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Computational Numerical Method for the Direct Solution of Highly Third Order Oscillatory Linear Initial Value Problems

Donald J. Zirra[a], **Adamu Wakili**[a] **and Omotayo S. Olumide**[a]
[a]*Department of Mathematics, Adamawa State University, Mubi, Adamawa State*

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ABSTRACT

This paper presents the development and analysis of a Computational Numerical Method (CNM) for the direct solution of highly third order oscillatory linear initial value problems (IVPs). The method was derived using a linear multistep framework based on power series approximation combined with interpolation and collocation techniques. The formulation avoids the reduction of higher-order problems to systems of first-order equations, thereby improving computational efficiency and reducing accumulated errors. The theoretical properties of the method, including order, error constant, consistency, zero-stability, convergence and region of absolute stability, are rigorously analyzed. The results show that the CNM is of uniform order eight, consistent, zero-stable, convergent and A-stable, making it suitable for both stiff and non-stiff differential equations. The effectiveness of the CNM is verified through numerical simulations of several third-order oscillatory initial value problems. The results are presented in tabular form and compared with existing methods in the literature. The comparisons demonstrate that the CNM provides highly accurate approximations and performs competitively or better than the referenced schemes for both stiff and non-stiff problems. The outcomes confirm the reliability, stability and computational efficiency of the CNM approach in solving third-order oscillatory IVPs.

1. Introduction

Scientists and engineers use mathematical models to create differential equations which serve as their results. The equations enable scientists to create models which show how physical quantities change over time and through different locations. The strict structure of the system enables users to understand multiple dynamic systems through different scientific laws which scientists express using derivatives (Kuboye, 2015; Skwame *et al.,* 2024). The mathematical representation of real-world problems will give rise to differential equations that connect the dependent variable(s) to one or more independent variables and their derivatives, thus making it possible to systematically model and analyze continuous change (Kuboye, 2015). The process of Initial Value Problems (IVPs) serves as the fundamental component of ordinary differential equations because it enables system state prediction through initial conditions while also representing natural and physical phenomena which evolve with time (Jimoh, 2024). Scientists and engineers use differential equations with initial value problems to conduct their work because these mathematical systems serve as essential methods to study complex systems in various fields including economics medical research biological studies psychological research operations research and social sciences which involve market trends and disease distribution and population dynamics and cultural evolution (Kuboye and Omar, (2016); Abdulsalam *et al.*, (2019); Abolarin *et al.*, (2020); Abdulrahim, (2021).

* **Corresponding author. Tel.:** +2348036153072
E-mail address: donaldjohnz@yahoo.com, (Zirra D. J.)
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This study consider the direct solution of third order initial value problem of the form

$$y'''(\xi)=f(\xi, y, y', y''),\ y(a_0)=\xi_0,\ y'(a_1)=\xi_1,\ y''(a_2)=\xi_2 \tag{1}$$

The numerical methods which people use for solving third-order higher-order IVPs start from the reduction method which transforms the initial problem into a collection of first-order initial value problems. The system requires standard numerical methods for its solution, yet existing literature has documented the system's associated problems which include excessive computing requirements, extended development times, potential precision loss through error buildup, and resulting increased workload for operators (Roba *et al.,* 2017; Tumba *et al.,* 2021; Raymond *et al.,* 2023). The numerical methods which solve third-order initial value problems without any reduction have completely replaced traditional methods that existed before. The new method which scientists developed require less computational resources to achieve their goals while delivering greater accuracy than traditional methods according to their evaluation of former research results; therefore, they serve as reliable options for scientific and engineering work (Raymond *et al.,* 2021; Kashif *et al.,* 2023; Sabo *et al.,* 2024).

Numerous studies have focused on the analysis of predictor-corrector methods because these numerical techniques produce better results through their iterative refinement process. The combination of linear multistep schemes with this method enables users to access multiple historical solution points which help them create their most accurate estimate (Omar *et al.,* 2016; Kayode and Adegboro, 2018; Donald *et al.,* 2022). The research achieved its goal through the development of high-order block predictor-corrector methods which solve third-order initial value problems, and the seventh-order method developed by Omar *et al.,* (2016) which demonstrated superior efficiency compared to standard methods. The research dedicated to numerical formulations which use higher-order derivatives has yielded significant advancements in both convergence and accuracy, thus researchers can establish that third derivatives improve solution efficiency for third-order IVPs when applied through linear multistep methods.

This study considers a general linear multistep method based on the approach presented in Sunday, (2018) of the form

$$\sum_{j=0}^{1}\alpha_j y_{n+j}=h^3\sum_{j=0}^{1}\beta_j f_{n+j} \tag{2}$$

2. Method

2.1 Derivation of Computational Numerical Method (CNM)

The Computational Numerical Method (CNM) of the form

$$\mathbf{A}^{(0)}\mathbf{Y}_m^{(i)}=\sum_{i=0}^{1}\frac{(jh)^{(i)}}{i!}e_i y_n^{(i)}+h^{(3-i)}\left[\mathbf{d}_i f(y_n)+\mathbf{b}_i\mathbf{F}(\mathbf{Y}_m)\right] \tag{3}$$

shall be derived for the direct solution of third order initial value problems of the form (1.1).

A power series basis function of the form

$$y(\xi)=\sum_{j=0}^{r+s-1}a_j\xi^j \tag{4}$$

is consider, where $r\ and\ s$ are the numbers of collocation and interpolation points respectively.

Differentiate equation (4) three times to obtain

$$y'''(\xi)=\sum_{j=0}^{r+s-1}j(j-1)(j-2)a_j\xi^{j-3} \tag{5}$$

Equation (5) is then substituted into (1) to obtain a differential system as

$$y'''(\xi)=\sum_{j=0}^{r+s-1} j(j-1)(j-2)a_j\xi^{j-3}=f\left(\xi,y,y',y''\right) \tag{6}$$

Interpolating (4) at point $\xi_{n+s},\ s=\frac{1}{4},\frac{1}{3},\frac{1}{2}$ and collocating (6) at points $\xi_{n+r},\ r=0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},1$, gives a system of nonlinear equation of the form,

$$TA=U \tag{7}$$

where

$$T=\begin{bmatrix}
1 & \xi_{n+\frac{1}{4}} & \xi^2_{n+\frac{1}{4}} & \xi^3_{n+\frac{1}{4}} & \xi^4_{n+\frac{1}{4}} & \xi^5_{n+\frac{1}{4}} & \xi^6_{n+\frac{1}{4}} & \xi^7_{n+\frac{1}{4}} & \xi^8_{n+\frac{1}{4}} & \xi^9_{n+\frac{1}{4}} \\
1 & \xi_{n+\frac{1}{3}} & \xi^2_{n+\frac{1}{3}} & \xi^3_{n+\frac{1}{3}} & \xi^4_{n+\frac{1}{3}} & \xi^5_{n+\frac{1}{3}} & \xi^6_{n+\frac{1}{3}} & \xi^7_{n+\frac{1}{3}} & \xi^8_{n+\frac{1}{3}} & \xi^9_{n+\frac{1}{3}} \\
1 & \xi_{n+\frac{1}{2}} & \xi^2_{n+\frac{1}{2}} & \xi^3_{n+\frac{1}{2}} & \xi^4_{n+\frac{1}{2}} & \xi^5_{n+\frac{1}{2}} & \xi^6_{n+\frac{1}{2}} & \xi^7_{n+\frac{1}{2}} & \xi^8_{n+\frac{1}{2}} & \xi^9_{n+\frac{1}{2}} \\
0 & 0 & 0 & 6 & 12\xi_n & 60\xi_n^2 & 120\xi_n^3 & 210\xi_n^4 & 336\xi_n^5 & 504\xi_n^6 \\
0 & 0 & 0 & 6 & 12\xi_{n+\frac{1}{4}} & 60\xi^2_{n+\frac{1}{4}} & 120\xi^3_{n+\frac{1}{4}} & 210\xi^4_{n+\frac{1}{4}} & 336\xi^5_{n+\frac{1}{4}} & 504\xi^6_{n+\frac{1}{4}} \\
0 & 0 & 0 & 6 & 12\xi_{n+\frac{1}{3}} & 60\xi^2_{n+\frac{1}{3}} & 120\xi^3_{n+\frac{1}{3}} & 210\xi^4_{n+\frac{1}{3}} & 336\xi^5_{n+\frac{1}{3}} & 504\xi^6_{n+\frac{1}{3}} \\
0 & 0 & 0 & 6 & 12\xi_{n+\frac{1}{2}} & 60\xi^2_{n+\frac{1}{2}} & 120\xi^3_{n+\frac{1}{2}} & 210\xi^4_{n+\frac{1}{2}} & 336\xi^5_{n+\frac{1}{2}} & 504\xi^6_{n+\frac{1}{2}} \\
0 & 0 & 0 & 6 & 12\xi_{n+\frac{2}{3}} & 60\xi^2_{n+\frac{2}{3}} & 120\xi^3_{n+\frac{2}{3}} & 210\xi^4_{n+\frac{2}{3}} & 336\xi^5_{n+\frac{2}{3}} & 504\xi^6_{n+\frac{2}{3}} \\
0 & 0 & 0 & 6 & 12\xi_{n+\frac{3}{4}} & 60\xi^2_{n+\frac{3}{4}} & 120\xi^3_{n+\frac{3}{4}} & 210\xi^4_{n+\frac{3}{4}} & 336\xi^5_{n+\frac{3}{4}} & 504\xi^6_{n+\frac{3}{4}} \\
0 & 0 & 0 & 6 & 12\xi_{n+1} & 60\xi^2_{n+1} & 120\xi^3_{n+1} & 210\xi^4_{n+1} & 336\xi^5_{n+1} & 504\xi^6_{n+1}
\end{bmatrix}$$

$$U=\begin{bmatrix} y_{n+\frac{1}{4}} & y_{n+\frac{1}{3}} & y_{n+\frac{1}{2}} & f_n & f_{n+\frac{1}{4}} & f_{n+\frac{1}{3}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{3}{4}} & \xi_{n+1}\end{bmatrix}^T,$$

$$A=\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9\end{bmatrix}^T$$

Solving equation (7) for $a_j,\ j=0(1)9$ which are constants to be determined and putting back into (4) gives a computational continuous scheme of the form

$$y(\xi)=\alpha_{\frac{1}{4}}(\xi)y_{n+\frac{1}{4}}+\alpha_{\frac{1}{3}}(\xi)y_{n+\frac{1}{3}}+\alpha_{\frac{1}{2}}(\xi)y_{n+\frac{1}{2}}+h^3\left[\sum_{j=0}^{1}\beta_j(\xi)f_{n+j}+\beta_s(\xi)f_{n+s}\right],\ s=0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},1 \tag{8}$$

Where the values of the continuous form

$\alpha_{\frac{1}{4}}(\xi),\ \alpha_{\frac{1}{3}}(\xi),\ \alpha_{\frac{2}{3}}(\xi),\ \beta_0(\xi),\ \beta_{\frac{1}{4}}(\xi),\ \beta_{\frac{1}{3}}(\xi),\ \beta_{\frac{1}{2}}(\xi),\ \beta_{\frac{2}{3}}(\xi),\ \beta_{\frac{3}{4}}(\xi),\ \beta_1(\xi)$ in equation (8) are obtained as

$$\left.\begin{array}{l}
\alpha_{\frac{1}{4}}(\xi)=8-40t+48t^{2}\\
\alpha_{\frac{1}{3}}(\xi)=-9+54t-72t^{2}\\
\alpha_{\frac{2}{3}}(\xi)=2-14t+24t^{2}\\
\beta_{0}(\xi)=-\frac{13842805793}{75931542650880}+\frac{978507901999}{25310514216960}t-\frac{1305470587619}{37965771325440}t^{2}+\frac{1}{6}t^{3}-\frac{696199}{1453104}t^{4}+\frac{151319}{181638}t^{5}-\frac{1240393}{1453104}t^{6}+\frac{427561}{908190}t^{7}-\frac{4309}{40364}t^{8}-\frac{20}{211911}t^{9}\\
\beta_{\frac{1}{4}}(\xi)=-\frac{889595201}{148303794240}+\frac{3592139287}{49434598080}t-\frac{2932674917}{105931281660}t^{2}+\frac{430336}{151365}t^{4}-\frac{774400}{90819}t^{5}+\frac{1720064}{151365}t^{6}-\frac{23197184}{3178665}t^{7}+\frac{642304}{353185}t^{8}+\frac{512}{211911}t^{9}\\
\beta_{\frac{1}{3}}(\xi)=\frac{67147079}{312475484160}-\frac{574349933}{20831698944}t+\frac{6025009297}{31247548416}t^{2}-\frac{272241}{80728}t^{4}+\frac{4626801}{403640}t^{5}-\frac{13328793}{807280}t^{6}+\frac{7879113}{706370}t^{7}-\frac{812025}{282548}t^{8}-\frac{324}{70637}t^{9}\\
\beta_{\frac{1}{2}}(\xi)=-\frac{1922101141}{1581907138560}+\frac{9042264611}{527302379520}t-\frac{63236223871}{790953569280}t^{2}+\frac{13428}{10091}t^{4}-\frac{765076}{151365}t^{5}+\frac{1247284}{151365}t^{6}-\frac{6506168}{1059555}t^{7}+\frac{119772}{70637}t^{8}+\frac{320}{70637}t^{9}\\
\beta_{\frac{2}{3}}(\xi)=-\frac{76471163}{312475484160}+\frac{389242573}{104158494720}t-\frac{2885037809}{156237742080}t^{2}+\frac{270297}{807280}t^{4}-\frac{269973}{201820}t^{5}+\frac{1885599}{807280}t^{6}-\frac{268029}{141274}t^{7}+\frac{790641}{1412740}t^{8}-\frac{324}{70637}t^{9}\\
\beta_{\frac{3}{4}}(\xi)=\frac{45253}{59321517696}-\frac{1229887}{98869196160}t+\frac{1944463}{29660758848}t^{2}-\frac{128}{90819}t^{4}+\frac{2944}{454095}t^{5}-\frac{1280}{90819}t^{6}+\frac{10496}{635733}t^{7}-\frac{704}{70637}t^{8}-\frac{512}{211911}\\
\beta_{1}(\xi)=-\frac{683132693}{75931542650880}+\frac{717659399}{5062102843392}t-\frac{27354093887}{37965771325440}t^{2}+\frac{3377}{242184}t^{4}-\frac{212831}{3632760}t^{5}+\frac{89021}{807280}t^{6}-\frac{639833}{6357330}t^{7}+\frac{10201}{282548}t^{8}-\frac{20}{211911}t^{9}
\end{array}\right\} \qquad (9)$$

The values of the unknown coefficients in equation (9) are obtained using the scientific work place software as presented in appendix A.

where

$$t=\frac{\xi-\xi_{n}}{h} \qquad (10)$$

Solving (8) for the independent solution will eventually give a continuous linear multistep method of the form,

$$y(\xi)=\sum_{i=0}^{1}\frac{(jh)^{i}}{i!}y_{n}^{(i)}+h^{3}\left[\sum_{j=0}^{1}\sigma_{j}(\xi)f_{n+j}+\sigma_{s}(\xi)f_{n+s}\right],\ s=0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},1 \qquad (11)$$

Where

Where the values of the continuous form

$\sigma_{0}(\xi),\ \sigma_{\frac{1}{4}}(\xi),\ \sigma_{\frac{1}{3}}(\xi),\ \sigma_{\frac{1}{2}}(\xi),\ \sigma_{\frac{2}{3}}(\xi),\ \sigma_{\frac{3}{4}}(\xi),\ \sigma_{1}(\xi)$ in equation (11) are obtained as

$$\left.\begin{array}{l}
\sigma_{0}(\xi)=\frac{1}{6}t^{3}+\frac{696199}{1453104}t^{4}-\frac{151319}{181638}t^{5}+\frac{1240393}{1453104}t^{6}-\frac{427561}{908190}t^{7}-\frac{4309}{40364}t^{8}-\frac{20}{211911}t^{9}\\
\sigma_{\frac{1}{4}}(\xi)=\frac{430336}{151365}t^{4}-\frac{774400}{90819}t^{5}+\frac{1720064}{151365}t^{6}-\frac{23197184}{3178665}t^{7}+\frac{642304}{353185}t^{8}-\frac{512}{211911}t^{9}\\
\sigma_{\frac{1}{3}}(\xi)=-\frac{272241}{80728}t^{4}+\frac{4626801}{403640}t^{5}-\frac{13328793}{807280}t^{6}+\frac{7879113}{706370}t^{7}-\frac{812025}{282548}t^{8}+\frac{324}{70637}t^{9}\\
\sigma_{\frac{1}{2}}(\xi)=\frac{13428}{10091}t^{4}-\frac{765076}{151365}t^{5}+\frac{1247284}{151365}t^{6}-\frac{6506168}{1059555}t^{7}-\frac{119772}{70637}t^{8}-\frac{320}{70637}t^{9}\\
\sigma_{\frac{2}{3}}(\xi)=-\frac{270297}{807280}t^{4}+\frac{269973}{201820}t^{5}-\frac{1885599}{807280}t^{6}+\frac{268029}{141274}t^{7}-\frac{790641}{1412740}t^{8}-\frac{324}{70637}t^{9}\\
\sigma_{\frac{3}{4}}(\xi)=-\frac{128}{90819}t^{4}+\frac{2944}{454095}t^{5}-\frac{1280}{90819}t^{6}+\frac{10496}{635733}t^{7}-\frac{704}{70637}t^{8}+\frac{512}{211911}t^{9}\\
\sigma_{1}(\xi)=\frac{3377}{242184}t^{4}-\frac{212831}{3632760}t^{5}+\frac{89021}{807280}t^{6}-\frac{639833}{6357330}t^{7}-\frac{10201}{282548}t^{8}+\frac{20}{211911}t^{9}
\end{array}\right\} \qquad (12)$$

and t is as defined in equation (10), the values are obtained using scientific workplace as presented in appendix B. Evaluating (11) at $\xi=\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},1$, gives a computational numerical method (CNM) of the form (3) where,

$$\mathbf{Y}_m^{(i)}=\begin{bmatrix} y_{n+\frac{1}{4}}^{(i)} & y_{n+\frac{1}{3}}^{(i)} & y_{n+\frac{1}{2}}^{(i)} & y_{n+\frac{2}{3}}^{(i)} & y_{n+\frac{3}{4}}^{(i)} & y_{n+1}^{(i)} \end{bmatrix}^T,\ \mathbf{F}(\mathbf{Y}_m)=\begin{bmatrix} f_{n+\frac{1}{4}} & f_{n+\frac{1}{3}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{4}{4}} & f_{n+1} \end{bmatrix}^T,$$

$$y_n^{(i)}=\begin{bmatrix} y_{n-\frac{1}{4}}^{(i)} & y_{n-\frac{1}{3}}^{(i)} & y_{n-\frac{1}{2}}^{(i)} & y_{n-\frac{2}{3}}^{(i)} & y_{n-\frac{3}{4}}^{(i)} & y_n^{(i)} \end{bmatrix}^T$$

and $\mathbf{A}^{(0)}$ is a 6×6 identity matrix.

When $i=0$

$$e_0=\begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix}, e_1=\begin{bmatrix} 0&0&0&0&0&\frac{1}{4}\\ 0&0&0&0&0&\frac{1}{3}\\ 0&0&0&0&0&\frac{1}{2}\\ 0&0&0&0&0&\frac{2}{3}\\ 0&0&0&0&0&\frac{3}{4}\\ 0&0&0&0&0&1 \end{bmatrix}, e_2=\begin{bmatrix} 0&0&0&0&0&\frac{1}{32}\\ 0&0&0&0&0&\frac{1}{18}\\ 0&0&0&0&0&\frac{1}{8}\\ 0&0&0&0&0&\frac{2}{9}\\ 0&0&0&0&0&\frac{9}{32}\\ 0&0&0&0&0&\frac{1}{2} \end{bmatrix}, d_0=\begin{bmatrix} 0&0&0&0&0&\frac{8917}{6881280}\\ 0&0&0&0&0&\frac{10583}{4133430}\\ 0&0&0&0&0&\frac{2069}{322560}\\ 0&0&0&0&0&\frac{24881}{2066715}\\ 0&0&0&0&0&\frac{17793}{1146880}\\ 0&0&0&0&0&\frac{1}{35} \end{bmatrix},$$

$$b_0=\begin{bmatrix} \frac{3181}{460800} & -\frac{25029}{2867200} & \frac{4507}{860160} & -\frac{43821}{11468800} & \frac{633}{358400} & -\frac{521}{10321920}\\ \frac{170272}{10333575} & -\frac{1937}{97200} & \frac{8108}{688905} & -\frac{1451}{170100} & \frac{40672}{10333575} & -\frac{3707}{33067440}\\ \frac{629}{12600} & -\frac{9963}{179200} & \frac{1}{30} & -\frac{4293}{179200} & \frac{139}{12600} & -\frac{101}{322560}\\ \frac{1055744}{10333575} & -\frac{4558}{42525} & \frac{47296}{688905} & -\frac{289}{6075} & \frac{226304}{10333575} & -\frac{1282}{2066715}\\ \frac{48519}{358400} & -\frac{1594323}{11468800} & \frac{26487}{286720} & -\frac{177147}{2867200} & \frac{1467}{51200} & -\frac{1863}{2293760}\\ \frac{416}{1575} & -\frac{729}{2800} & \frac{4}{21} & -\frac{81}{700} & \frac{32}{525} & \frac{1}{720} \end{bmatrix}$$

When $i=1$

$$e_1=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}, e_2=\begin{bmatrix}0&0&0&0&0&\frac{1}{4}\\0&0&0&0&0&\frac{1}{3}\\0&0&0&0&0&\frac{1}{2}\\0&0&0&0&0&\frac{2}{3}\\0&0&0&0&0&\frac{3}{4}\\0&0&0&0&0&1\end{bmatrix}, d_1=\begin{bmatrix}0&0&0&0&0&\frac{129271}{10321920}\\0&0&0&0&0&\frac{32741}{1837080}\\0&0&0&0&0&\frac{2293}{80640}\\0&0&0&0&0&\frac{8968}{229635}\\0&0&0&0&0&\frac{50871}{1146880}\\0&0&0&0&0&\frac{151}{2520}\end{bmatrix},$$

$$b_1=\begin{bmatrix}\frac{111}{1280}&-\frac{120447}{1146880}&\frac{13253}{215040}&-\frac{51111}{1146880}&\frac{1657}{80640}&-\frac{2011}{3440640}\\\frac{6592}{45927}&-\frac{22}{135}&\frac{7268}{76545}&-\frac{517}{7560}&\frac{7232}{229635}&-\frac{821}{918540}\\\frac{9}{35}&-\frac{2349}{8960}&\frac{1}{6}&-\frac{1053}{8960}&\frac{17}{315}&-\frac{41}{26880}\\\frac{84992}{229635}&-\frac{338}{945}&\frac{19904}{76545}&-\frac{22}{135}&\frac{17408}{229635}&-\frac{494}{229635}\\\frac{765}{1792}&-\frac{465831}{1146880}&\frac{22167}{71680}&-\frac{203391}{1146880}&\frac{111}{1280}&-\frac{2817}{1146880}\\\frac{64}{105}&-\frac{81}{140}&\frac{52}{105}&-\frac{81}{280}&\frac{64}{315}&0\end{bmatrix}$$

When $i=2$

$$e_2=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}, d_2=\begin{bmatrix}0&0&0&0&0&\frac{41059}{645120}\\0&0&0&0&0&\frac{12967}{204120}\\0&0&0&0&0&\frac{2573}{40320}\\0&0&0&0&0&\frac{541}{8505}\\0&0&0&0&0&\frac{4563}{71680}\\0&0&0&0&0&\frac{151}{2520}\end{bmatrix}, b_2=\begin{bmatrix}\frac{3313}{5040}&-\frac{51273}{71680}&\frac{1357}{3360}&-\frac{4131}{14336}&\frac{667}{5040}&-\frac{2411}{645120}\\\frac{5888}{8505}&-\frac{1667}{2520}&\frac{3368}{8505}&-\frac{143}{504}&\frac{3328}{25515}&-\frac{251}{68040}\\\frac{315}{212}&-\frac{2511}{4480}&\frac{52}{105}&-\frac{1377}{4480}&\frac{44}{315}&-\frac{517}{40320}\\\frac{17408}{25515}&-\frac{184}{315}&\frac{5056}{8505}&-\frac{13}{63}&\frac{1024}{8505}&-\frac{92}{25515}\\\frac{381}{560}&-\frac{41553}{71680}&\frac{657}{1120}&-\frac{2187}{14336}&\frac{87}{560}&-\frac{267}{71680}\\\frac{256}{315}&-\frac{280}{243}&\frac{104}{105}&-\frac{243}{280}&\frac{315}{256}&\frac{151}{2520}\end{bmatrix}$$

3. Analysis of Basic Properties of the Computational Numerical Method (CNM)

In this section, the order, error constant, consistency, zero-stability and region of absolute stability of the CNM are numerically analyzed.

3.1 Order and Error Constant of CNM

Definition 1: Order of the CNM

The linear operator associated with the CNM is defined as;

$$\ell\{y(t):h\}=\mathbf{A}^{(0)}\mathbf{Y}_m^{(i)}-\sum_{i=0}^{1}\frac{(jh)^{(i)}}{i!}e_i y_n^{(i)}+h^{(3-i)}\left[\mathbf{d}_i f(y_n)+\mathbf{b}_i\mathbf{F}(\mathbf{Y}_m)\right] \tag{13}$$

Assuming that $y(t)$ is sufficiently differentiable, we write the terms in (13) as a Taylor series expansion about the point t to obtain the expression;

$$\ell\{y(t):h\}=C_0y(t)+C_1y'(t)+...+C_ph^py^p(t)+C_{p+1}h^{p+1}y^{p+1}(t)+C_{p+2}h^{p+2}y^{p+2}(t)+... \quad (14)$$

where the constant coefficients $c_p, p=0,1,2,...$ are given by;

$$\left.\begin{aligned} c_0 &= \sum_{j=0}^{k}\alpha_j \\ c_1 &= \sum_{j=0}^{k}\left(j\alpha_j-\beta_j\right) \\ &. \\ &. \\ &. \\ c_p &= \sum_{j=0}^{k}\left[\frac{1}{q!}j^q\alpha_j-\frac{1}{(q-1)!}j^{q-1}\beta_j\right],\ q=2,3,... \end{aligned}\right\} \quad (15)$$

The computational numerical method and the associated linear difference operators are said to have order p if $C_0=C_1=...=C_p=C_{p+1}=C_{p+2}=0,\ C_{p+3}\neq 0.$ The order is also defined as the largest positive real number p that quantifies the rate of convergence of a numerical approximation of a differential equation to that of the exact solution (Sunday, 2018).

Definition 2: Error Constant

The term C_{p+3} is called the error constant and implies that the local truncation error for (3) is given by,

$$T_{n+k}=C_{p+3}h^{p+3}y^{p+3}(t)+O(h^{p+4}) \quad (16)$$

The error constant is the accumulated error when the order of a CNM has been computed (Sunday, 2018). Applying Definition 1 on the CNM, we obtain

$$\left.\begin{array}{l}
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}h\right)^{j}}{j!}-y_{n}-\frac{1}{4}hy_{n}^{'}-\frac{1}{32}h^{2}y_{n}^{''}-\frac{8917}{6881280}h^{3}y_{n}^{'''}-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_{n}^{j+3}\left[\begin{array}{l}\frac{3181}{460800}\left(\frac{1}{4}\right)^{j}-\frac{25029}{2867200}\left(\frac{1}{3}\right)^{j}+\frac{4507}{860160}\left(\frac{1}{2}\right)^{j}\\ -\frac{43821}{11468800}\left(\frac{2}{3}\right)^{j}+\frac{633}{358400}\left(\frac{3}{4}\right)^{j}-\frac{521}{10321920}(1)^{j}\end{array}\right]=0\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}h\right)^{j}}{j!}-y_{n}-\frac{1}{3}hy_{n}^{'}-\frac{1}{18}h^{2}y_{n}^{''}-\frac{10583}{4133430}h^{3}y_{n}^{'''}-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_{n}^{j+3}\left[\begin{array}{l}\frac{170272}{10333575}\left(\frac{1}{4}\right)^{j}-\frac{1937}{97200}\left(\frac{1}{3}\right)^{j}+\frac{8108}{688905}\left(\frac{1}{2}\right)^{j}\\ -\frac{1451}{170100}\left(\frac{2}{3}\right)^{j}+\frac{40672}{10333575}\left(\frac{3}{4}\right)^{j}-\frac{3707}{33067440}(1)^{j}\end{array}\right]=0\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}h\right)^{j}}{j!}-y_{n}-\frac{1}{2}hy_{n}^{'}-\frac{1}{8}h^{2}y_{n}^{''}-\frac{2069}{322560}h^{3}y_{n}^{'''}-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_{n}^{j+3}\left[\begin{array}{l}\frac{629}{12600}\left(\frac{1}{4}\right)^{j}-\frac{9963}{179200}\left(\frac{1}{3}\right)^{j}+\frac{1}{30}\left(\frac{1}{2}\right)^{j}\\ -\frac{4293}{179200}\left(\frac{2}{3}\right)^{j}+\frac{139}{12600}\left(\frac{3}{4}\right)^{j}-\frac{101}{322560}(1)^{j}\end{array}\right]=0\\
\sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}h\right)^{j}}{j!}-y_{n}-\frac{2}{3}hy_{n}^{'}-\frac{2}{9}h^{2}y_{n}^{''}-\frac{24881}{2066715}h^{3}y_{n}^{'''}-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_{n}^{j+3}\left[\begin{array}{l}\frac{1055744}{10333575}\left(\frac{1}{4}\right)^{j}-\frac{4558}{42525}\left(\frac{1}{3}\right)^{j}+\frac{47296}{688905}\left(\frac{1}{2}\right)^{j}\\ -\frac{289}{6075}\left(\frac{2}{3}\right)^{j}+\frac{226304}{10333575}\left(\frac{3}{4}\right)^{j}-\frac{1282}{2066715}(1)^{j}\end{array}\right]=0\\
\sum_{j=0}^{\infty}\frac{(h)^{j}}{j!}-y_{n}-hy_{n}^{'}-\frac{1}{2}h^{2}y_{n}^{''}-\frac{1}{35}h^{3}y_{n}^{'''}-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_{n}^{j+3}\left[\frac{416}{1575}\left(\frac{1}{4}\right)^{j}-\frac{729}{2800}\left(\frac{1}{3}\right)^{j}+\frac{4}{21}\left(\frac{1}{2}\right)^{j}-\frac{81}{700}\left(\frac{2}{3}\right)^{j}+\frac{32}{525}\left(\frac{3}{4}\right)^{j}-\frac{1}{720}(1)^{j}\right]=0
\end{array}\right\} \qquad (17)$$

Thus, $\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=\bar{c}_6=\bar{c}_7=\bar{c}_8=\bar{c}_9=\bar{c}_{10}=0,\ \bar{c}_{11}\neq 0$; by definition 2, it imply that the order of the CNM is $p=\begin{bmatrix}8 & 8 & 8 & 8 & 8 & 8\end{bmatrix}^{T}$. That is, the CNM is of uniform order 8. The error constant is given by

$$\begin{bmatrix}1.9483\times10^{-10} & 4.2906\times10^{-10} & 1.1880\times10^{-09} & 2.3480\times10^{-09} & 3.0654\times10^{-09} & 5.7411\times10^{-09}\end{bmatrix}^{T}.$$

3.2 Consistency of the CNM

The CNM is consistent since it has order $p\geq 1$. Consistency controls the magnitude of the local truncation error committed at each stage of the computation (Kuboye *et al.*, 2020).

3.3 Zero Stability of the CNM

Definition 5: Zero-Stability

A CNM is said to be zero-stable if the roots $z_s,\ s=1,2,\ldots,n$ of the first characteristic polynomial $\bar{\rho}(z)$, defined by

$$\bar{\rho}(z)=\det\left[zA^{(0)}-E\right] \qquad (18)$$

Equation (18) satisfies $|z_s|\leq 1$ and every root with $|z_s|=1$ has multiplicity not exceeding the order of the differential equation as $h\rightarrow 0$. Moreover, as $h\rightarrow 0$, $\rho(z)=z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ *and* E (Kuboye *et al.*, 2020). The main consequence of zero-stability is to control the propagation of the error as the integration proceeds.

Applying definition 5 on the CNM, the first characteristic polynomial is given by,

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\right|=\begin{bmatrix}z&0&0&0&0&-1\\0&z&0&0&0&-1\\0&0&z&0&0&-1\\0&0&0&z&0&-1\\0&0&0&0&z&-1\\0&0&0&0&0&z-1\end{bmatrix}=z^5(z-1)$$

Thus, solving for z in

$$z^5(z-1)=0 \qquad (19)$$

Solving for (19), gives $z_1=z_2=z_3=z_4=z_5=0$ and $z_6=1$. Hence, the CNM is zero-stable (Sunday, 2018).

The zero-stability analysis shows that the developed block CNM satisfies the root condition, with the characteristic roots lying within or on the unit circle. This implies that errors generated during computation do not grow uncontrollably from one block to another, ensuring stable and reliable numerical solutions. Consequently, the method is suitable for solving ordinary differential equations and, together with consistency, guarantees convergence.\

3.4 Convergence of the Computational Method (CNM)

The CNM is said to be convergent since it is consistent and zero-stable by Dhalquiest theorem that stated as the necessary and sufficient conditions for a CNM to be convergent are that it be consistent and zero-stable (Adoghe & Omole, 2019).

3.5 Region of Absolute Stability of the CNM

Definition 6: Region of Absolute Stability

Region of absolute stability is a region in the complex z plane, where $z=\lambda h$. It is defined as those values of z such that the numerical solutions of $y'''=-\lambda^3 y$ satisfy $y_j \to 0 \ as \ j \to \infty$ for any initial condition (Sunday 2018). To determine the regions of absolute stability of an algorithm, a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. This method according to Lambert is called the Boundary Locus Method (BLM). Applying the boundary locus method, we obtain the stability polynomial for the computational method as;

Applying the boundary locus method, we obtain the stability polynomial for the CNM as;

$$\left.\begin{aligned}\bar{h}(z) &= h^6\left(-\frac{1}{241920}w^5+\frac{1}{241920}w^6\right)+h^5\left(-\frac{11}{103680}w^5-\frac{11}{103680}w^6\right)+h^4\left(-\frac{7}{4320}w^5+\frac{7}{4320}w^6\right)\\ &+h^3\left(-\frac{29}{1728}w^5-\frac{29}{1728}w^6\right)+h^2\left(-\frac{101}{864}w^5+\frac{101}{864}w^6\right)+h\left(-\frac{1}{2}w^6-\frac{1}{2}w^5\right)-w^5+w^6\end{aligned}\right\} \qquad (20)$$

The region of absolute stability of the computational method (CNM) is shown in Figure 1.

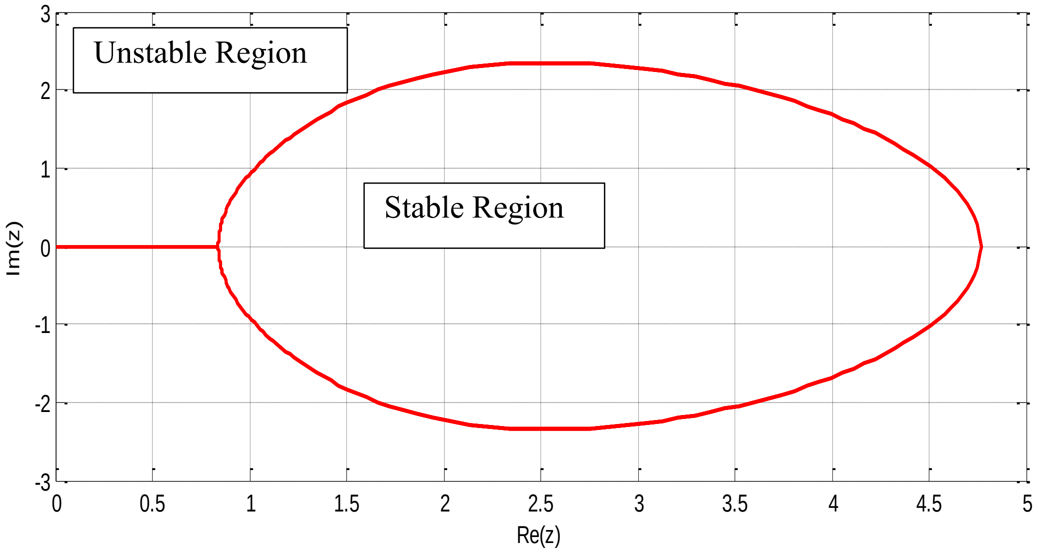

Figure 1: Stability region of the CNM

The stability region obtained in Figure 1 is A-stable because the stability region is the interior.

4. Numerical Simulation of the CNM

The numerical simulation of the CNM was carried out in this section by testing its efficiency on some third-order oscillatory initial value problems of the form (1.1). The results are presented in tabular form and compared with those obtained using existing methods reported in the literature.

Problem 1: Consider the third order oscillatory differential equation given as

$$y'''(\xi) = y'' - y' + y,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -1 \tag{21}$$

With exact solution is

$$y(\xi) = \cos\xi \tag{22}$$

Source: [Adoghe and Omole, (2019), Tumba, (2021)].

Problem 2: Consider the highly non- stiff third order oscillatory linear problem of the form

$$y'''(\xi) = 3\sin(\xi), \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = -2 \tag{23}$$

with the exact solution given by

$$y(\xi) = 3\cos(\xi) + \frac{\xi^2}{2} - 2 \tag{24}$$

Source: [Adoghe and Omole, (2019), Aigbiremhon and Omole, (2020)].

Problem 3: Consider the highly stiff third order oscillatory linear problem of the form

$$y'''(\xi) = \exp(\xi),\ y(0) = 3,\ y'(0) = 1,\ y''(0) = 5,\ h = 0.1 \tag{25}$$

With the exact solution given by

$$y(\xi) = 2(1 + \xi^2) + e^{\xi} \tag{26}$$

Source: [Adeyeye and Omar, (2018), Aigbiremhon and Omole, (2020)].

Problem 4: Consider the highly third order oscillatory linear problem of the form

$$y'''(\xi) = -5y'' - 7y' - 3y,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = -1, \tag{27}$$

with the exact solution given by

$$y(\xi) = \exp(-\xi) + \xi \exp(-\xi) \tag{28}$$

Source: [Adoghe and Omole, (2019)].

The following notations were used in the tables.

Notations	Meaning
ES	Exact Solution
NS	Numerical Solution in CNM
AECNM	Absolute Error in CNM
AEAO18	Absolute Error in Adeyeye and Omar, (2018)
AEAO19	Absolute Error in Adoghe and Omole, (2019)
AEAO20	Absolute Error in Aigbiremhon and Omole, (2020)
AETe21	Absolute Error in Tumba et al., (2021)

Table 1: Numerical Simulation of problem 1

ξ	**ES**	**CS**	**AECNM**	**AETe21**	**AEAO19**
0.01	0.99995000041666527778	0.99995000041666527775	3.0000e-20	6.1000e-20	0.0000e00
0.02	0.99980000666657777841	0.99980000666657777838	3.0000e-20	1.2000e-19	1.1102e-16
0.03	0.99955003374898751627	0.99955003374898751630	3.0000e-20	1.9000e-19	4.4409e-16
0.04	0.99920010666097794031	0.99920010666097794028	3.0000e-20	2.5000e-19	5.8842e-15
0.05	0.99875026039496624656	0.99875026039496624659	3.0000e-20	3.2000e-19	2.6201e-14
0.06	0.99820053993520416555	0.99820053993520416558	3.0000e-20	3.9000e-19	8.3822e-14
0.07	0.99755100025327957462	0.99755100025327957464	2.0000e-20	4.5000e-19	2.0750e-13
0.08	0.99680170630261938498	0.99680170630261938501	3.0000e-20	5.1000e-19	4.4142e-13
0.09	0.99595273301199425309	0.99595273301199425327	1.8000e-19	5.6000e-19	4.4743e-13
0.10	0.99500416527802576610	0.99500416527802576672	6.2000e-19	6.3000e-19	1.5086e-12

Table 2: Numerical Simulation of problem 2

ξ	**ES**	**CS**	**AECNM**	**AEAO19**	**AEAO20**
0.1	0.99001249583407729830	0.99001249583407730000	9.7200e-18	2.2204e-16	4.8906e-10
0.2	0.96019973352372489340	0.96019973352372490023	3.2589e-16	4.4409e-16	3.2663e-09
0.3	0.91100946737681805890	0.91100946737681807431	2.0202e-15	1.3323e-15	1.0296e-08
0.4	0.84318298200865524840	0.84318298200865527561	7.1190e-15	3.8858e-15	2.3509e-08
0.5	0.75774768567111814840	0.75774768567111819060	1.8583e-14	9.2149e-15	4.4764e-08
0.6	0.65600684472903489170	0.65600684472903495210	4.0277e-14	1.8985e-14	7.5847e-08
0.7	0.53952656185346527880	0.53952656185346536017	7.6933e-14	3.4084e-14	1.1844e-07
0.8	0.41012012804149626280	0.41012012804149636793	1.3410e-13	5.7343e-14	1.7411e-07
0.9	0.26982990481199336940	0.26982990481199350091	2.1809e-13	9.0095e-14	2.4429e-07
1.0	0.12090691760441915220	0.12090691760441931224	3.3593e-13	1.3678e-13	3.3028e-07

Table 3: Numerical Simulation of problem 3

ξ	**ES**	**CS**	**AECNM**	**AETe21**	**AEAO18**
0.1	3.12517091807564762480	3.12517091807564762430	5.0000e-19	1.7009e-15	6.3427e-13
0.2	3.30140275816016983390	3.30140275816016983160	2.3000e-18	1.9180e-14	2.3288e-12
0.3	3.52985880757600310400	3.52985880757600309850	5.5000e-18	7.2513e-14	5.4435e-12
0.4	3.81182469764127031780	3.81182469764127030760	1.0200e-17	1.8391e-13	9.8532e-12
0.5	4.14872127070012814680	4.14872127070012813030	1.6500e-17	3.7788e-13	1.5997e-11
0.6	4.54211880039050897490	4.54211880039050895020	2.4700e-17	6.8153e-13	2.3722e-11
0.7	4.99375270747047652160	4.99375270747047648670	3.4900e-17	1.1243e-12	3.3568e-11
0.8	5.50554092849246760460	5.50554092849246755740	4.7200e-17	1.7409e-12	4.5344e-11
0.9	6.07960311115694966380	6.07960311115694960200	6.1800e-17	2.5663e-12	5.9708e-11
1.0	6.71828182845904523540	6.71828182845904515630	7.9100e-17	3.6415e-12	7.6432e-11

Table 4: Numerical Simulation of problem 4

ξ	**ES**	**CS**	**AECNM**	**AEAO19**
0.0101563	0.99994877266765112344	0.99994877266765112343	0.0000e00	7.9936e-15
0.0109375	0.99994061991025034788	0.99994061991025034788	0.0000e00	1.1546e-14
0.0117188	0.99993186896184807675	0.99993186896184807674	0.0000e00	1.5543e-14
0.0125000	0.99992252300005494592	0.99992252300005494592	0.0000e00	2.2093e-14
0.0132813	0.99991258056872248302	0.99991258056872248303	0.0000e00	3.0309e-14
0.0140625	0.99990204514826226392	0.99990204514826226392	0.0000e00	4.0079e-14
0.0148437	0.99989091643831168876	0.99989091643831168877	0.0000e00	5.1625e-14
0.0156250	0.99987919383361791233	0.99987919383361791233	0.0000e00	6.4615e-14
0.0164063	0.99986687965306953639	0.99986687965306953639	0.0000e00	8.2045e-14
0.0171875	0.99985397651731844212	0.99985397651731844212	0.0000e00	1.0258e-13

5. Discussion of Results

The derivation of the Computational Numerical Method (CNM) is based on the direct solution approach for third order initial value problems of the form (1). The method begins with the construction of an approximate solution using a power series basis function as given in equation (4), where the parameters represent the number of interpolation and collocation points used in the formulation. By differentiating the assumed power series three times, equation (5) is obtained and subsequently substituted into the original differential equation to form the system expressed in equation (6). Interpolation and collocation procedures are then applied at selected grid points, which generate a system of nonlinear equations shown in equation (7). Solving this system for the unknown constants and substituting the resulting coefficients back into the assumed series produces a continuous computational scheme as presented in equation (8). Further employment of the continuous formulation yields the coefficients in equations (9) and (10), which are then used to obtain a continuous linear multistep representation shown in equation (11). By evaluating the continuous scheme at specific points, the final discrete computational numerical method (CNM) in equation (3) is obtained for practical implementation.

The basic properties of the CNM were analyzed to establish its reliability and effectiveness. Using the linear operator formulation and Taylor series expansion, the order and error constant of the method were determined. The analysis shows that the CNM possesses a uniform order of eight with a corresponding error constant derived from the truncation error expression, indicating a high level of accuracy in the numerical approximation. Since the order of the method is greater than one, the CNM satisfies the condition for consistency. The zero-stability of the method was also examined using the first characteristic polynomial, where the roots satisfy the required stability condition, confirming that the propagation of numerical errors is well controlled. Consequently, based on the Dahlquist theorem, the CNM is convergent because it is both consistent and zero-stable. Furthermore, the region of absolute stability was determined using the Boundary Locus Method, which produced the stability polynomial and the graphical stability region illustrated in Figure 1. The obtained region indicates that the CNM is A-stable, demonstrating that the method remains stable for a wide range of step sizes and is therefore suitable for solving third–order oscillatory differential equations.

The efficiency of the CNM was examined using four third-order oscillatory initial value problems whose results are summarized in Tables 1 to 4. In Table 1, the numerical simulation of Problem 4.1 shows that the numerical solution obtained using CNM closely matches the exact solution across all step values. The error values ECNM remain extremely small, which demonstrates the very high accuracy of the method. When compared with the existing methods reported in references Tumba *et al.*, (2021), Adoghe and Omole, (2019), the CNM produces smaller or comparable errors throughout the interval considered. This indicates that the CNM is capable of maintaining high precision while solving oscillatory problems, even when the step size increases gradually. The consistency between the exact solution (ES) and the computed solution (CS) further confirms the reliability of the scheme for third-order oscillatory differential equations.

Table 2 presents the numerical simulation of Problem 2, which represents a highly non-stiff third-order oscillatory linear problem. The numerical solutions generated by the CNM remain very close to the exact solution as the independent variable increases. A comparison with the results obtained using the methods in references Adoghe and Omole, (2019), Aigbiremhon and Omole, (2020) shows that the CNM maintains better consistency with the exact solution over the entire integration range. This behavior demonstrates the capability of the method to handle non-stiff oscillatory problems effectively while maintaining a high level of accuracy.

The performance of the method for more demanding problems is further illustrated in Tables 3 and 4, corresponding to Problems 3 and 4 respectively. In Table 3, which represents a highly stiff oscillatory problem, the CNM produces numerical results that closely follow the exact solution and show better agreement than the results reported in references Adeyeye and Omar, (2018); Tumba *et al.,* (2021). Similarly, Table 4 indicates that the numerical solutions obtained using the CNM almost coincide with the exact solution across the entire interval considered, while the results from reference Adoghe and Omole, (2019) show relatively larger deviations. The results presented in these tables confirm that the CNM is accurate, stable, and efficient for solving both stiff and non-stiff third-order oscillatory initial value problems.

7. Conclusion

This study presents the development of a Computational Numerical Method (CNM) for the direct solution of highly third order oscillatory linear initial value problems. The method was formulated using a power series basis function combined with interpolation and collocation techniques to construct a continuous linear multistep scheme, which was subsequently transformed into a discrete computational form for practical implementation. The theoretical analysis shows that the CNM possesses a uniform order of eight, indicating high accuracy, and its local truncation error confirms strong convergence characteristics. The method was further examined for consistency, zero-stability, and convergence, and it was established using the relevant stability criteria and Dahlquist theorem that the CNM is consistent, zero-stable, and therefore convergent. In addition, the stability analysis using the Boundary Locus Method revealed that the method is A-stable, making it suitable for solving both stiff and non-stiff oscillatory third-order problems. Numerical simulations conducted on four test problems demonstrated that the CNM produces highly accurate results when compared with existing methods in the literature.

The Computational Numerical Method developed in this study provides an efficient and reliable approach for solving third-order initial value problems directly without reducing them to systems of first-order equations. The method combines high-order accuracy with strong stability properties, as confirmed by the theoretical analysis and numerical experiments. The results obtained from the test problems show excellent agreement with the exact solutions and outperform or compare favorably with several existing methods. The A-stability property further enhances its applicability to stiff and oscillatory problems, ensuring stable performance for a wide range of step sizes. Therefore, the CNM is a robust, accurate, and computationally efficient method suitable for solving third-order differential equations arising in science and engineering applications.

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Appendix A:

$$
\begin{array}{cccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\left(\frac{1}{4}\right)^{1}}{1!} & \frac{\left(\frac{1}{3}\right)^{1}}{1!} & \frac{\left(\frac{1}{2}\right)^{1}}{1!} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\left(\frac{1}{4}\right)^{2}}{2!} & \frac{\left(\frac{1}{3}\right)^{2}}{2!} & \frac{\left(\frac{1}{2}\right)^{2}}{2!} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\left(\frac{1}{4}\right)^{3}}{3!} & \frac{\left(\frac{1}{3}\right)^{3}}{3!} & \frac{\left(\frac{1}{2}\right)^{3}}{3!} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{\left(\frac{1}{4}\right)^{4}}{4!} & \frac{\left(\frac{1}{3}\right)^{4}}{4!} & \frac{\left(\frac{1}{2}\right)^{4}}{4!} & 0 & \frac{\left(\frac{1}{4}\right)^{1}}{1!} & \frac{\left(\frac{1}{3}\right)^{1}}{1!} & \frac{\left(\frac{1}{2}\right)^{1}}{1!} & \frac{\left(\frac{2}{3}\right)^{1}}{1!} & \frac{\left(\frac{3}{4}\right)^{1}}{1!} & \frac{\left(1\right)^{1}}{1!} \\
\frac{\left(\frac{1}{4}\right)^{5}}{5!} & \frac{\left(\frac{1}{3}\right)^{5}}{5!} & \frac{\left(\frac{1}{2}\right)^{5}}{5!} & 0 & \frac{\left(\frac{1}{4}\right)^{2}}{2!} & \frac{\left(\frac{1}{3}\right)^{2}}{2!} & \frac{\left(\frac{1}{2}\right)^{2}}{2!} & \frac{\left(\frac{2}{3}\right)^{2}}{2!} & \frac{\left(\frac{3}{4}\right)^{2}}{2!} & \frac{\left(1\right)^{2}}{2!} \\
\frac{\left(\frac{1}{4}\right)^{6}}{6!} & \frac{\left(\frac{1}{3}\right)^{6}}{6!} & \frac{\left(\frac{1}{2}\right)^{6}}{6!} & 0 & \frac{\left(\frac{1}{4}\right)^{3}}{3!} & \frac{\left(\frac{1}{3}\right)^{3}}{3!} & \frac{\left(\frac{1}{2}\right)^{3}}{3!} & \frac{\left(\frac{2}{3}\right)^{3}}{3!} & \frac{\left(\frac{3}{4}\right)^{3}}{3!} & \frac{\left(1\right)^{3}}{3!} \\
\frac{\left(\frac{1}{4}\right)^{7}}{7!} & \frac{\left(\frac{1}{3}\right)^{7}}{7!} & \frac{\left(\frac{1}{2}\right)^{7}}{7!} & 0 & \frac{\left(\frac{1}{4}\right)^{4}}{4!} & \frac{\left(\frac{1}{3}\right)^{4}}{4!} & \frac{\left(\frac{1}{2}\right)^{4}}{4!} & \frac{\left(\frac{2}{3}\right)^{4}}{4!} & \frac{\left(\frac{3}{4}\right)^{4}}{4!} & \frac{\left(1\right)^{4}}{4!} \\
\frac{\left(\frac{1}{4}\right)^{8}}{8!} & \frac{\left(\frac{1}{3}\right)^{8}}{8!} & \frac{\left(\frac{1}{2}\right)^{8}}{8!} & 0 & \frac{\left(\frac{1}{4}\right)^{5}}{5!} & \frac{\left(\frac{1}{3}\right)^{5}}{5!} & \frac{\left(\frac{1}{2}\right)^{5}}{5!} & \frac{\left(\frac{2}{3}\right)^{5}}{5!} & \frac{\left(\frac{3}{4}\right)^{5}}{5!} & \frac{\left(1\right)^{5}}{5!} \\
\frac{\left(\frac{1}{4}\right)^{9}}{9!} & \frac{\left(\frac{1}{3}\right)^{9}}{9!} & \frac{\left(\frac{1}{2}\right)^{9}}{9!} & 0 & \frac{\left(\frac{1}{4}\right)^{6}}{6!} & \frac{\left(\frac{1}{3}\right)^{6}}{6!} & \frac{\left(\frac{1}{2}\right)^{6}}{6!} & \frac{\left(\frac{2}{3}\right)^{6}}{6!} & \frac{\left(\frac{3}{3}\right)^{6}}{6!} & \frac{\left(1\right)^{6}}{6!}
\end{array}
$$

✍1

$$
\begin{array}{c}
1 \\
\frac{\left(l\right)^{1}}{1!} \\
\frac{\left(l\right)^{2}}{2!} \\
\frac{\left(l\right)^{3}}{3!} \\
\frac{\left(l\right)^{4}}{4!} \\
\frac{\left(l\right)^{5}}{5!} \\
\frac{\left(l\right)^{6}}{6!} \\
\frac{\left(l\right)^{7}}{7!} \\
\frac{\left(l\right)^{8}}{8!} \\
\frac{\left(l\right)^{9}}{9!}
\end{array}
$$

Appendix B:

$$\begin{matrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \frac{\left(\frac{1}{4}\right)^{1}}{1!} & \frac{\left(\frac{1}{3}\right)^{1}}{1!} & \frac{\left(\frac{1}{2}\right)^{1}}{1!} & \frac{\left(\frac{2}{3}\right)^{1}}{1!} & \frac{\left(\frac{3}{4}\right)^{1}}{1!} & \frac{(1)^{1}}{1!} \\
0 & \frac{\left(\frac{1}{4}\right)^{2}}{2!} & \frac{\left(\frac{1}{3}\right)^{2}}{2!} & \frac{\left(\frac{1}{2}\right)^{2}}{2!} & \frac{\left(\frac{2}{3}\right)^{2}}{2!} & \frac{\left(\frac{3}{4}\right)^{2}}{2!} & \frac{(1)^{2}}{2!} \\
0 & \frac{\left(\frac{1}{4}\right)^{3}}{3!} & \frac{\left(\frac{1}{3}\right)^{3}}{3!} & \frac{\left(\frac{1}{2}\right)^{3}}{3!} & \frac{\left(\frac{2}{3}\right)^{3}}{3!} & \frac{\left(\frac{3}{4}\right)^{3}}{3!} & \frac{(1)^{3}}{3!} \\
0 & \frac{\left(\frac{1}{4}\right)^{4}}{4!} & \frac{\left(\frac{1}{3}\right)^{4}}{4!} & \frac{\left(\frac{1}{2}\right)^{4}}{4!} & \frac{\left(\frac{2}{3}\right)^{4}}{4!} & \frac{\left(\frac{3}{4}\right)^{4}}{4!} & \frac{(1)^{4}}{4!} \\
0 & \frac{\left(\frac{1}{4}\right)^{5}}{5!} & \frac{\left(\frac{1}{3}\right)^{5}}{5!} & \frac{\left(\frac{1}{2}\right)^{5}}{5!} & \frac{\left(\frac{2}{3}\right)^{5}}{5!} & \frac{\left(\frac{3}{4}\right)^{5}}{5!} & \frac{(1)^{5}}{5!} \\
0 & \frac{\left(\frac{1}{4}\right)^{6}}{6!} & \frac{\left(\frac{1}{3}\right)^{6}}{6!} & \frac{\left(\frac{1}{2}\right)^{6}}{6!} & \frac{\left(\frac{2}{3}\right)^{6}}{6!} & \frac{\left(\frac{3}{3}\right)^{6}}{6!} & \frac{(1)^{6}}{6!}
\end{matrix}^{✍1}
\qquad
\begin{matrix}
\frac{(l)^{3}}{3!} \\
\frac{(l)^{4}}{4!} \\
\frac{(l)^{5}}{5!} \\
\frac{(l)^{6}}{6!} \\
\frac{(l)^{7}}{7!} \\
\frac{(l)^{8}}{8!} \\
\frac{(l)^{9}}{9!}
\end{matrix}$$