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Three-step Collocation Method for Solution of Third Derivative Initial Value Problem

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ABSTRACT

In this research paper, we introduce a novel three-step block method designed to directly tackle third-order initial value problems. This method is crafted through an interpolation and collocation approach, leveraging power series analysis. We conduct a comprehensive examination of the proposed method, ensuring it meets all requisite conditions for rigorous analysis. To assess its efficacy and validity, we employ the method on both highly stiff linear and nonlinear initial value problems, juxtaposing our findings with established approaches in the literature. Our results highlight that the new method exhibits faster convergence compared to existing methods, underscoring its superior performance.

1. Introduction

In the realm of differential equations, oscillatory systems pose notable hurdles, a theme underscored by (Sabo, *et al*. 2021). Solutions to such equations manifest as smoothly varying, "nearly periodic" functions, featuring oscillations with slowly evolving waveforms and periods relative to time. Handling these solutions across a significant number of cycles often renders tracking precise trajectories impractical and superfluous. Instead, researchers commonly opt for approximative solutions or computing the quasi-envelopes of these oscillatory systems, a strategy elucidated by (Adeyeye and Omar, 2018). Eigenvalues in oscillatory problems typically hover close to the imaginary axis, yielding solutions characterized by oscillatory behavior with gradually shifting amplitudes. The complexity of solving these problems lies in the accurate determination of both amplitude and phase angle over numerous periods, as highlighted by (Kuboye and Omar, (2015); Kayode and Obaruha, (2017); Sabo, *et al*. (2021); Raymond *et al*. (2023)).

A differential equation refers to an equation involving one or more independent variables, a dependent variable, and one or more derivatives of the dependent variable with respect to the independent variable(s), as defined by (Omar, 1999). Essentially, it encapsulates the relationship between the derivatives of one or more independent variables and one or more dependent variables. The fundamental categorizations of differential equations are ordinary differential equations (ODEs) and partial differential equations (PDEs), delineated by (Omar and Suleiman, 1999). In an ordinary differential equation, the unknown function is a function of a solitary independent variable, whereas in a partial differential equation, it is a function of two or more independent variables, as explicated by (Fatunla, 1994). Further classification of a differential equation can be based on its order and degree. The order signifies the highest derivative incorporated in the equation (Sharp and Fine, 1992), while the degree corresponds to the power to which the highest derivative occurs in the equation. To elucidate these principles, let's examine the following examples of differential equations:

$$\frac{d^3y}{dx^3} - xy\left(\frac{dy}{dx}\right)^2 = 0 \tag{1}$$

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dz^3} - \frac{dx}{dz} = \sin z \tag{2}$$

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$$\frac{\partial v}{\partial s}-\frac{\partial v}{\partial \omega}=v \tag{3}$$

$$\frac{\partial^3 \omega}{\partial x^3}-\frac{\partial^2 \omega}{\partial y^2}-\frac{\partial \omega}{\partial z}=1 \tag{4}$$

Equations (1) and (2) fall under the category of ordinary differential equations (ODEs), whereas equations (3) and (4) belong to partial differential equations (PDEs). Equation (1) exhibits an order of 3 with a degree of 1, whereas equation (2) is characterized by both an order and degree of 4 (Sabo, 2021).

In scientific and engineering contexts, the mathematical representation of diverse physical phenomena often leads to distinctive third-order oscillatory differential equations expressed as:

$$y''' = f(x,y), y(a) = y_0, y'(a) = \eta_0 , y''(a) = \eta_1 \tag{5}$$

However, the pool of analytical methods available for directly solving equation (5) without reducing it to a first-order system of initial value problems is limited. Various authors, including Awoyemi (2003), Awoyemi and Idowu (2005), and Adeyeye and Omar (2019), have proposed different strategies for addressing higher-order initial value problems in ordinary differential equations. Awoyemi and Idowu (2005) specifically devised a hybrid collocation approach tailored for third-order ordinary differential equations. Awoyemi (2003) introduced a predictor-corrector formulation for a general linear multistep technique targeting third-order initial value problems, emphasizing its p-stability. These methods typically rely on initial values obtained from Runge-Kutta techniques or other one-step methods, aligning with the conventions of linear multistep methods. The predictor-corrector integration procedure advocated by Fatunla (1994) for non-stiff special second-order ordinary differential equations is akin to those used in block methods. While akin to other linear multistep methods, these approaches lack self-starting properties and incrementally advance numerical integration, leading to the overlapping of piecewise polynomial solution models (Sunday (2018), Adeyeye and Omar (2019)).

This numerical technique demonstrates the application of symmetry reduction to approximate solutions for third-order differential equations represented by equation (5). Instead of employing the conventional approach of converting a high-order ordinary differential equation into a system of first-order ordinary differential equations, this method, as proposed by (Butcher, 1965), directly solves third-order differential equations by expressing them in terms of their differential equations.

2. Subject and Methodology

2.1. Formulation of the methodology

This section outlines the formulation of a novel hybrid block method tailored for specialized third-order ordinary differential equations, employing an interpolation and collocation methodology.

The power series polynomial of the form

$$y(x)=\sum_{j=0}^{k} \alpha_j x^j \tag{6}$$

is utilized to approximate the solution of equation (5). This method is formulated by introducing off-mesh points via a three-step scheme, inspired by the approach detailed in (Sabo *et al.* 2021).

We consider the approximate solution of (6) as

$$y(x)=\sum_{j=0}^{9} a_j x^j \tag{7}$$

Differentiating (7) three times, yield

$$y'''(x)=\sum_{j=0}^{9} j(j-1)(j-2)a_j\, x^{j-3} \tag{8}$$

Putting (8) into (5) yield

$$\sum_{j=0}^{7} j(j-1)(j-2)a_j\, x^{j-3} = f(x, y, y'') \tag{9}$$

Now, interpolating (7) at point x_{n+p}, $p=1, 2 \; and \; 3$ and collocating (8) at x_{n+q}, $q=0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2},$ and 3 lead to a system of equation (10) as

$$\begin{bmatrix}
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 & 504x_n^6 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{1}{2}} & 60x^2_{n+\frac{1}{2}} & 120x^3_{n+\frac{1}{2}} & 210x^3_{n+\frac{1}{2}} & 336x^5_{n+\frac{1}{2}} & 504x^6_{n+1} \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^3 & 336x_{n+1}^5 & 504x_{n+1}^6 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{3}{2}} & 60x^2_{n+\frac{3}{2}} & 120x^3_{n+\frac{3}{2}} & 210x^3_{n+\frac{3}{2}} & 336x^5_{n+\frac{3}{2}} & 504x^6_{n+\frac{3}{2}} \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^3 & 336x_{n+2}^5 & 504x_{n+2}^6 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{5}{2}} & 60x^2_{n+\frac{5}{2}} & 120x^3_{n+\frac{5}{2}} & 210x^3_{n+\frac{5}{2}} & 336x^5_{n+\frac{5}{2}} & 504x^6_{n+\frac{5}{2}} \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^3 & 336x_{n+3}^5 & 504x_{n+3}^6
\end{bmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix}
=
\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{pmatrix} \tag{10}$$

using Gaussian elimination method, (10) is solved for the $a_j{}'s$. The values of the $a_j{}'s$ obtained are then substituted into (7), after some manipulations, this gives a continuous hybrid linear multistep method of the form;

$$y(x)=\alpha_1(x)y_{n+1}+\alpha_2(x)y_{n+2}+\alpha_3(x)y_{n+3}+h^3\left[\sum_{j=0}^{3}\beta_j(x)f_{n+j}+\beta_{v_i}(x)f_{n+v_i}\right], v_i=0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3 \tag{11}$$

the coefficient $\alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_{\frac{1}{2}}, \beta_1, \beta_{\frac{3}{2}}, \beta_2, \beta_{\frac{5}{2}}, \beta_3$ are given by;

$$
\left.\begin{aligned}
\alpha_1 &= 3-\frac{5}{2}t+\frac{1}{2}t^2 \\
\alpha_2 &= 3+4t-t^2 \\
\alpha_3 &= 1-\frac{3}{2}t+\frac{1}{2}t^2 \\
\beta_0 &= -\frac{1}{945}+\frac{929}{56700}t-\frac{115}{1512}t^2+\frac{1}{6}t^3-\frac{49}{240}t^4+\frac{203}{1350}t^5-\frac{49}{720}t^6+\frac{1}{54}t^7-\frac{1}{360}t^8+\frac{1}{5670}t^9 \\
\beta_{\frac{1}{2}} &= -\frac{19}{315}+\frac{2521}{9450}t-\frac{23}{63}t^2+\frac{1}{2}t^4-\frac{29}{50}t^5+\frac{29}{90}t^6-\frac{31}{315}t^7+\frac{1}{63}t^8-\frac{1}{945}t^9 \\
\beta_1 &= -\frac{157}{630}+\frac{3553}{7560}t-\frac{567}{5040}t^2-\frac{5}{8}t^4-\frac{39}{40}t^5-\frac{461}{720}t^6+\frac{137}{630}t^7-\frac{19}{504}t^8+\frac{1}{378}t^9 \\
\beta_{\frac{3}{2}} &= -\frac{359}{945}+\frac{253}{405}t-\frac{319}{945}t^2+\frac{5}{9}t^4-\frac{127}{135}t^5+\frac{31}{45}t^6-\frac{242}{945}t^7+\frac{1}{21}t^8-\frac{2}{567}t^9 \\
\beta_2 &= -\frac{157}{630}+\frac{97}{270}t-\frac{151}{2520}t^2-\frac{5}{16}t^4-\frac{11}{20}t^5-\frac{307}{720}t^6+\frac{107}{630}t^7-\frac{17}{504}t^8+\frac{1}{378}t^9 \\
\beta_{\frac{5}{2}} &= -\frac{19}{315}+\frac{901}{9450}t-\frac{16}{315}t^2+\frac{1}{10}t^4-\frac{9}{50}t^5+\frac{13}{90}t^6-\frac{19}{315}t^7+\frac{4}{315}t^8-\frac{1}{945}t^9 \\
\beta_3 &= -\frac{1}{945}+\frac{103}{113400}t+\frac{1}{432}t^2+\frac{1}{10}t^4+\frac{137}{5400}t^5-\frac{1}{72}t^6+\frac{17}{1890}t^7-\frac{1}{504}t^8+\frac{1}{5670}t^9
\end{aligned}\right\} \quad (12)
$$

Evaluate (12) at non interpolating points to obtain the continuous form as,

$$
y_n = 3y_{n+1} - 3y_{n+2} + y_{n+3} - \frac{1}{945}h^3 f_n - \frac{19}{315}h^3 f_{n+\frac{1}{2}} - \frac{157}{630}h^3 f_{n+1} - \frac{358}{945}h^3 f_{n+\frac{3}{2}} - \frac{157}{630}h^3 f_{n+2} - \frac{19}{315}h^3 f_{n+\frac{7}{2}} - \frac{1}{945}h^3 f_{n+3}
$$

$$
y_{n+\frac{1}{2}} = \frac{15}{8}y_{n+1} - \frac{5}{4}y_{n+2} + \frac{3}{8}y_{n+3} - \frac{73}{1935360}h^3 f_n - \frac{121}{161280}h^3 f_{n+\frac{1}{2}} - \frac{7675}{129024}h^3 f_{n+1} - \frac{3305}{24192}h^3 f_{n+\frac{3}{2}} - \frac{11917}{129024}h^3 f_{n+2} - \frac{3691}{161280}h^3 f_{n+\frac{7}{2}} - \frac{703}{1935360}h^3 f_{n+3}
$$

$$
y_{n+\frac{3}{2}} = \frac{3}{8}y_{n+1} + \frac{3}{4}y_{n+2} - \frac{1}{8}y_{n+3} - \frac{61}{1935360}h^3 f_n + \frac{41}{161280}h^3 f_{n+\frac{1}{2}} - \frac{347}{645120}h^3 f_{n+1} + \frac{179}{7560}h^3 f_{n+\frac{3}{2}} + \frac{20443}{645120}h^3 f_{n+2} + \frac{235}{32256}h^3 f_{n+\frac{7}{2}} + \frac{317}{1935360}h^3 f_{n+3}
$$

$$
y_{n+\frac{5}{2}} = -\frac{1}{8}y_{n+1} + \frac{3}{4}y_{n+2} + \frac{3}{8}y_{n+3} - \frac{13}{387072}h^3 f_n + \frac{43}{161280}h^3 f_{n+\frac{1}{2}} - \frac{703}{645120}h^3 f_{n+1} - \frac{659}{120960}h^3 f_{n+\frac{3}{2}} - \frac{21913}{645120}h^3 f_{n+2} - \frac{3527}{161280}h^3 f_{n+\frac{7}{2}} - \frac{139}{387072}h^3 f_{n+3}
$$

Evaluate (12) at all the points to obtain the hybrid block method as

$$
y_{n+\frac{1}{2}} = y_n + \frac{1}{2}h y'_n + \frac{1}{8}h^2 y''_n + \frac{343801}{29030400}h^3 f_n + \frac{6031}{345600}h^3 f_{n+\frac{1}{2}} - \frac{32981}{1935360}h^3 f_{n+1}
$$

$$+\frac{5177}{362880}h^3 f_{n+\frac{3}{2}}-\frac{15107}{1935360}h^3 f_{n+2}+\frac{5947}{2419200}h^3 f_{n+\frac{5}{2}}-\frac{9809}{29030400}h^3 f_{n+3}$$

$$y_{n+1}=y_n+h y'_n+\frac{1}{2}h^2 y''_n+\frac{6887}{113400}h^3 f_n+\frac{1499}{9450}h^3 f_{n+\frac{1}{2}}-\frac{233}{2160}h^3 f_{n+1}+\frac{52}{567}h^3 f_{n+\frac{3}{2}}$$

$$-\frac{379}{7560}h^3 f_{n+2}+\frac{149}{9450}h^3 f_{n+\frac{5}{2}}-\frac{491}{226800}h^3 f_{n+3}$$

$$y_{n+\frac{3}{2}}=y_n+\frac{3}{2}h y'_n+\frac{9}{8}h^2 y''_n+\frac{52893}{358400}h^3 f_n$$

$$+\frac{43173}{89600}h^3 f_{n+\frac{1}{2}}-\frac{14499}{71680}h^3 f_{n+1}+\frac{9}{40}h^3 f_{n+\frac{3}{2}}$$

$$-\frac{8829}{71680}h^3 f_{n+2}+\frac{3483}{89600}h^3 f_{n+\frac{5}{2}}-\frac{1917}{358400}h^3 f_{n+3}$$

$$y_{n+2}=y_n+2h y'_n+2h^2 y''_n+\frac{3863}{14175}h^3 f_n+\frac{4664}{4725}h^3 f_{n+\frac{1}{2}}-\frac{226}{945}h^3 f_{n+1}+\frac{272}{567}h^3 f_{n+\frac{3}{2}}$$

$$-\frac{31}{135}h^3 f_{n+2}+\frac{344}{4725}h^3 f_{n+\frac{5}{2}}-\frac{142}{14175}h^3 f_{n+3}$$

$$y_{n+\frac{5}{2}}=y_n+\frac{5}{2}h y'_n+\frac{25}{8}h^2 y''_n+\frac{505625}{1161216}h^3 f_n+\frac{162125}{96768}h^3 f_{n+\frac{1}{2}}-\frac{85625}{387072}h^3 f_{n+1}$$

$$+\frac{66875}{72576}h^3 f_{n+\frac{3}{2}}-\frac{119375}{387072}h^3 f_{n+2}+\frac{1625}{13824}h^3 f_{n+\frac{5}{2}}-\frac{18625}{1161216}h^3 f_{n+3}$$

$$y_{n+3}=y_n+3h y'_n+\frac{9}{2}h^2 y''_n+\frac{891}{1400}h^3 f_n+\frac{891}{350}h^3 f_{n+\frac{1}{2}}-\frac{81}{560}h^3 f_{n+1}+\frac{54}{35}h^3 f_{n+\frac{3}{2}}$$

$$-\frac{81}{280}h^3 f_{n+2}+\frac{81}{350}h^3 f_{n+\frac{5}{2}}-\frac{9}{400}h^3 f_{n+3}$$

$$y'_{n+\frac{1}{2}}=y'_n+\frac{1}{2}h y''_n+\frac{28549}{483840}h^2 f_n+\frac{275}{2304}h^2 f_{n+\frac{1}{2}}-\frac{5717}{53760}h^2 f_{n+1}+\frac{10621}{120960}h^2 f_{n+\frac{3}{2}}$$

$$-\frac{7703}{161280}h^2 f_{n+2}+\frac{403}{26880}h^2 f_{n+\frac{5}{2}}-\frac{199}{96768}h^2 f_{n+3}$$

$$y'_{n+1}=y'_n+h y''_n+\frac{1027}{7560}h^2 f_n+\frac{97}{210}h^2 f_{n+\frac{1}{2}}-\frac{2}{9}h^2 f_{n+1}+\frac{197}{945}h^2 f_{n+\frac{3}{2}}-\frac{97}{840}h^2 f_{n+2}$$

$$+\frac{23}{630}h^2 f_{n+\frac{5}{2}}-\frac{19}{3780}h^2 f_{n+3}$$

$$y'_{n+\frac{3}{2}}=y'_n+\frac{3}{2}h y''_n+\frac{759}{3584}h^2 f_n+\frac{1485}{1792}h^2 f_{n+\frac{1}{2}}-\frac{2403}{17920}h^2 f_{n+1}+\frac{45}{128}h^2 f_{n+\frac{3}{2}}$$

$$-\frac{3267}{17920}h^2 f_{n+2}+\frac{513}{8960}h^2 f_{n+\frac{5}{2}}-\frac{141}{17920}h^2 f_{n+3}$$

$$y'_{n+2}=y'_n+2hy''_n+\frac{272}{945}h^2 f_n+\frac{376}{315}h^2 f_{n+\frac{1}{2}}-\frac{2}{105}h^2 f_{n+1}+\frac{656}{945}h^2 f_{n+\frac{3}{2}}$$

$$-\frac{2}{9}h^2 f_{n+2}+\frac{8}{105}h^2 f_{n+\frac{5}{2}}-\frac{2}{189}h^2 f_{n+3}$$

$$y'_{n+\frac{5}{2}}=y'_n+\frac{5}{2}hy''_n+\frac{35225}{96768}h^2 f_n+\frac{8375}{5376}h^2 f_{n+\frac{1}{2}}+\frac{3125}{32256}h^2 f_{n+1}+\frac{25625}{24192}h^2 f_{n+\frac{3}{2}}$$

$$-\frac{625}{10752}h^2 f_{n+2}+\frac{275}{2304}h^2 f_{n+\frac{5}{2}}-\frac{1375}{96768}h^2 f_{n+3}$$

$$y'_{n+3}=y'_n+3hy''_n+\frac{123}{280}h^2 f_n+\frac{27}{14}h^2 f_{n+\frac{1}{2}}+\frac{27}{140}h^2 f_{n+1}+\frac{51}{35}h^2 f_{n+\frac{3}{2}}$$

$$+\frac{27}{280}h^2 f_{n+2}+\frac{27}{70}h^2 f_{n+\frac{5}{2}}$$

$$y''_{n+\frac{1}{2}}=y''_n+\frac{19087}{120960}hf_n+\frac{2713}{5040}hf_{n+\frac{1}{2}}-\frac{15487}{40320}hf_{n+1}+\frac{293}{945}hf_{n+\frac{3}{2}}-\frac{6737}{40320}hf_{n+2}$$

$$+\frac{263}{5040}hf_{n+\frac{5}{2}}-\frac{863}{120960}hf_{n+3}\frac{11}{2520}hf_{n+1}+\frac{166}{945}hf_{n+\frac{3}{2}}-\frac{269}{2520}hf_{n+2}$$

$$+\frac{11}{315}hf_{n+\frac{5}{2}}-\frac{37}{7560}hf_{n+3}$$

$$y''_{n+\frac{3}{2}}=y''_n+\frac{137}{896}hf_n+\frac{81}{112}hf_{n+\frac{1}{2}}+\frac{1161}{4480}hf_{n+1}+\frac{17}{35}hf_{n+\frac{3}{2}}-\frac{729}{4480}hf_{n+2}$$

$$+\frac{27}{560}hf_{n+\frac{5}{2}}-\frac{29}{4480}hf_{n+3}$$

$$y''_{n+2}=y''_n+\frac{143}{945}hf_n+\frac{232}{315}hf_{n+\frac{1}{2}}+\frac{64}{315}hf_{n+1}+\frac{752}{945}hf_{n+\frac{3}{2}}+\frac{29}{315}hf_{n+2}$$

$$+\frac{8}{315}hf_{n+\frac{5}{2}}-\frac{4}{945}hf_{n+3}$$

$$y''_{n+\frac{5}{2}}=y''_n+\frac{3715}{24192}hf_n+\frac{725}{1008}hf_{n+\frac{1}{2}}+\frac{2125}{8064}hf_{n+1}+\frac{125}{189}hf_{n+\frac{3}{2}}+\frac{3875}{8064}hf_{n+2}$$

$$+\frac{235}{1008}hf_{n+\frac{5}{2}}-\frac{275}{24192}hf_{n+3}$$

$$y''_{n+3}=y''_n+\frac{41}{280}hf_n+\frac{27}{35}hf_{n+\frac{1}{2}}+\frac{27}{280}hf_{n+1}+\frac{34}{35}hf_{n+\frac{3}{2}}+\frac{27}{280}hf_{n+2}+\frac{27}{35}hf_{n+\frac{5}{2}}$$

$$+\frac{41}{280}hf_{n+3} \tag{13}$$

3. Results

3.1 Analysis of the method

In this section, the basic properties of the method such as, order and error constant, consistency, zero-stability and region of absolute stability) are obtained.

According to Butcher (1965), the three-step block method (13), is of uniform order $p=[6\ \ 6\ \ 6\ \ 6\ \ 6\ \ 6]^T$, with its error constant is given by $C_{p+6}=[-3.1206\times10^{-3}\ \ 1.8519\times10^{-3}\ \ 9.7307\times10^{-3}\ \ 1.7196\times10^{-2}\ \ 3.2143\times10^{-2}\ \ 1.4166\times10^{-2}]$

In the spirit of [17], the method is consistent since it is of order 6.

Applying the method of [16], the first characteristic polynomial is given by,

$$\rho(z)==\left|z\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}=\begin{bmatrix}z&0&0&0&0&-1\\0&z&0&0&0&-1\\0&0&z&0&0&-1\\0&0&0&z&0&-1\\0&0&0&0&z&-1\\0&0&0&0&0&z-1\end{bmatrix}\right|=z^6(z-1)$$

Thus, solving for Z in

$z^6(z-1)$

gives $z=0,0,0,0,0,1$. Hence, (13) is zero-stable. Applying the theorem in (Abolarin *et al.*, 2020), the method is said to be convergent since it is consistent and zero-stable. According to (Wend, 1969), we obtain the stability polynomial as;

$$\bar{h}(w)=\left(-\frac{53}{9909043200}w^4-\frac{1}{52848230400}w^5\right)h^{10}+\left(-\frac{2402251}{356725555200}w^4-\frac{23}{11324620800}w^5\right)h^8$$
$$+\left(-\frac{8882591}{7431782400}w^4-\frac{41}{141557760}w^5\right)h^6+\left(-\frac{469453}{7741440}w^4+\frac{11}{114688}w^5\right)h^4+$$
$$\left(-\frac{151}{180}w^4-\frac{1}{56}w^5\right)h^2-2w^4+w^5 \tag{14}$$

The region of absolute stability of (13) is shown in Figure 3.1 as

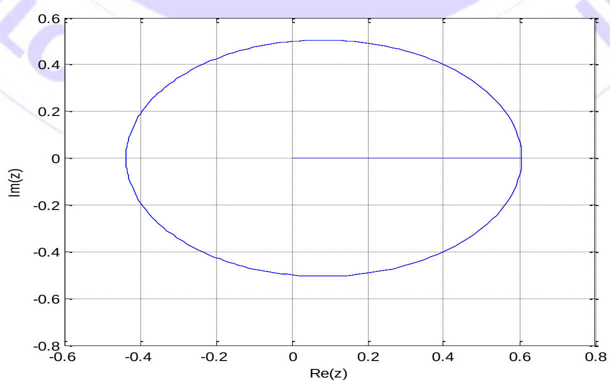

Figure 1 Stability region of (13).

3.2. Numerical Implementation

This study introduces a novel block hybrid technique designed specifically for addressing third-order initial value problems in ordinary differential equations (ODEs), as described by equation (1), without necessitating their conversion into equivalent first-order systems. Subsequently, this newly developed method will be employed to tackle select second and third-order ODEs, as outlined below. The accuracy of the obtained solutions will be assessed by computing the absolute error and juxtaposing the outcomes against established methodologies, particularly those delineated by Kayode and Obaruha, (2017), Adeyeye and Omar, (2018); Taparki *et al.* (2020).

Problem 1. Let's ponder over the considerably non-stiff third-order linear problem

$$y''' = 3\cos(x) = 0, \quad y(0) = 1, \quad y'(0) = 0, \; y''(0) = 2, \; h = 0.1$$

with the exact solution given by

$$y(x) = x^2 - 3\sin(x) + 3x + 1$$

Source: (Taparki *et al.*, 2020).

Problem 2. Let's examine the significantly non-stiff third-order linear problem

$$y''' = 3\sin(x) = 0, \quad y(0) = 1, \; y'(0) = 0, \; y''(0) = -2$$

with the exact solution given by

$$y(x) = 3\cos(x) + \frac{x^2}{2} - 2$$

Source: (Adeyeye and Omar, 2018).

Problem 3. Let's explore the notably stiff third-order linear problem

$y''' = -\exp(x) = 0, \; y(0) = 1, \; y'(0) = -1, \; y''(0) = 3, \; h = 0.1$ with the exact solution given by

$$y(x) = 2x^2 - \exp(x) + 2$$

Source (Kayode and Obaruha, 2017).

Problem 4. Let's explore the notably stiff third-order linear problem

$$y''' = -y', \; y(0) = 0, \; y'(0) = 1, \; y''(0) = 2, \; h = 0.1$$

with the exact solution given by

$$y(x) = 2(1 - \cos x) + \sin x$$

Source (Kayode and Obaruha, 2017).

The following notations shall be used in Tables below.

ES	Exact Solution
CS	Computed Solution
ENM	Absolute error in Method
AETGS10	Absolute error in Taparki *et al.* (2020)
AEAO18	Absolute error in Adeyeye and Omar, 2018)
AEKO17	Absolute error in Kayode and Obaruha, (2017)

Table 1 Showing the results for problem 1.

x	ES	CS	ENM	AEKO17
0.1	1.01049975005951554310	1.01049975005951499670	3.5610e-16	2.4800e-07
0.2	1.04399200761481635360	1.04399200761481381080	4.3204e-15	7.3740e-06
0.3	1.10343938001598127470	1.10343938001597528630	2.3646e-14	6.0542e-05
0.4	1.19174497307404852500	1.19174497307403705530	6.6494e-14	2.5479e-04
0.5	1.31172338418739099920	1.31172338418736982850	1.3989e-13	7.7602e-04
0.6	1.46607257981489392840	1.46607257981485883770	2.5271e-13	1.9261e-03
0.7	1.65734693828692683900	1.65734693828687393890	4.1371e-13	4.1505e-03
0.8	1.88793172730143171510	1.88793172730135439280	6.2811e-13	8.3637e-03
0.9	2.16001927111754983460	2.16001927111744147820	9.0322e-13	1.4774e-02
1.0	2.47558704557631048000	2.47558704557616569370	1.2469e-13	2.4702e-02

Table 2 Showing the results for problem 2.

x	ES	CS	ENM	AEKO17
0.1	0.99001249583407729830	0.99001249583408137232	4.0820e-15	1.7282e-12
0.2	0.96019973352372489340	0.96019973352374389240	1.9318e-14	6.3179e-12
0.3	0.91100946737681805890	0.91100946737686283420	4.6780e-14	1.4295e-11
0.4	0.84318298200865524840	0.84318298200873364027	8.5484e-14	2.5020e-11
0.5	0.75774768567111814840	0.75774768567124015081	1.4054e-13	3.8928e-11
0.6	0.65600684472903489170	0.65600684472921049945	2.1582e-13	5.5360e-11
0.7	0.53952656185346527880	0.53952656185370143660	3.1301e-13	7.4644e-11
0.8	0.41012012804149626280	0.41012012804180132748	4.3906e-13	9.6128e-11
0.9	0.26982990481199336940	0.26982990481237569884	6.0029e-13	1.2002e-10
1.0	0.12090691760441915220	0.12090691760488428778	8.0090e-13	1.4570e-10

Table 3 Showing the results for problem 3.

X	ES	CS	ENM	AEKO17
0.1	0.9148290819243523752	0.91482908192435394257	1.5677e-15	1.8241e-13
0.2	0.8585972418398301661	0.85859724183983747358	7.3075e-15	1.6708e-12
0.3	0.8301411924239968960	0.83014119242401411575	1.7220e-14	6.0014e-12
0.4	0.8281753023587296822	0.82817530235876040460	3.0722e-14	1.4860e-11
0.5	0.8512787292998718532	0.85127872929992156489	4.9712e-14	3.0121e-11
0.6	0.8978811996094910251	0.89788119960956521199	7.4187e-14	5.3842e-11
0.7	0.9662472925295234784	0.96624729252962684100	1.0336e-13	8.8316e-11
0.8	1.0544590715075323954	1.05445907150767219400	1.3980e-13	1.3606e-10
0.9	1.1603968888430503362	1.16039688884323383010	1.8350e-13	1.9987e-10
1.0	1.2817181715409547646	1.28171817154118815290	2.3339e-13	2.8281e-10

Table 4 Showing the results for problem 4.

x	ES	CS	ENM	AEKO17
0.1	0.10982508609077662011	0.10982508609077408792	2.5322e-15	1.1177e-10
0.2	0.23853617511257795326	0.23853617511256616970	1.1784e-14	9.3348e-10
0.3	0.38484722841012753581	0.38484722841009987366	2.7662e-14	3.2775e-09
0.4	0.54729635430288032607	0.54729635430283251642	4.7810e-14	8.0524e-09
0.5	0.72426041482345756807	0.72426041482338483741	7.2731e-14	1.6249e-08
0.6	0.91397124357567876270	0.91397124357557658567	1.0218e-13	2.8912e-08
0.7	1.11453331266871420120	1.11453331266858026760	1.3393e-13	4.7125e-08
0.8	1.32394267220519191980	1.32394267220502420390	1.6772e-13	7.1985e-08
0.9	1.54010697308615447550	1.54010697308595128830	2.0319e-13	1.0458e-07
1.0	1.76086637307161707180	1.76086637307137854920	2.3852e-13	1.4596e-07

4. Discussion

In this study, we have introduced a three-step block method tailored for the direct resolution of higher-order initial value problems within ordinary differential equations, specifically targeting third-order ODEs. Our assessment focused on evaluating the efficacy and reliability of these techniques, particularly in handling highly stiff linear problems, comparing the outcomes with existing literature. Among these, problems 1 through 4 exemplify third-order highly stiff linear scenarios. The comparative analysis presented in Tables 1 through 4 juxtaposes our new approach against methodologies proposed by Kayode and Obaruha (2017), Adeyeye and Omar (2018), and Taparki *et al.* (2020). Our results consistently reveal that our method surpasses the referenced approaches in terms of convergence rate. The evident superiority of our methodology, as depicted by the obtained results, underscores its enhanced efficiency and effectiveness in addressing higher-order initial value problems within ordinary differential equations.

5. Conclusions

To conclude, this study detailed the creation of a three-step block method designed to directly tackle third-order initial value problems. Constructed through interpolation and collocation techniques, the method underwent rigorous analysis to ensure accuracy and convergence by meeting all requisite conditions. Validation occurred through testing on highly stiff linear initial value problems, affirming the efficacy and reliability of the proposed approach. Furthermore, comparison with existing methodologies from the literature unequivocally illustrated the superior convergence speed of the new method. In essence, the introduced block method presents a promising avenue for effectively resolving third-order initial value problems.

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Competing Interests

Authors have declared that there is no conflict of interest reported in this work.

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