'(t) = y'_n + h \begin{pmatrix} \sum_{j=0}^2 \sigma_j(t) f_{n+j} + \sigma_{\frac{1}{4}}(t) f_{n+\frac{1}{4}} + \sigma_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} \\ + \sigma_{\frac{3}{4}}(t) f_{n+\frac{3}{4}} + \sigma_{\frac{4}{3}}(t) f_{n+\frac{4}{3}} + \sigma_{\frac{5}{3}}(t) f_{n+\frac{5}{3}} \end{pmatrix} \tag{10}$$

where

$$\left. \begin{aligned}
 \sigma_0(t) &= -\frac{1}{2520} \left(756t^8 - 6480t^7 + 23093t^6 - 44415t^5 + 50043t^4 - 33495t^3 + 12831t^2 - 2520t \right) \\
 \sigma_{\frac{1}{4}}(t) &= \frac{256}{487305} \left(3780t^8 - 31320t^7 + 106330t^6 - 190176t^5 + 190785t^4 - 103880t^3 + 25200t^2 \right) \\
 \sigma_{\frac{1}{2}}(t) &= -\frac{8}{315} \left(216t^8 - 1728t^7 + 5590t^6 - 9336t^5 + 8463t^4 - 3928t^3 + 720t^2 \right) \\
 \sigma_{\frac{3}{4}}(t) &= \frac{256}{24255} \left(756t^8 - 5832t^7 + 17990t^6 - 28224t^5 + 23583t^4 - 9912t^3 + 1680t^2 \right) \\
 \sigma_1(t) &= -\frac{1}{2520} \left(\begin{array}{l} 15120t^8t - 112320t^7 + 330820t^6 - 491316t^5 + 386715t^4 \\ -154280t^3 + 25200t^2 \end{array} \right) \\
 \sigma_{\frac{4}{3}}(t) &= \frac{243}{25480} \left(252t^8 - 1776t^7 + 4935t^6 - 6909t^5 + 5166t^4 - 1981t^3 + 315t^2 \right) \\
 \sigma_{\frac{5}{3}}(t) &= -\frac{243}{52360} \left(144t^8 - 960t^7 + 2532t^6 - 3396t^5 + 2457t^4 - 920t^3 + 144t^2 \right) \\
 \sigma_2(t) &= \frac{1}{17640} \left(1512t^8 - 9504t^7 + 24010t^6 - 31206t^5 + 22071t^4 - 8134t^3 + 1260t^2 \right)
 \end{aligned} \right\} \tag{11}$$

Evaluating equations (8) and (10) at $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{4}{3}, \frac{5}{3}, 2$ gives a discrete ATBHM with five off-grid points of the forms,

$$\begin{aligned}
y_{n+\frac{1}{4}} &= y_n + \frac{1}{4}hy'_n + h^2 \left(\frac{11952373}{825753600}f_n + \frac{1972571}{62375040}f_{n+\frac{1}{4}} - \frac{1413697}{45158400}f_{n+\frac{1}{2}} + \frac{219053}{7761600}f_{n+\frac{3}{4}} \right. \\
&\quad \left. - \frac{90817}{5898240}f_{n+1} + \frac{37715787}{8349286400}f_{n+\frac{4}{3}} - \frac{29759967}{30025318400}f_{n+\frac{5}{3}} + \frac{303427}{2890137600}f_{n+2} \right) \\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + h^2 \left(\frac{13417}{403200}f_n + \frac{58427}{487305}f_{n+\frac{1}{4}} - \frac{1483}{22050}f_{n+\frac{1}{2}} + \frac{8209}{121275}f_{n+\frac{3}{4}} \right. \\
&\quad \left. - \frac{605}{16128}f_{n+1} + \frac{22599}{2038400}f_{n+\frac{4}{3}} - \frac{143127}{58643200}f_{n+\frac{5}{3}} + \frac{731}{2822400}f_{n+2} \right) \\
y_{n+\frac{3}{4}} &= y_n + \frac{3}{4}hy'_n + h^2 \left(\frac{679893}{13107200}f_n + \frac{74187}{346528}f_{n+\frac{1}{4}} - \frac{260019}{5017600}f_{n+\frac{1}{2}} + \frac{193578}{1724800}f_{n+\frac{3}{4}} \right. \\
&\quad \left. - \frac{271017}{4587520}f_{n+1} + \frac{145004661}{8349286400}f_{n+\frac{4}{3}} - \frac{114692841}{30025318400}f_{n+\frac{5}{3}} + \frac{130149}{321126400}f_{n+2} \right) \\
y_{n+1} &= y_n + hy'_n + h^2 \left(\frac{3553}{50400}f_n + \frac{150016}{487305}f_{n+\frac{1}{4}} - \frac{328}{11025}f_{n+\frac{1}{2}} + \frac{512}{2475}f_{n+\frac{3}{4}} \right. \\
&\quad \left. - \frac{187}{2520}f_{n+1} + \frac{243}{10400}f_{n+\frac{4}{3}} - \frac{9477}{1832600}f_{n+\frac{5}{3}} + \frac{97}{176400}f_{n+2} \right) \\
y_{n+\frac{4}{3}} &= y_n + \frac{4}{3}hy'_n + h^2 \left(\frac{140936}{1476225}f_n + \frac{1380319232}{3197208105}f_{n+\frac{1}{4}} + \frac{249856}{72335025}f_{n+\frac{1}{2}} + \frac{264372224}{795685275}f_{n+\frac{3}{4}} \right. \\
&\quad \left. - \frac{4544}{413343}f_{n+1} + \frac{56848}{1289925}f_{n+\frac{4}{3}} - \frac{146752}{18555075}f_{n+\frac{5}{3}} + \frac{58736}{72335025}f_{n+2} \right) \\
y_{n+\frac{5}{3}} &= y_n + \frac{5}{3}hy'_n + h^2 \left(\frac{1589825}{13226976}f_n + \frac{356480000}{639441621}f_{n+\frac{1}{4}} + \frac{89000}{2893401}f_{n+\frac{1}{2}} + \frac{14848000}{31827411}f_{n+\frac{3}{4}} \right. \\
&\quad \left. + \frac{184375}{3306744}f_{n+1} + \frac{260875}{1651104}f_{n+\frac{4}{3}} - \frac{3475}{5937624}f_{n+\frac{5}{3}} + \frac{35125}{46294416}f_{n+2} \right) \\
y_{n+2} &= y_n + 2hy'_n + h^2 \left(\frac{461}{3150}f_n + \frac{65536}{97461}f_{n+\frac{1}{4}} + \frac{1024}{11025}f_{n+\frac{1}{2}} + \frac{65536}{121275}f_{n+\frac{3}{4}} + \frac{8}{45}f_{n+1} + \frac{8019}{31850}f_{n+\frac{4}{3}} \right. \\
&\quad \left. + \frac{25272}{229075}f_{n+\frac{5}{3}} + \frac{89}{11025}f_{n+2} \right)
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
y'_{n+\frac{1}{4}} &= y'_n + h \left(\begin{aligned} &\frac{3179503}{41287680} f_n + \frac{8773889}{31187520} f_{n+\frac{1}{4}} - \frac{24109}{107520} f_{n+\frac{1}{2}} + \frac{306893}{1552320} f_{n+\frac{3}{4}} \\ &-\frac{1103141}{10321920} f_{n+1} + \frac{13008033}{417464320} f_{n+\frac{4}{3}} - \frac{208737}{30638080} f_{n+\frac{5}{3}} + \frac{11561}{16056320} f_{n+2} \end{aligned} \right) \\
y'_{n+\frac{1}{2}} &= y'_n + h \left(\begin{aligned} &\frac{149}{2016} f_n + \frac{3652}{9555} f_{n+\frac{1}{4}} - \frac{1}{45} f_{n+\frac{1}{2}} + \frac{2924}{24255} f_{n+\frac{3}{4}} \\ &-\frac{323}{4480} f_{n+1} + \frac{8991}{407680} f_{n+\frac{4}{3}} - \frac{243}{49280} f_{n+\frac{5}{3}} + \frac{149}{282240} f_{n+2} \end{aligned} \right) \\
y'_{n+\frac{3}{4}} &= y'_n + h \left(\begin{aligned} &\frac{68619}{917504} f_n + \frac{1290153}{3465280} f_{n+\frac{1}{4}} + \frac{519}{5120} f_{n+\frac{1}{2}} + \frac{48309}{172480} f_{n+\frac{3}{4}} \\ &-\frac{116157}{1146880} f_{n+1} + \frac{11763873}{417464320} f_{n+\frac{4}{3}} - \frac{1305639}{214466560} f_{n+\frac{5}{3}} + \frac{10281}{16056320} f_{n+2} \end{aligned} \right) \\
y'_{n+1} &= y'_n + h \left(\begin{aligned} &\frac{187}{2520} f_n + \frac{184064}{487305} f_{n+\frac{1}{4}} + \frac{8}{105} f_{n+\frac{1}{2}} + \frac{10496}{24255} f_{n+\frac{3}{4}} + \frac{61}{2520} f_{n+1} \\ &+\frac{243}{12740} f_{n+\frac{4}{3}} - \frac{243}{52360} f_{n+\frac{5}{3}} + \frac{1}{1960} f_{n+2} \end{aligned} \right) \\
y'_{n+\frac{4}{3}} &= y'_n + h \left(\begin{aligned} &\frac{3478}{45927} f_n + \frac{129630208}{355245345} f_{n+\frac{1}{4}} + \frac{4096}{32805} f_{n+\frac{1}{2}} + \frac{5636096}{17681895} f_{n+\frac{3}{4}} \\ &+\frac{68224}{229635} f_{n+1} + \frac{522}{3185} f_{n+\frac{4}{3}} - \frac{256}{19635} f_{n+\frac{5}{3}} + \frac{1832}{1607445} f_{n+2} \end{aligned} \right) \\
y'_{n+\frac{5}{3}} &= y'_n + h \left(\begin{aligned} &\frac{26555}{367416} f_n + \frac{2144000}{5465313} f_{n+\frac{1}{4}} + \frac{200}{6561} f_{n+\frac{1}{2}} + \frac{1772800}{3536379} f_{n+\frac{3}{4}} \\ &+\frac{33625}{367416} f_{n+1} + \frac{275}{588} f_{n+\frac{4}{3}} + \frac{1185}{10472} f_{n+\frac{5}{3}} - \frac{5275}{2571912} f_{n+2} \end{aligned} \right) \\
y'_{n+2} &= y'_n + h \left(\begin{aligned} &\frac{11}{126} f_n + \frac{45056}{162435} f_{n+\frac{1}{4}} + \frac{128}{315} f_{n+\frac{1}{2}} - \frac{4096}{24255} f_{n+\frac{3}{4}} + \frac{26}{35} f_{n+1} \\ &+\frac{243}{6370} f_{n+\frac{4}{3}} + \frac{486}{935} f_{n+\frac{5}{3}} + \frac{214}{2205} f_{n+2} \end{aligned} \right)
\end{aligned}
\tag{13}$$

Equations (12) and (13) collectively form the ATBHM.

3. Analysis of Basic Properties of the ATBHM

3.1 Order and Error Constant of the ATBHM

Let the linear difference operator L associated with the continuous linear multistep method (LMM) be defined by,

$$L\{y(x); h\} = \sum_{j=0}^k [\alpha_j y(x_n + jh) - h^2 \beta_j y''(x_n + jh)] \quad (14)$$

where $y(x)$ is the arbitrary test function that is continuously differentiable in the interval $[a, b]$. Expanding $y(x_n + jh)$ and $y''(x_n + jh)$, $j = 0, 1, 2, \dots, m$ in Taylor series about the point x_n and collecting like terms in h and y gives

$$L\{y(x); h\} = c_0 y_0 + c_1 h y'(x) + c_2 h^2 y^{(2)}(x) + \dots + c_p h^p y^{(p)}(x) \quad (15)$$

Note that, the constants c_p are defined as

$$\left. \begin{aligned} c_0 &= \sum_{j=0}^k \alpha_j \\ c_1 &= \sum_{j=0}^k (j\alpha_j - \beta_j) \\ &\cdot \\ &\cdot \\ &\cdot \\ c_p &= \sum_{j=0}^k \left[\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!} j^{p-1} \beta_j \right], p = 2, 3, \dots \end{aligned} \right\} \quad (16)$$

The difference operator L and the continuous LMM (8) are said to be of order p if in (16), $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \dots = \bar{c}_p = \bar{c}_{p+1} = 0$, $\bar{c}_{p+2} \neq 0$, Fatunla (1988). The term $\bar{c}_{p+2} \neq 0$ is called the error constant of the method and the local truncation error is given by

$$t_{n+k} = c_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3}).$$

The Taylor series expansion of the ATBHM in equations (12) and (13) is given by

$$\left[\begin{array}{l}
 \sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}h\right)^j}{j!} y_n^j - y_n - \frac{1}{4}hy_n' - \frac{11952373}{825753600}h^2y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \begin{array}{l} \frac{1972571}{62375040}\left(\frac{1}{4}\right)^j - \frac{1413697}{45158400}\left(\frac{1}{2}\right)^j + \frac{219053}{7761600}\left(\frac{3}{4}\right)^j - \frac{90817}{5898240}(1)^j \\ + \frac{37715787}{8349286400}\left(\frac{4}{3}\right)^j - \frac{29759967}{30025318400}\left(\frac{5}{3}\right)^j + \frac{303427}{2890137600}(2)^j \end{array} \right\} \\
 \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}h\right)^j}{j!} y_n^j - y_n - \frac{1}{2}hy_n' - \frac{13417}{403200}h^2y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \begin{array}{l} \frac{58427}{487305}\left(\frac{1}{4}\right)^j - \frac{1483}{22050}\left(\frac{1}{2}\right)^j + \frac{8209}{121275}\left(\frac{3}{4}\right)^j - \frac{605}{16128}(1)^j \\ + \frac{22599}{2038400}\left(\frac{4}{3}\right)^j - \frac{143127}{58643200}\left(\frac{5}{3}\right)^j + \frac{731}{2822400}(2)^j \end{array} \right\} \\
 \sum_{j=0}^{\infty} \frac{\left(\frac{3}{4}h\right)^j}{j!} y_n^j - y_n - \frac{3}{4}hy_n' - \frac{679893}{13107200}h^2y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \begin{array}{l} \frac{74187}{346528}\left(\frac{1}{4}\right)^j - \frac{260019}{5017600}\left(\frac{1}{2}\right)^j + \frac{193578}{1724800}\left(\frac{3}{4}\right)^j - \frac{271017}{4587520}(1)^j \\ + \frac{145004661}{8349286400}\left(\frac{4}{3}\right)^j - \frac{114692841}{30025318400}\left(\frac{5}{3}\right)^j + \frac{130149}{321126400}(2)^j \end{array} \right\} \\
 \sum_{j=0}^{\infty} \frac{(h)^j}{j!} y_n^j - y_n - hy_n' - \frac{3553}{50400}h^2y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \begin{array}{l} \frac{150016}{487305}\left(\frac{1}{4}\right)^j - \frac{328}{11025}\left(\frac{1}{2}\right)^j + \frac{512}{2475}\left(\frac{3}{4}\right)^j - \frac{187}{2520}(1)^j \\ + \frac{243}{10400}\left(\frac{4}{3}\right)^j - \frac{9477}{1832600}\left(\frac{5}{3}\right)^j + \frac{97}{176400}(2)^j \end{array} \right\} \\
 \sum_{j=0}^{\infty} \frac{\left(\frac{4}{3}h\right)^j}{j!} y_n^j - y_n - \frac{4}{3}hy_n' - \frac{140936}{1476225}h^2y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \begin{array}{l} \frac{1380319232}{3197208105}\left(\frac{1}{4}\right)^j + \frac{249856}{72335025}\left(\frac{1}{2}\right)^j + \frac{264372224}{795685275}\left(\frac{3}{4}\right)^j - \frac{4544}{413343}(1)^j \\ + \frac{56848}{1289925}\left(\frac{4}{3}\right)^j - \frac{146752}{18555075}\left(\frac{5}{3}\right)^j + \frac{58736}{72335025}(2)^j \end{array} \right\} \\
 \sum_{j=0}^{\infty} \frac{\left(\frac{5}{3}h\right)^j}{j!} y_n^j - y_n - \frac{5}{3}hy_n' - \frac{1589825}{13226976}h^2y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \begin{array}{l} \frac{356480000}{639441621}\left(\frac{1}{4}\right)^j + \frac{89000}{2893401}\left(\frac{1}{2}\right)^j + \frac{14848000}{31827411}\left(\frac{3}{4}\right)^j + \frac{184375}{3306744}(1)^j \\ + \frac{260875}{1651104}\left(\frac{4}{3}\right)^j - \frac{3475}{5937624}\left(\frac{5}{3}\right)^j + \frac{35125}{46294416}(2)^j \end{array} \right\} \\
 \sum_{j=0}^{\infty} \frac{(2h)^j}{j!} y_n^j - y_n - 2hy_n' - \frac{461}{3150}h^2y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \begin{array}{l} \frac{65536}{97461}\left(\frac{1}{4}\right)^j + \frac{1024}{11025}\left(\frac{1}{2}\right)^j + \frac{65536}{121275}\left(\frac{3}{4}\right)^j + \frac{8}{45}(1)^j \\ + \frac{8019}{31850}\left(\frac{4}{3}\right)^j + \frac{25272}{229075}\left(\frac{5}{3}\right)^j + \frac{89}{11025}(2)^j \end{array} \right\}
 \end{array} \right] = \begin{array}{l} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \tag{17}$$

From equation (17), we obtain equation (16) as

$$\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \dots = \bar{c}_9 = 0$$

and

$$\bar{c}_{10} = \begin{bmatrix} -7.2739 \times 10^{-09} & -1.8059 \times 10^{-08} & -2.8286 \times 10^{-08} & -3.8547 \times 10^{-08} \\ -5.4152 \times 10^{-08} & -6.5172 \times 10^{-08} & -1.0498 \times 10^{-07} & \end{bmatrix}^T$$

Therefore, the ATBHM in equations (12) and (13) is of uniform order eight with error constant \bar{c}_{10} .

3.2 Consistency of the ATBHM

The ATBHM is consistent since its order $p \geq 1$, Lambert (1973).

3.3 Zero-stability of the ATBHM

For the ATBHM given in equations (12) and (13), the first characteristic polynomial is given by,

$$\rho(z) = z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} z & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & z & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & z & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & z-1 \end{vmatrix} = z^6(z-1)$$

Solving for z in

$$z^6(z-1) = 0 \tag{18}$$

gives $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = 0$ and $z_7 = 1$. Hence, the ATBHM is also zero-stable.

3.4 Convergence of the ATBHM

The ATBHM is convergent since it is consistent and zero-stable, Lambert (1991).

3.5 Regions of Absolute Stability of the ATBHM

The stability polynomial of the ATBHM is obtained as,

$$\begin{aligned} \bar{h}(w) = & -h^{14} \left(\frac{1}{42138206208} w^7 + \frac{1793}{1264146186240} w^6 \right) - h^{12} \left(\frac{57329}{25282923724800} w^7 + \frac{1572533}{3033950846976} w^6 \right) \\ & - h^{10} \left(\frac{883843}{25282923724800} w^7 + \frac{486451241}{8427641241600} w^6 \right) - h^8 \left(\frac{4019492713}{1404606873600} w^6 - \frac{3293}{10404495360} w^7 \right) \\ & - h^6 \left(\frac{595559}{23410114560} w^7 + \frac{4823246353}{70230343680} w^6 \right) - h^4 \left(\frac{3017123}{3981312} w^6 - \frac{605513}{418037760} w^7 \right) \\ & - h^2 \left(\frac{979}{18144} w^7 + \frac{19181}{6048} w^6 \right) + w^7 - 3w^6 \end{aligned} \tag{19}$$

MATLAB (R2018a) software was used to obtain the region of absolute stability shown in Figures 1.

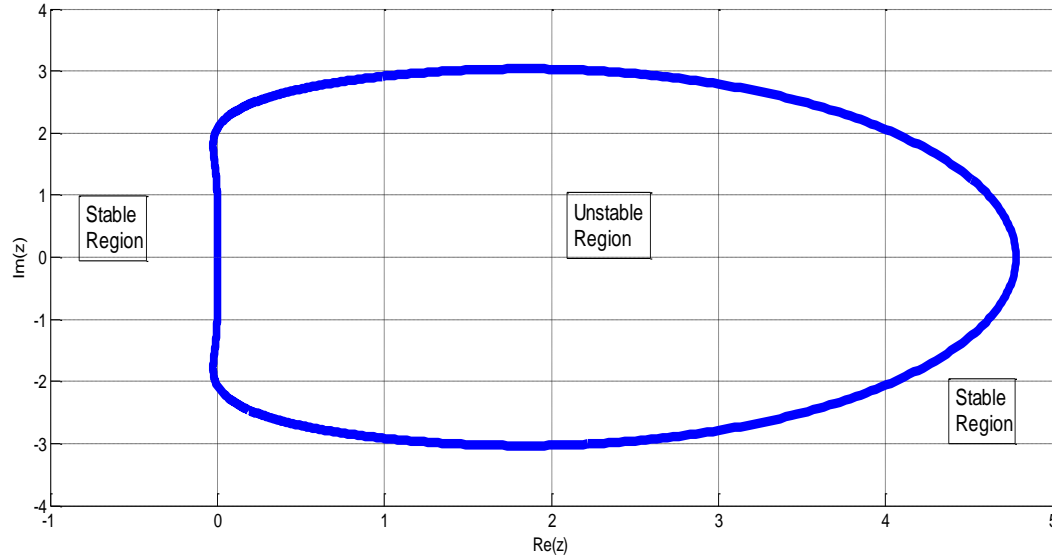


Figure 1. Region of Absolute Stability of the ATBHM

From the plot generated in Figures 1, it is clear that the ATBHM is A-Stable. Note that the stability region is the exterior of the contour.

4. Results and Discussion

The ATBHM is applied in solving second-order differential equations of the form (1). The following notations and acronyms will be used in presenting the results.

t or x : point of evaluation

h : step-size

NS: number of steps taken

MaxErr: maximum error

AbsErr: absolute error

CPUT: central processing unit time (in seconds)

ode45: inbuilt Matlab solver

ode15s: inbuilt Matlab solver

HBM: fourteenth-order hybrid block method developed by Jator *et al.* (2023)

TSHBM: three-step hybrid block method of order 8 developed by Rasedee *et al.* (2017)

ATBHM: the newly derived two-step Adams-type block hybrid method with five off-grid points. Let the exact and approximate solutions at the point x_n be denoted by $y(x_n)$ and y_n respectively, then the absolute error is defined as,

$$AbsErr = |y(x_n) - y_n|,$$

maximum error as,

$$MaxErr = \max_{0,1,\dots,N} |y(x_n) - y_n|,$$

and average error as,

$$AvrErr = |y(x_n) - y_n| / N.$$

4.1 Numerical Experiments

Problem 1

Consider the second-order differential equation,

$$y''(x) + 2y'(x) + y(x) + 8y^3(x) = e^{-3x}, \quad y(0) = \frac{1}{2}, \quad y'(0) = -\frac{1}{2} \quad (20)$$

with exact solution is given by,

$$y(x) = \frac{1}{2}e^{-x} \quad (21)$$

The ATBHM is applied on Problem 1 and the results obtained were compared with that of the Three-Step Hybrid Block Method (TSHBM) derived by Rasedee *et al.* (2017). See Table 1 and Figure 2 for numerical results and solution curves respectively.

Table 1: Maximum and Average Errors for Problem 1

Tol	Method	NS	MaxErr	AvrErr	CPUT
10^{-2}	ATBHM	179	1.0027×10^{-05}	2.1246×10^{-05}	1.1789
	TSHBM	324	1.8129×10^{-01}	7.2336×10^{-02}	-
10^{-4}	ATBHM	237	2.7201×10^{-09}	1.4990×10^{-09}	2.2018
	TSHBM	331	1.3584×10^{-03}	4.3135×10^{-04}	-
10^{-6}	ATBHM	272	2.7246×10^{-11}	2.7273×10^{-11}	3.1484
	TSHBM	333	1.9814×10^{-05}	2.2591×10^{-06}	-
10^{-8}	ATBHM	308	3.7289×10^{-13}	3.8910×10^{-13}	3.8198
	TSHBM	347	1.5230×10^{-07}	6.7281×10^{-08}	-
10^{-10}	ATBHM	347	2.6889×10^{-16}	3.1939×10^{-16}	4.2112
	TSHBM	376	4.9294×10^{-09}	8.2086×10^{-10}	-

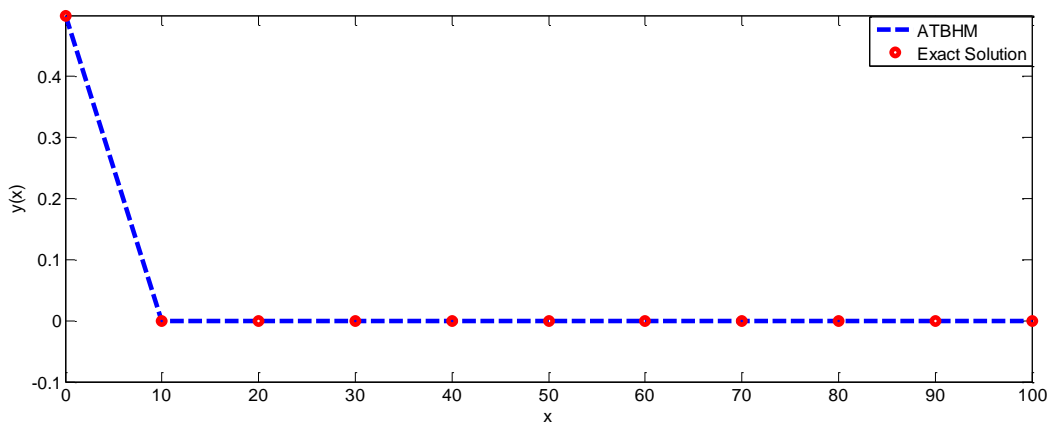


Figure 2. Solution Curves for Problem 1

Problem 2

Consider the second-order problem,

$$y''(x) - 2\varepsilon(1 - y^2(x))y'(x) + y(x) = 0, y(0) = 0, y'(0) = 0.5 \quad (22)$$

defined on $0 \leq x \leq 10$, where the parameter ε is the nonlinearity coefficient and strength of damping with the value 0.005. This problem, called the Van der Pol oscillator does not have an exact solution. We compared the approximate solutions obtained using the ATBHM with those of the ode15s and fourteenth-order hybrid block method derived by Jator *et al.* (2023). Table 2 showed the approximate solutions obtained over the domain at the varying end points of x while Figure 3 presented the solution curves.

Table 2. Numerical Solutions of Problem 2 using the ATBHM in Relation to ode15s and HBM derived by Jator *et al.* (2023)

x	ode15s	HBM	ATBHM
0.00	0.00000000	0.00000000	0.00000000
0.50	0.24030705	0.24030707	0.24030691
1.00	0.42277361	0.42277361	0.42277349
1.50	0.50228043	0.50228038	0.50228032
2.00	0.45888175	0.45888176	0.45888165
2.50	0.30272752	0.30272754	0.30272741
3.50	-0.17829330	-0.17829325	-0.17829339
4.50	-0.49920275	-0.49920270	-0.49920285
5.50	-0.36193145	-0.36193145	-0.36193155
6.50	0.11085614	0.11085605	0.11085601
7.50	0.48578445	0.48578440	0.48578435
8.50	0.41539059	0.41539062	0.41539047
9.50	-0.03923235	-0.03923225	-0.03923245
10.00	-0.28502438	-0.28502430	-0.28502449

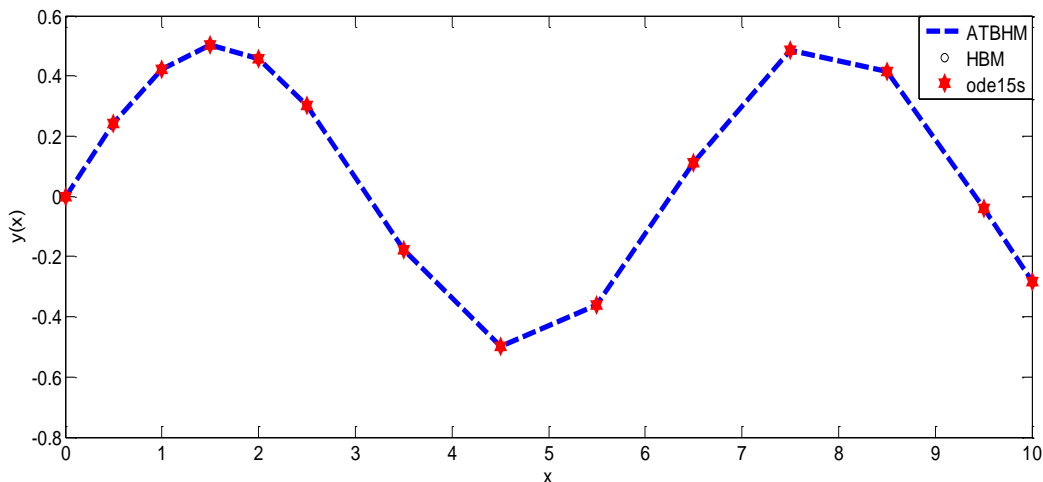


Figure 3. Solution Curves for Problem 2

Problem 3

Consider the following second-order nonlinear system which describes the aerodynamic damping of a pendulum,

$$\theta''(t) + \frac{g}{L} \sin \theta + \frac{c_a}{mL} (\theta'(t))^2 \operatorname{sgn}(\theta'(t)) = 0 \quad (23)$$

and the linear system below which describes the viscous damping of a pendulum

$$\theta''(t) + \frac{c_d}{m} (\theta'(t))^2 + \frac{g}{L} \theta = 0 \quad (24)$$

To determine the quadratic damping in equations (23) and (24) respectively, the ATBHM is employed in simulating the two equations and the simulation results were compared with that of ode 45 used by Soraya (2016). Let $g = 981 \text{cms}^{-2}$, $c_a = c_d = 14$, $L = 5 \text{cm}$, $m = 10 \text{g}$, $\theta(0) = 1.57$ and $\theta'(0) = 0$. The initial value θ is in radians. Also, sgn in equation (23) represents signum function; c_a and c_d are the coefficients of aerodynamic damping and viscous damping respectively. Figure 4 shows a simple pendulum with two forces acting on the mass; that is, the weight mg and the tension T . The net force is found to be

$$F = mg \sin \theta.$$

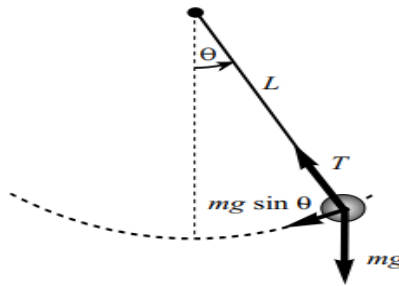


Figure 4. A simple pendulum (Soraya, 2016)

From simulation results presented in Figure 5, it is obvious that quadratic damping causes a rapid initial amplitude reduction but is less effective as time increases and amplitude decreases. On the other hand, linear damping is less effective initially but is more effective as time increases. The plots confirmed that the ATBHM is computationally reliable as they agree with those of ode 45.

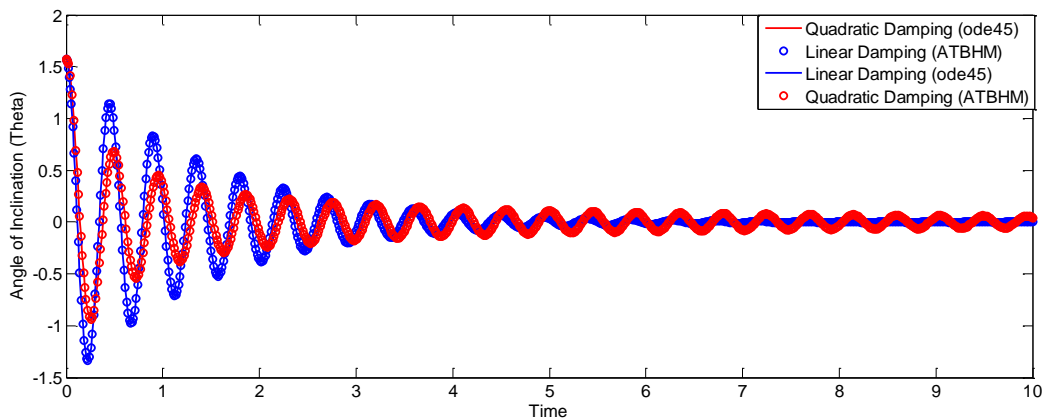


Figure 5. Solution Curves of Problem 3 showing Quadratic and Linear Damping

4.2 Discussion of Results

A second-order differential equation was considered in Problem 1. Table 1 showed the maximum and average errors over some selected tolerance levels. It was noted that the ATBHM performed better than the TSHBM of order 8 derived

by Rasedee *et al.* (2017) as they had smaller maximum and average errors. It was also observed that fewer numbers of step (NS) were taken in achieving those results in contrast to the number of steps of the TSHBM. Table 1 further showed that the smaller the tolerance level, the better the accuracy of the integrators. The solutions curves obtained in Figure 2 showed the convergence of the solutions obtained using the ATBHM to that of the exact solution. In Problem 2, the Van der Pol problem was considered. The ATBHM was applied in solving this problem and the numerical results obtained were compared with those of ode15s and HBM of order 14 derived by Jator *et al.* (2023), see Table 2. It is also clear from the solution curves obtained in Figure 3 that the solutions of the ATBHM converge to those of ode15s. The application of the ATBHM on Problem 3, which is a pendulum problem with aerodynamic damping, shows that quadratic damping causes a rapid initial amplitude reduction while linear damping is less effective initially. This is in consonance with the result of ode45 as shown in Figure 5. Overall, these results show that the ATBHM is computationally reliable.

5. Conclusion

A method, the ATBHM, has been derived for the in this research for the solution of second-order differential equations of the form (1). From the results obtained, it is obvious that the method is computationally reliable as it performed better than existing methods which we compared our results with. The method has also been shown to be consistent, convergent, stable and accurate. Furthermore, the stability region of the method shows that it is A-stable.

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