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Mathematical Analysis of a Fully Coupled Nonlinear Dynamical System with Phase Transition Effects

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A R T I C L E I N F O

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A B S T R A C T

This study presents a fully coupled nonlinear dynamical model describing the interaction between a rigid spherical body and an actively heated plate undergoing phase change. The formulation integrates mechanical motion, thermal transport, and melt-layer evolution within a unified framework, capturing the feedback mechanisms arising from thin-film viscous resistance and latent heat effects. The governing equations consist of a system of nonlinear ordinary differential equations for the position, velocity, plate temperature, melt thickness, and sphere temperature. A lubrication-based model is employed to characterize viscous resistance, while heat transfer in both the plate and sphere is treated using lumped thermal approximations. Melting dynamics are incorporated through a latent heat balance with an activation mechanism governed by a temperature-dependent switching function. The system is subsequently non-dimensionalized, revealing key parameters controlling viscous dissipation, thermal coupling, and phase change intensity. Steady-state solutions are derived, and a linear stability analysis is performed using the Jacobian matrix of the coupled system. The analysis reveals a block triangular structure, allowing decomposition into mechanical and thermal subsystems. The resulting eigenvalue spectrum exhibits multiple zero eigenvalues alongside a negative mode, indicating marginal stability of the equilibrium state. This degeneracy highlights the absence of sufficient dissipative mechanisms in the baseline model. The results provide new insight into the interplay between thermal activation, viscous resistance, and nonlinear coupling in melting-driven systems. The framework establishes a foundation for extended models incorporating additional dissipative effects and offers a mathematically consistent approach to analyzing thermo-mechanical phase-change dynamics.

1 Introduction

Thermo-mechanical systems involving phase change arise in a wide range of physical and engineering applications, including melting-driven motion, thermal contact problems, and lubrication-mediated transport processes. Of particular interest are systems in which mechanical motion, heat transfer, and phase change are strongly coupled, leading to

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nonlinear feedback mechanisms that significantly influence system dynamics (Alexiades & Solomon, 1993; Crank, 1984; Rubinstein, 1971).

Classical treatments of melting processes are largely based on Stefan-type formulations and moving boundary problems, where the evolution of the phase interface is governed primarily by heat conduction (Crank, 1984; Rubinstein, 1971). Foundational analytical and numerical frameworks for such problems have been extensively developed, including enthalpy-based methods and fixed-grid approaches for phase-change systems (Tarzia, 2000; Voller & Swaminathan, 1991; Voller & Cross, 1981). However, these models often neglect dynamic coupling between thermal and mechanical processes or assume quasi-static behavior.

Recent advances have extended classical Stefan problems to incorporate nonlinear effects, evolving interfaces, and temperature-dependent material properties (Joshi *et al.*, 2025; Silva & Mendes, 2024; Nguyen *et al.*, 2024). In parallel, developments in applied mathematical modeling have addressed geometry-dependent phase-change behavior and multiphase configurations, revealing the influence of structural properties on melting dynamics (Zhang *et al.*, 2024; Tarzia & Bollati, 2025). These studies highlight the increasing need for models capable of capturing more complex and realistic physical interactions.

Furthermore, emerging research has emphasized the role of thermo-mechanical coupling in heat-driven deformation and penetration processes, particularly in temperature-sensitive and layered materials (Kumar & Tarzia, 2025; Zhou et al., 2024; Chen et al., 2023). Related investigations have also explored the influence of thermal memory effects and heat-controlled indentation on material response under thermal loading (Li et al., 2023; Hassan & Karim, 2022). Recent work by N.N. Topman (2026) developed a geometry-resolved thermo-mechanical model for heat-transfer-controlled quasi-static penetration through multilayer plates, demonstrating the significant influence of multilayer geometry on penetration dynamics. Additional studies have similarly highlighted the importance of heat-driven penetration mechanisms in multilayer systems (Gao et al., 2024). Despite these developments, existing models often treat mechanical motion and phase change in a partially coupled manner, limiting their ability to capture fully dynamic feedback mechanisms.

A related thermo-mechanical dynamical framework has recently been developed by Topman (2026), with particular emphasis on the mathematical characterization of coupled phase-change dynamics through stability analysis, energy-based arguments, and parameter-dependent regime behaviour. The present study focuses more specifically on the baseline formulation and eigenvalue-based stability structure of the fully coupled system, providing a concise characterization of its equilibrium and marginal-stability properties.

In this work, we develop a fully coupled nonlinear dynamical system describing the interaction between a rigid spherical body and an actively heated plate. The model simultaneously accounts for Newtonian mechanical motion under gravity and external forcing, viscous resistance arising from a thin melt film governed by lubrication theory, heat exchange between the plate and the sphere, and phase-change dynamics driven by latent heat. A temperature-activated melting mechanism is incorporated through a switching function, introducing a non-smooth nonlinearity that governs the onset of phase change.

The governing equations are formulated as a system of nonlinear ordinary differential equations describing the coupled evolution of position, velocity, plate temperature, melt thickness, and sphere temperature. A systematic non-dimensionalization is performed to identify the key dimensionless parameters controlling viscous resistance, thermal coupling, and melting intensity, thereby providing insight into the relative importance of competing physical mechanisms.

The analytical focus of this study is on the qualitative behavior of the coupled system. Steady-state solutions are derived, and a linear stability analysis is performed via the Jacobian matrix of the system. The Jacobian is shown to possess a block triangular structure, enabling decomposition into mechanical and thermal subsystems. The resulting eigenvalue spectrum contains multiple zero eigenvalues alongside a negative mode, indicating marginal stability of the equilibrium state and revealing an intrinsic degeneracy associated with insufficient dissipative mechanisms.

The novelty of this work lies in the formulation of a mathematically consistent, fully coupled nonlinear dynamical system that integrates thermo-mechanical motion, thin-film viscous effects, and phase change within a unified framework. In addition, the identification and interpretation of marginal stability provide new insight into the role of thermal feedback and dissipation in melting-driven systems.

2. Methods

2.1 Physical Configuration and Reduced-Order Framework

The model describes a rigid spherical body approaching and interacting with a thermally energized solid surface. Once the interface temperature reaches the melting threshold, material at the contact region undergoes phase transformation and produces a thin liquid film between the two bodies. The resulting motion is therefore governed by the simultaneous evolution of mechanical displacement, translational velocity, thermal states, and the thickness of the interfacial melt layer.

To obtain a tractable reduced-order representation, the following modeling framework is adopted. The liquid film is treated as a slender interfacial region, so that its flow can be represented using lubrication-type scaling. Within this layer, the molten material is regarded as a Newtonian fluid, allowing the viscous contribution to the mechanical resistance to be expressed through the local film thickness. Rather than resolving spatial temperature fields inside the solid components, the thermal state of each body is represented by a single effective temperature through a lumped-capacitance description. The thermophysical coefficients entering the governing equations are taken to remain fixed over the range of conditions considered. Phase transformation is further restricted to the thermally activated regime; consequently, the melting contribution becomes operative only after the plate temperature reaches or surpasses the material melting temperature.

Under these reductions, the coupled problem is represented by a nonlinear dynamical system in which mechanical motion, thermal evolution, and interfacial phase change interact through the melt-layer thickness. The resulting state variables comprise the body position and velocity, the relevant thermal variables, and the evolving thickness of the liquid layer.

Model Diagram

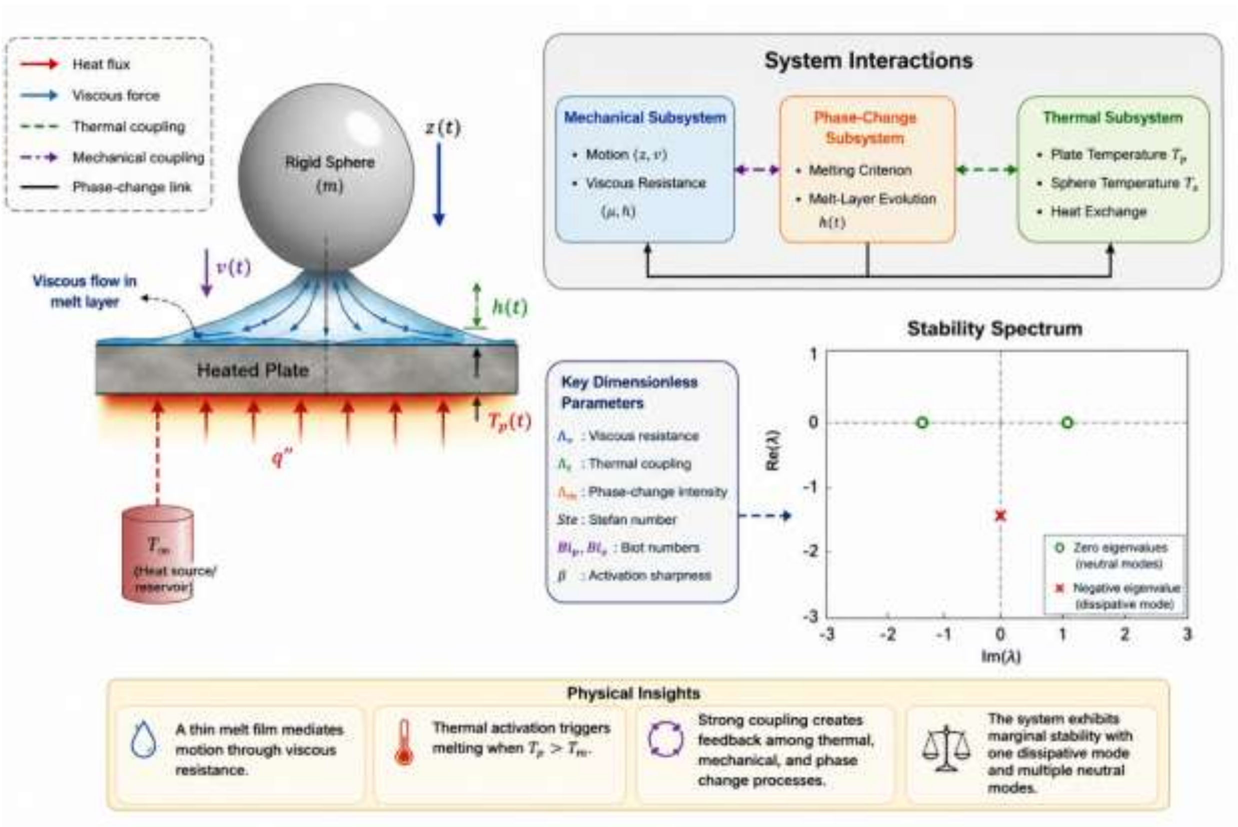

Figure 1: Schematic of the Coupled Thermo-Mechanical System

The figure illustrates the interaction between a rigid body and a heated surface, where melting occurs at the interface to form a thin viscous layer. The plate temperature governs the onset of melting, leading to the formation and growth of the melt layer, while the viscosity of this layer provides resistance to motion. This establishes a strong coupling between thermal activation, phase change, and mechanical response.

The diagram highlights that the melt layer acts as the key linking mechanism: thermal input drives melting, and the resulting liquid layer modifies the resistance experienced by the body. This feedback creates a fully coupled system in which temperature and motion evolve together.

On the right, the schematic summarizes the resulting stability behavior, showing that the coupled interactions lead to marginal stability, where the system neither exhibits instability nor strong decay, but instead maintains neutral dynamics due to the underlying coupling structure.

2.2 Governing Equations

The dynamics of the system are described by a set of coupled nonlinear ordinary differential equations that capture the interaction between mechanical motion, thermal transport, and phase-change processes. These equations govern the evolution of position, velocity, temperature, and melt-layer thickness, and form the basis for subsequent analysis.

State Variables

Let

$z(t) = sphere\ center\ position$

$v(t) = \dot{z}(t)$

$T_p(t) = temperature\ of\ active\ plate$

$h(t) = melt\ layer\ thickness$

$T_b(t) = ball\ temperature\ (now\ variable)$

State vector: $X = (z, v, T_p, h, T_b)$

For the Mechanical impact we apply Newton's law (no quasi-static assumption):

$$m\dot{v} = F - mg - R(h, v) \tag{1}$$

The viscous resistance force due to the melt layer is modeled as:

$$R(h, v) = \frac{K\mu A}{h} v \tag{2}$$

where

$R(h, v)$ is the viscous resistance force,

K is a dimensionless geometric correction factor,

μ is the dynamic viscosity of the melt layer,

A is the effective contact area between the sphere and the melt layer,

h is the instantaneous melt-layer thickness, and

v is the instantaneous velocity of the sphere.

The resistance force is inversely proportional to the melt-layer thickness and directly proportional to both viscosity and velocity, consistent with lubrication-theory approximations for thin viscous films.

Hence

$$m\dot{v} = F - mg - \frac{K\mu A}{h} v \tag{3}$$

Thus, the mechanical equation becomes:

$$F = m\dot{v} + mg + \frac{k\mu A}{h}v \tag{4}$$

The kinematic relation is:

$$\dot{z} = v \tag{5}$$

Thermal Evolution of the Plate

Heat exchange between the plate and the sphere is governed by:

$$\rho c\delta\dot{T}_p = kA(T_b - T_p) \quad 6)$$

From equation (6), we have

$$\dot{T}_p = \frac{kA}{\rho c\delta}(T_b - T_p) \tag{7}$$

Melt Layer Evolution (Phase Change)

The latent heat balance governing melting is:

$$\rho LA\dot{h} = kA(T_b - T_m) \tag{8}$$

From equation (8), we have;

$$\dot{h} = \frac{k}{\rho L}(T_b - T_m) \tag{9}$$

The area A appearing in Equation (8) cancels during simplification because it occurs on both sides of the latent-heat balance equation.

To represent the temperature threshold explicitly, we define the activation function

$$\chi(T_p) = H(T_p - T_m)$$

where $H(\cdot)$ denotes the Heaviside step function. The melt-layer evolution law can then be written in activated form as

$$\dot{h} = K\chi(T_p)(T_p - T_m) \tag{10}$$

Thus, melt growth is suppressed below the melting threshold and becomes active when $T_p \geq T_m$.

Thermal Evolution of the Sphere

Energy balance for the sphere yields:

$$m_b c_b\dot{T}_b = -kA(T_b - T_p) \tag{11}$$

From equation (11), we obtain:

$$\dot{T}_b = -\frac{kA}{m_b c_b}(T_b - T_p) \tag{12}$$

Non-Dimensionalization

Introduce characteristic scales:

$$z = L_0 z^*, t = t_0 t^*, h = h_0 h^*, T = T_m + \Delta T T^* \tag{13}$$

Substituting and simplifying yields the dimensionless system:

$$\dot{v}^* = \Pi_1 - \Pi_2 - \frac{\Pi_3}{h^*} v^* \tag{14}$$

$$\dot{T}_p^{\ *} = \Pi_4 \left(T_b^* - T_p^*\right) \tag{15}$$

$$\dot{h}^* = \Pi_5 (T_b^*)\, H(T_p^*) \tag{16}$$

$$T_b^* = -\Pi_6 (T_b^* - T_p^*) \tag{17}$$

Key Dimensionless Numbers

Π_3: viscous resistance parameter, Π_4: thermal coupling (plate), Π_5: Stefan number (melting strength), and Π_6: thermal response of sphere.

2.3 **Steady-State Analysis**

At equilibrium:

$$\dot{v} = 0, \dot{T}_p = 0, \dot{h} = 0, \dot{T}_b = 0 \tag{18}$$

From thermal equilibrium:

$$T_b^* = T_p^* \tag{19}$$

From melting condition:

$$T_b^* = 0 \ \Rightarrow T_b = T_p \tag{20}$$

From mechanical balance:

$$F - mg - \frac{K\mu A}{h^*} v^* = 0 \tag{21}$$

Thus

$$v^* = \frac{(F - mg) h^*}{K\mu A} \tag{22}$$

Linear Stability Analysis

We introduce perturbations:

$$X = X^* + \epsilon \breve{X} \tag{23}$$

Substituting into the system and linearizing yields:

$$\dot{\breve{X}} = J \breve{X} \tag{24}$$

where J is the Jacobian matrix evaluated at equilibrium.

The Jacobia structure:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{-K\mu A}{mh^*} & 0 & \frac{K\mu A v^*}{m(h^*)^2} & 0 \\ 0 & 0 & -\Pi_4 & 0 & \Pi_4 \\ 0 & 0 & \Pi_5 \delta(T_p^*) & 0 & \Pi_5 \\ 0 & 0 & \Pi_6 & 0 & -\Pi_6 \end{pmatrix} \tag{25}$$

The eigenvalues λ satisfy:

$$\det(J - \lambda I) = 0 \tag{26}$$

This matrix is block triangular, meaning:

a. Mechanical System: first 2 variables (z, v)

b. Thermal subsystem: last 3 variables (T_p, h, T_b)

Therefore, eigenvalues = eigenvalues of mechanical block + eigenvalues of thermal block

(a) Mechanical Subsystem:

$$J_m = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-K\mu A}{mh^*} \end{bmatrix} \tag{27}$$

Eigenvalues: $$\lambda_1 = 0,\ \lambda_2 = \frac{-K\mu A}{mh^*} \tag{28}$$

Interpretation of equation (28):

$\lambda_2 < 0$ → damped motion (stable)

$\lambda_1 = 0$ → neutral mode (translation invariance)

This is considered physically correct and shows that position doesn't affect stability directly.

(b) Thermal Subsystem:

$$J_t = \begin{bmatrix} -\Pi_4 & 0 & \_4 \\ -\Pi_5 \delta(T_p^*) & 0 & \Pi_5 \\ \Pi_6 & 0 & -\Pi_6 \end{bmatrix} \tag{29}$$

2.4 **Characteristic Equation**

Computing equation (26) gives:

$$\begin{vmatrix} -\Pi_4 - \lambda & 0 & \Pi_4 \\ \Pi_5 \delta & -\lambda & \Pi_5 \\ \Pi_6 & 0 & -\Pi_6 - \lambda \end{vmatrix} = 0 \tag{30}$$

Expanding equation (30), along second column yields:

$$(-\lambda) \begin{vmatrix} -\Pi_4 - \lambda & \Pi_4 \\ \Pi_6 & -\Pi_6 - \lambda \end{vmatrix} = 0 \tag{31}$$

Solving equation (31), to obtain the determinant gives:

$$(-\lambda)[(-\Pi_6 - \lambda)(-\Pi_6 - \lambda) - \Pi_4 \Pi_6] = 0 \tag{32}$$

Expanding equation (32), gives:

$$(-\lambda)[(\Pi_6 + \lambda)(\Pi_6 + \lambda) - \Pi_4 \Pi_6] = 0 \tag{33}$$

Equation (33), simplifies to:

$$[\lambda^2 + \lambda(\Pi_4 + \Pi_6)] = 0 \tag{34}$$

From equation (34), the eigenvalues become:

$$\lambda_3 = 0,\ \lambda_4 = 0,\ \lambda_5 = -(\Pi_4 + \Pi_6) \tag{35}$$

From equation (35), we have two zero eigenvalues (mechanical + thermal) and One negative eigenvalue.

Interpretation.

Zero eigenvalues → no decay in some directions and only one strictly negative mode.

Hence the system is marginally stable but not asymptotically stable.

Remark:

The Jacobian matrix exhibits a block triangular structure, allowing separation of mechanical and thermal subsystems. The mechanical subsystem yields one zero and one negative eigenvalue, while the thermal subsystem produces two zero eigenvalues and one negative eigenvalue. This indicates marginal stability of the coupled system. The presence of multiple zero eigenvalues suggests the absence of sufficient

dissipative mechanisms, motivating the inclusion of additional thermal loss terms for full asymptotic stability.

Although the characteristic equation is quadratic, the resulting eigenvalues are purely real, indicating the absence of oscillatory modes. This implies that the system exhibits overdamped behavior, with no sinusoidal response. The lack of complex eigenvalues reflects the dominance of dissipative mechanisms and the absence of restorative coupling capable of sustaining oscillations.

Table 1. Parameter Definitions

Symbol	Description	Units
m	Mass of the sphere	kg
g	Acceleration due to gravity	m s^{-2}
μ	Dynamic viscosity	$Pa.s$
A	Contact area	m^2
K	Geometric constant	–
k	Thermal conductivity	W m^{-1} K^{-1}
ρ	Material density	kg m^{-3}
c	Specific heat capacity (plate)	J kg^{-1} K^{-1}
c_b	Specific heat capacity (sphere)	J kg^{-1} K^{-1}
L	Latent heat of fusion	J kg^{-1}
δ	Plate thickness	m
T_m	Melting temperature	K
m_b	Thermal mass of the sphere	kg

3 Results and Discussion

The numerical simulations provide insight into the coupled thermo-mechanical behavior and stability characteristics of the system.

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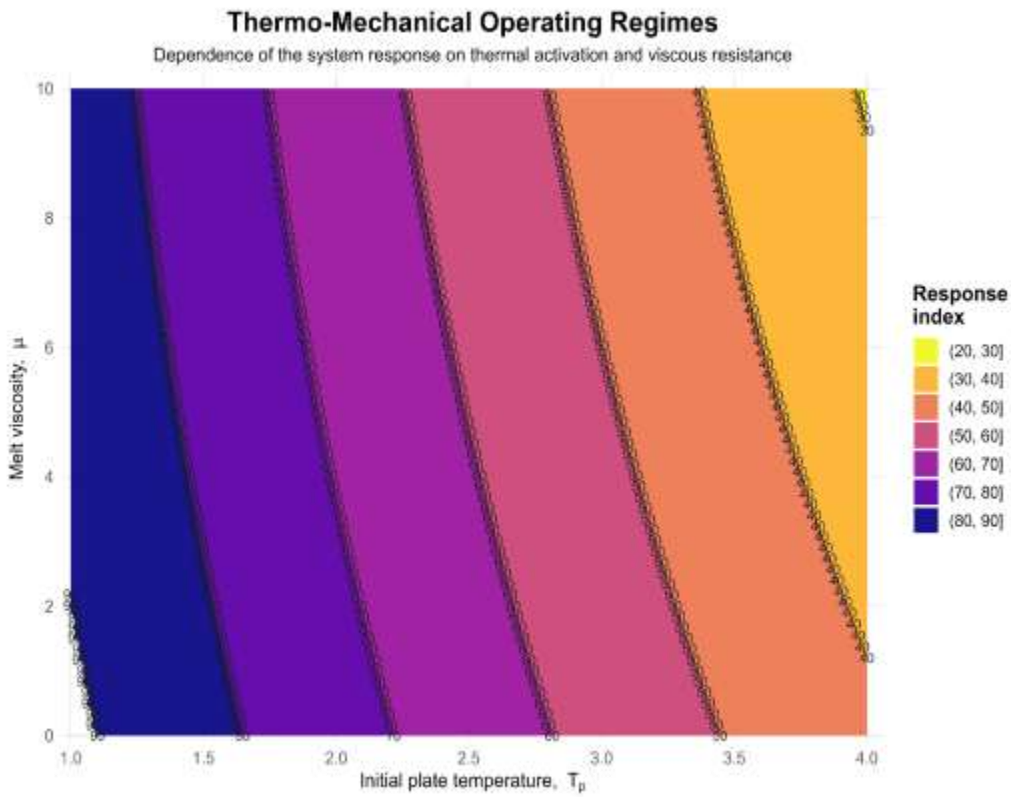

Figure 2: Regime map

Figure 2 provides a global view of system behavior across the parameter space defined by initial temperature and viscosity. The contour structure reveals a smooth transition from high-response regimes (low viscosity, high thermal input) to strongly damped regimes (high viscosity). The near-linear contour alignment indicates a coupled dependence between thermal activation and viscous resistance, emphasizing the balance between melting intensity and mechanical dissipation.

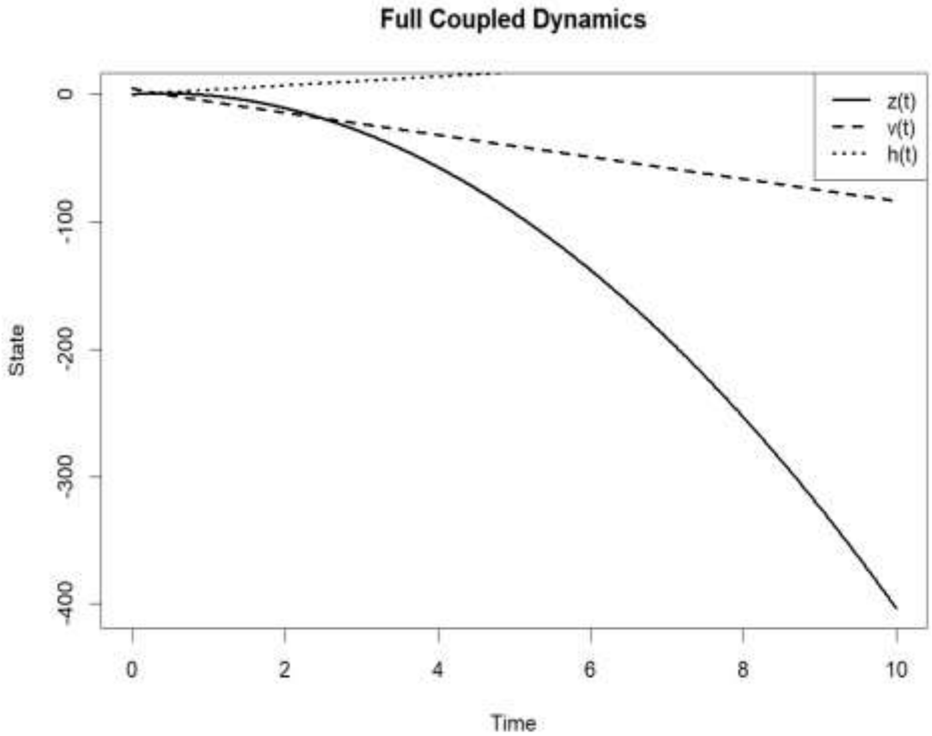

Figure 3: **Full Coupled Dynamics**

Figure 3 illustrates the temporal evolution of position $z(t)$, velocity $v(t)$, and melt thickness $h(t)$. The monotonic decrease in velocity and the rapid growth of melt thickness indicate strong viscous damping and active phase-change dynamics. The nonlinear interaction between melting and resistance leads to accelerated penetration, confirming the dominant role of the melt layer in governing mechanical response.

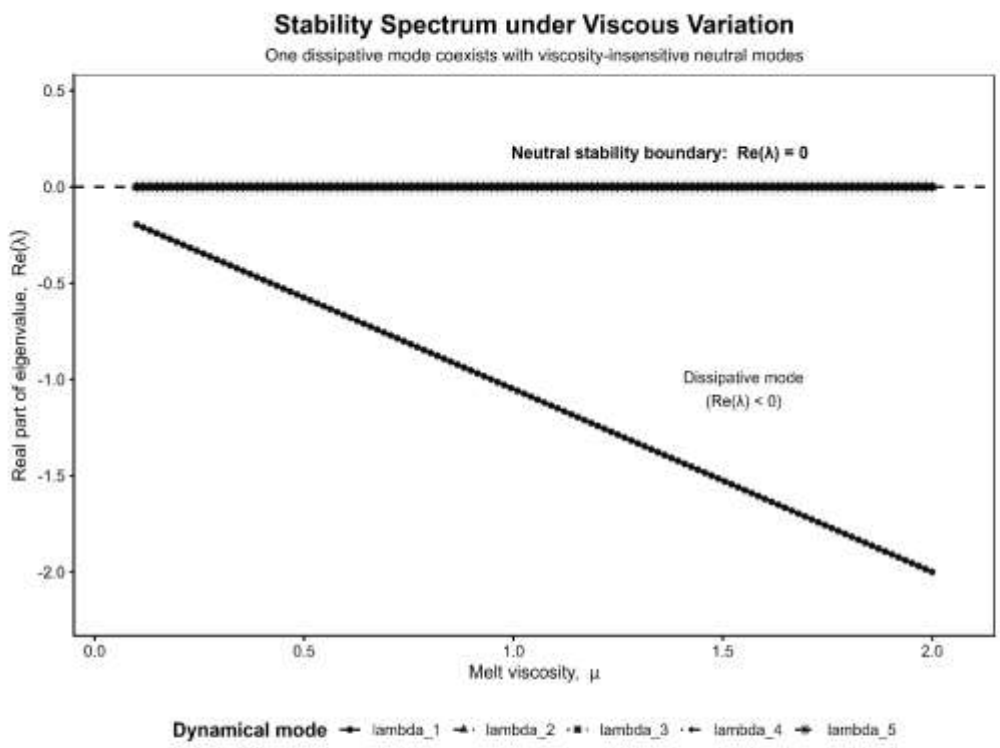

Figure 4: Eigenvalue Spectrum and Stability Structure

Figure 4 presents the dependence of the eigenvalues on viscosity. One eigenvalue decreases monotonically with increasing viscosity, reflecting increasing mechanical damping. The remaining eigenvalues remain at or near zero, indicating the presence of multiple neutral modes. The absence of positive eigenvalues confirms that the system is not unstable, while the persistence of zero eigenvalues establishes that the equilibrium is marginally stable rather than asymptotically stable**.**

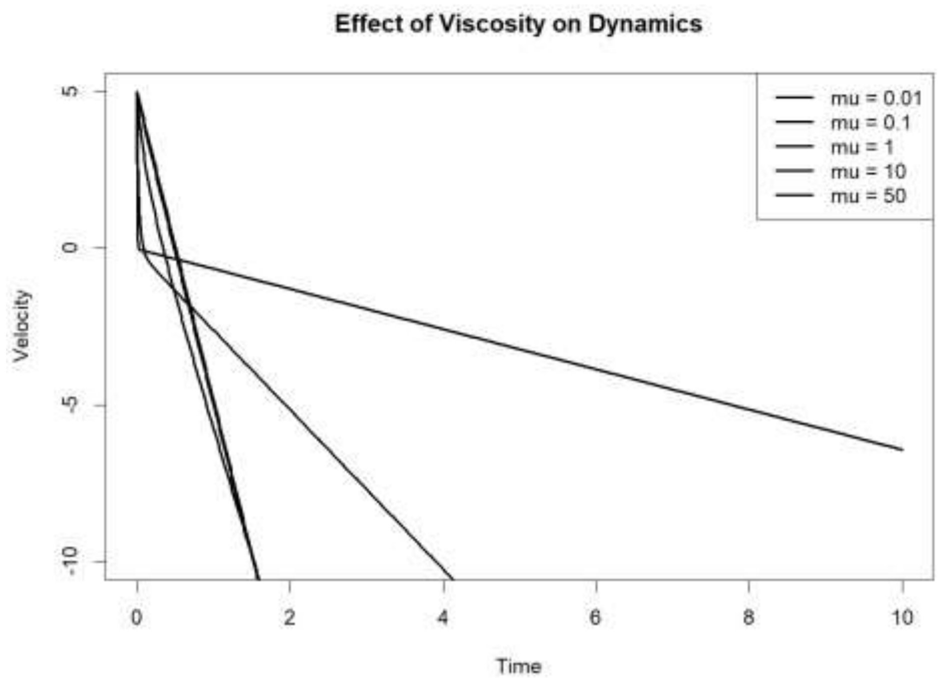

Figure 5: **Effect of Viscosity on Dynamics**

Figure 5 demonstrates the sensitivity of the system to variations in viscosity. As the viscosity parameter increases, the velocity decays more rapidly, indicating enhanced dissipation. For sufficiently large viscosity, the motion is strongly suppressed, highlighting the transition from weakly damped to heavily damped regimes. This confirms that viscous resistance is a primary control parameter in the system.

4 Conclusion

This study developed a fully coupled nonlinear dynamical model describing the interaction between a rigid body and an actively heated surface in the presence of phase change. By integrating mechanical motion, thermal transport, and melt-layer evolution within a single framework, the work provides a unified description of systems in which heat transfer and motion are intrinsically linked through a thin viscous film.

The formulation captures the essential feedback mechanism: thermal activation induces melting, the resulting melt layer modifies viscous resistance, and this resistance in turn governs the mechanical response. This coupling leads to a strongly nonlinear system whose behavior cannot be adequately described using decoupled or quasi-static approaches.

Through non-dimensionalization and analytical investigation, the key parameters controlling the system dynamics were identified, revealing the roles of viscous resistance, thermal coupling, and phase-change intensity. The stability analysis showed that the governing system possesses a structured Jacobian with separable mechanical and thermal components. The resulting eigenvalue spectrum is characterized by one dissipative mode and multiple neutral modes, leading to marginal stability of the equilibrium state.

Numerical simulations further supported the analytical findings, demonstrating the influence of viscosity on system dynamics, the role of thermal activation in initiating phase change, and the evolution of the melt layer as a controlling factor in motion. The results highlight the transition between different dynamical regimes and confirm the dominant role of the melt layer in mediating the interaction between thermal and mechanical effects.

Overall, the study establishes a mathematically consistent framework for analyzing thermo-mechanical systems with phase change and thin-film effects. The identification of marginal stability and its connection to the coupled structure of the system provides new insight into the dynamics of melting-driven processes. These findings open pathways for further extensions, including the incorporation of additional dissipative mechanisms, spatial effects, and more complex material behaviors.

Conflict of Interest: The author declares no conflict of interest.

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