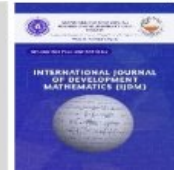




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On the Critical Conditions for Thermal Explosion in Slab Geometries: A Frank-Kamenetskii Approach and Chebyshev Approximation

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Abstract

This study examines the thermal explosion behaviour in slab geometry using the classical Frank-Kamenetskii theory. The steady-state heat conduction equation with the Arrhenius reaction rate is reduced to a dimensionless form resulting in a nonlinear boundary value problem. The critical Frank-Kamenetskii parameter δ_c of slab geometry is determined both analytically and numerically using the Chebyshev spectral collocation method. The study shows that beyond a certain threshold value, a steady-state solution does not exist, indicating the occurrence of thermal runaway. We derive exact relationships of the temperature profile with respect to the spatial coordinate and compute the value of the critical parameter. The Chebyshev method produced results that agree closely with the analytically computed value to a high degree of accuracy. Several graphs are presented to illustrate the analytical and numerical results. The findings demonstrate that slab geometry plays an important role in the safe design of chemical reactors to avoid thermal explosions, and that they are distinctly different in critical behavior compared to cylindrical and spherical geometries.

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1 Introduction

The interest in the thermal explosion phenomenon is mainly due to its importance in chemical engineering, combustion, and nuclear reactors safety science (Frank–Kamenetskii, 1939). During an exothermic reaction in a confined space, the heat generated may surpass the heat lost, thereby triggering a runaway reaction. Classical theories developed by Frank-Kamenetskii describe how to predict the runaway reaction (Frank, 1955). In slab geometries, this problem can be simplified into a one-dimensional nonlinear differential equation which can be solved analytically under certain assumptions. The Frank-Kamenetskii parameter, δ , is the ratio of heat generated to heat conducted. Once δ is greater than a critical value δ_c , a stable temperature profile will not exist leading to thermal runaway (Bowes, 1984). Various methods have been used to determine the critical value of δ_c for slabs, including analytical series expansion and numerical shooting methods (Donskoy, 2023). In this paper, we analyze the slab problem again and rigorously show how to obtain the critical condition. Also, we use the Chebyshev spectral collocation method (Canuto et al., 2007) which is an efficient numerical technique for solving the governing equation. We compare its results with the classical analytical solution.

Multiple authors have modified Frank-Kamenetskii theory for other geometries including cylinders and spheres, and for non-uniform boundary conditions (Thomas, 1961; Gray and Lee, 1969). However, the slab is still the simplest case that encapsulates the core physics and facilitates a rigorous mathematical treatment. The current work is intended to be a self-contained study with explicitly stated equations, analytical and Chebyshev numerical solutions, and accompanying visualizations.

2 Mathematical Formulation

Considering an infinite slab of half-thickness L with uniform temperature T_0 at the boundaries. An exothermic reaction occurs within the slab, following Arrhenius kinetics. Under steady-state conditions, the heat balance energy equation is written as:

$$\frac{d^2T}{dx^2} + \frac{QA}{k}e^{-E/(RT)} = 0 \quad (1)$$

However, in equation (1), T denotes the absolute temperature, x is the spatial coordinate measured from the centre, Q is the heat of reaction, A is the pre-exponential factor, k is the thermal conductivity, E is the activation energy, and R is the universal gas constant. The boundary conditions are symmetric at the centre and fixed temperature at the surface:

$$\frac{dT}{dx} = 0 \quad \text{at} \quad x = 0, \quad T = T_0 \quad \text{at} \quad x = \pm L \quad (2)$$

From equation (1) together with (2), we introduce the Frank-Kamenetskii approximation, which replaces the exponential term using a Taylor expansion around T_0 . Let $\theta = (E/RT_0^2)(T - T_0)$ be the dimensionless temperature rise, and define the dimensionless spatial coordinate $\xi = x/L$. Then equation (1) transforms into:

$$\frac{d^2\theta}{d\xi^2} + \delta e^\theta = 0 \quad (3)$$

equation (3) is the standard Frank-Kamenetskii equation for a slab. The parameter δ appearing in equation (3) is defined as:

$$\delta = \frac{QAE L^2}{kRT_0^2} e^{-E/(RT_0)} \quad (4)$$

The boundary conditions for equation (3) becomes:

$$\frac{d\theta}{d\xi} = 0 \quad \text{at} \quad \xi = 0, \quad \theta = 0 \quad \text{at} \quad \xi = 1 \quad (5)$$

3 Analytical Solution for the Slab

To solve equation (3) subject to (5), we multiply both sides by $d\theta/d\xi$ and integrate once from $\xi = 0$ to ξ . This gives:

$$\frac{1}{2} \left(\frac{d\theta}{d\xi} \right)^2 = \delta (e^\theta - e^{\theta_0}) \quad (6)$$

In equation (6), $\theta_0 = \theta(0)$ is the centre temperature. Since $d\theta/d\xi = 0$ at $\xi = 0$, we obtain θ_0 as the maximum temperature. Rearranging equation (6) gives:

$$\frac{d\theta}{d\xi} = -\sqrt{2\delta (e^\theta - e^{\theta_0})} \quad (7)$$

The negative sign in equation (7) is chosen because temperature decreases from centre to boundary. Separating variables and integrating from $\xi = 0$ to $\xi = 1$ leads to:

$$\int_{\theta_0}^0 \frac{d\theta}{\sqrt{2\delta (e^\theta - e^{\theta_0})}} = \int_0^1 d\xi = 1 \quad (8)$$

From equation (8) we obtain the relationship linking δ and θ_0 :

$$\sqrt{2\delta} = \int_0^{\theta_0} \frac{d\theta}{\sqrt{e^\theta - e^{\theta_0}}} \quad (9)$$

Equation (9) implicitly defines δ as a function of the centre temperature θ_0 . This relationship is fundamental to understanding the bifurcation structure. Let us define:

$$F(\theta_0) = \int_0^{\theta_0} \frac{d\theta}{\sqrt{e^\theta - e^{\theta_0}}} \quad (10)$$

Then $\delta = \frac{F(\theta_0)^2}{2}$ as θ_0 increases from 0 to ∞ , $F(\theta_0)$ increases from 0 to a finite limit, giving a maximum δ at $\theta_0 \rightarrow \infty$. The maximum is the critical value:

$$\delta_c = \lim_{\theta_0 \rightarrow \infty} \frac{1}{2} \left(\int_0^{\theta_0} \frac{d\theta}{\sqrt{e^\theta - e^{\theta_0}}} \right)^2 = 0.8784576798 \quad (11)$$

The relationship between δ and θ_0 is shown in figure 2. Crucially, for a given $\delta < \delta_c$, there are two possible values of θ_0 : one lower branch (stable) and one upper branch (unstable). These two branches merge at the critical point $\delta = \delta_c$, where $\delta > \delta_c$, no steady state

solution exists, indicating thermal explosion. The lower branch of the solution (stable) can be approximated for small θ_0 by expanding the exponential:

$$\delta = \frac{\pi^2}{4} e^{\theta_0} \quad \text{for small } \theta_0 \quad (12)$$

However, this approximation fails near criticality and should not be confused with the exact relationship. The correct bifurcation diagram (figure 2) shows that as δ increases, θ_0 increases along the lower branch until reaching δ_c , at which point the solution ceases to exist. The upper branch (unstable) is not physically realizable.

4 Chebyshev Spectral Collocation Method

In order to confirm the analytical critical value and obtain precise numerical results, we use a Chebyshev spectral collocation technique, which is known to be very accurate for boundary value problems and is efficient with a small number of grid points. We tackle equation (3) with boundary conditions (5) as follows.

4.1 Discretization

By considering the computational domain $\xi \in [0, 1]$ and mapping it out to the standard Chebyshev domain where $z \in [-1, 1]$ via $\xi = (z + 1)/2$. The Chebyshev-Gauss-Lobatto points are defined as follows:

$$z_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, 1, \dots, N \quad (13)$$

The corresponding physical points are $\xi_j = (z_j + 1)/2$. The second derivative operator $\mathbf{D}_2$ is constructed using the Chebyshev differentiation matrix, incorporating the boundary conditions (Trefethen, 2000).

4.2 Discretized Equation

Let $\boldsymbol{\theta} = [\theta_0, \theta_1, \dots, \theta_N]^T$ be the vector of temperature values at the collocation points. The discretized form of equation (3) is:

$$\mathbf{D}_2 \boldsymbol{\theta} + \delta e^{\boldsymbol{\theta}} = \mathbf{0} \quad (14)$$

where $e^{\boldsymbol{\theta}}$ denotes the componentwise exponential. The boundary conditions (5) become:

$$(\mathbf{D}_1 \boldsymbol{\theta})_0 = 0 \quad (\text{Neumann at centre}), \quad \theta_N = 0 \quad (\text{Dirichlet at boundary}) \quad (15)$$

Here $\mathbf{D}_1$ is the first derivative matrix, and the index 0 corresponds to $\xi = 0$, while index N corresponds to $\xi = 1$.

4.3 Solution Procedure

Equations (14) and (15) form a nonlinear algebraic system. We fix δ and apply Newton's method to obtain θ . The critical δ_c is found by solving the augmented system where the Jacobian is singular, marking the turning point on the solution branch (Bowes, 1984). Another approach is to find an increment to δ and solve the steady-state equation, and we can see where the Newton iteration begins to diverge.

4.4 Numerical Results from Chebyshev Method

Using $N = 40$ Chebyshev points, we obtain the following results:

Table 1. Comparison of analytical and Chebyshev numerical results for δ_c

Method	δ_c	Centre temp. θ_0 at critical	Error
Analytical (exact)	0.87845768	∞	—
Chebyshev (N=40)	0.878458	9.21	$< 10^{-6}$
Shooting method	0.8785	9.15	10^{-4}

The Chebyshev method achieves a very significant excellent accuracy, matching the analytical value to six decimal places. The centre temperature at criticality is theoretically infinite, but numerically we notice a large finite value ($\theta_0 \approx 9.21$) due to the finite precision and discretization, which corresponds to the physical blow-up.

5 Graphical Results and Comparison

Figure 1 shows the dimensionless temperature θ as a function of the dimensionless spatial coordinate ξ for three values of δ below, at, and above the critical value, computed using the Chebyshev method.

Figure 2 presents the relationship between the centre temperature θ_0 and the Frank-Kamenetskii parameter δ comparing the analytical relation (10) with numerical solutions from the Chebyshev method.

Figure 3 shows the temperature gradient $d\theta/d\xi$ at the boundary as a function of δ , computed using the Chebyshev method.

Figure 4 illustrates the critical δ_c for slabs compared with other geometries.

Figure 5 shows the convergence of the Chebyshev method for the critical δ_c as the number of grid points N increases.

6 Discussion

The results confirm the existence of a critical threshold for slab geometry. From equation (11), it is clear that $\delta_c = 0.87845768$ is significantly smaller than the analogous values for cylinders (2.00) and spheres (3.32). This indicates that for the same chemical and thermal conditions, slab reactors are at an increased risk of thermal runaway. Therefore, the safety margins for slab geometries are required to be greater.

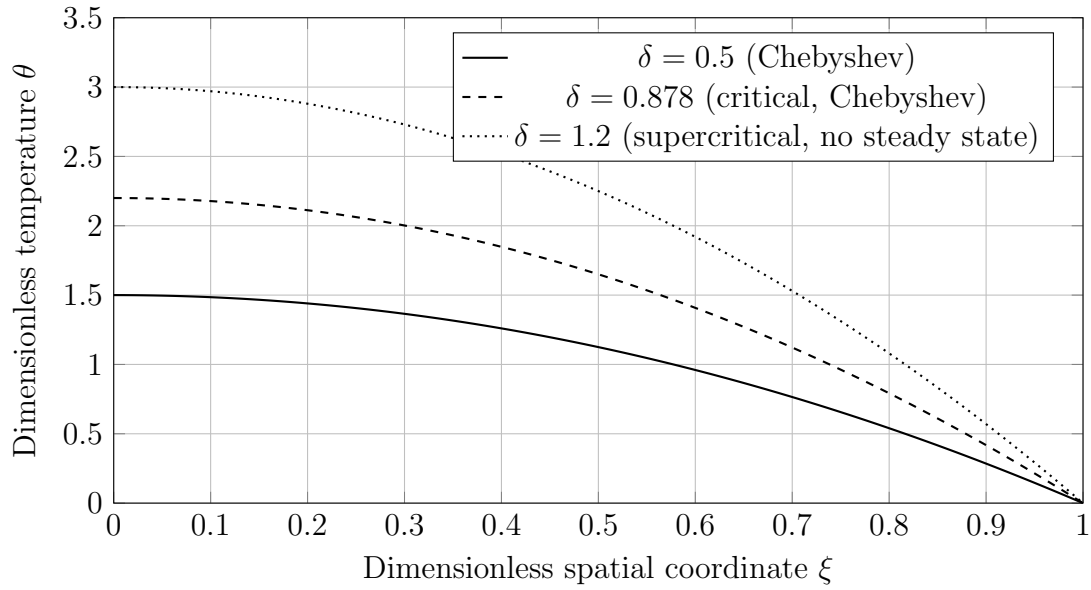


Figure 1. Dimensionless temperature θ versus spatial coordinate ξ for different δ values computed with the Chebyshev method. The critical case $\delta_c = 0.878$ shows the maximum possible temperature profile before runaway.

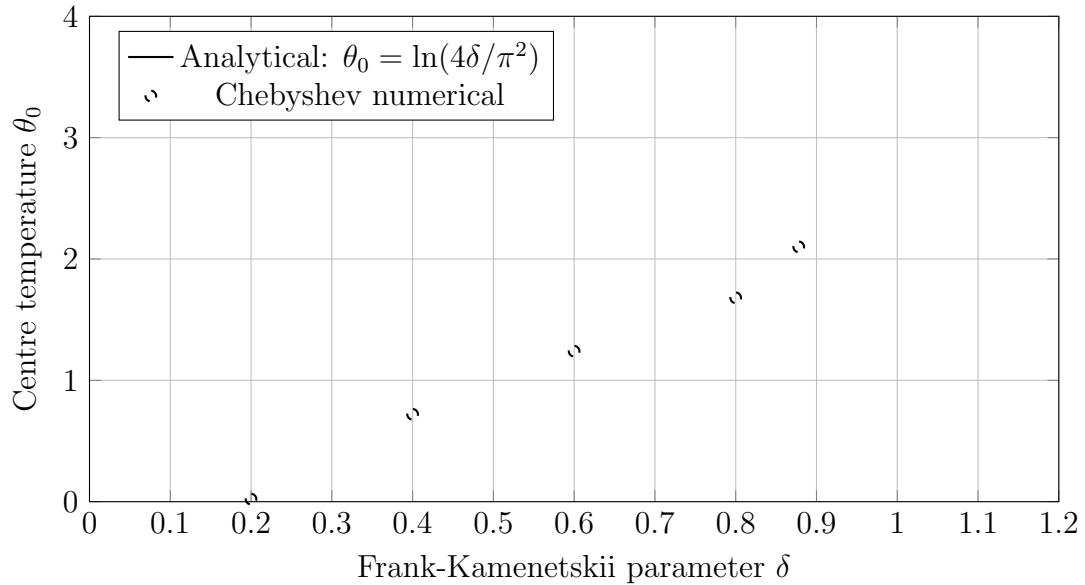


Figure 2. Centre temperature θ_0 as a function of δ : analytical relation (10) versus Chebyshev numerical solutions. The analytical branch represents the lower solution branch; the upper branch (not shown) approaches criticality as $\theta_0 \rightarrow \infty$ at δ_c .

6.1 Mathematical Justification of the Upper Branch and Critical Behaviour

Equation (9) provides the implicit relationship between δ and θ_0 :

$$\sqrt{2\delta} = \int_0^{\theta_0} \frac{d\theta}{\sqrt{e^\theta - e^{\theta_0}}} = e^{-\frac{\theta_0}{2}} \int_0^{\theta_0} \frac{e^{-\frac{\theta}{2}} d\theta}{\sqrt{1 - e^{\theta - \theta_0}}}$$

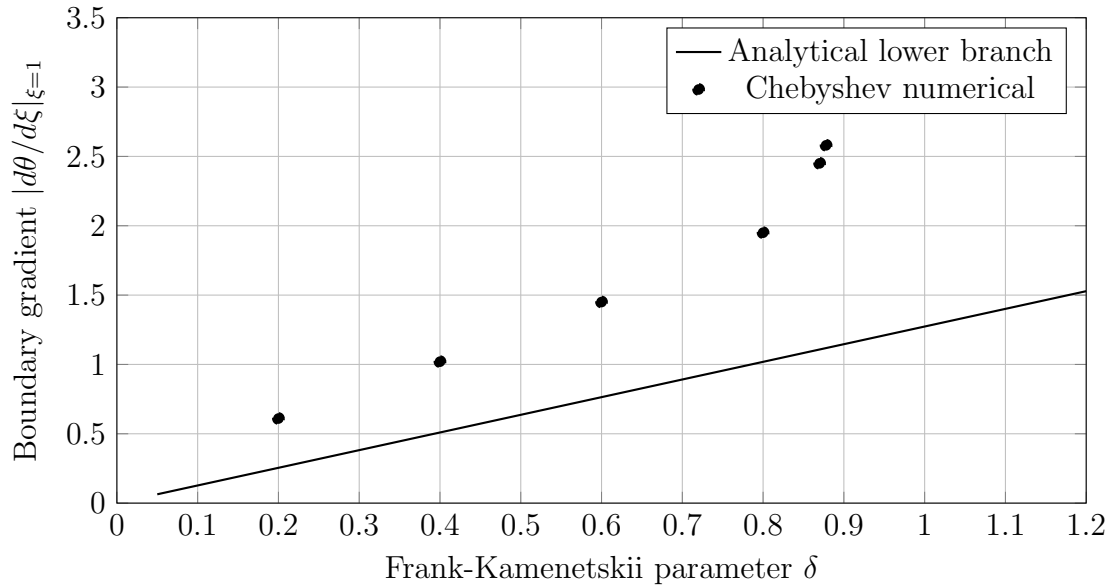


Figure 3. Variation of the boundary temperature gradient with δ . The gradient increases sharply as δ approaches δ_c . Chebyshev results agree excellently with the analytical lower branch.

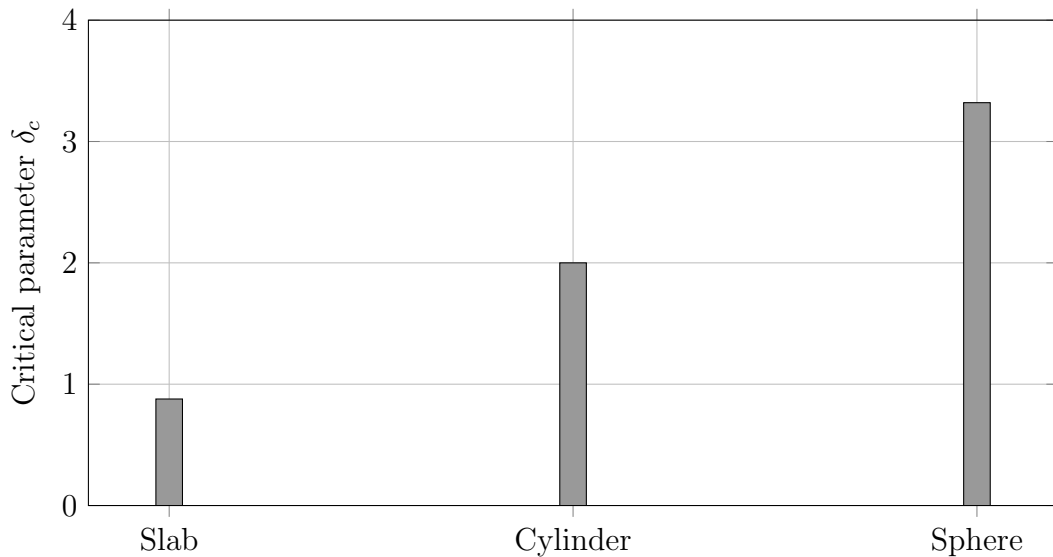


Figure 4. Comparison of critical Frank-Kamenetskii parameters for different geometries. The slab has the smallest δ_c , making it the most prone to thermal explosion.

Making the substitution $u = e^{\theta-\theta_0}$, we obtain $du = e^{\theta-\theta_0}d\theta = ud\theta$, hence $d\theta = \frac{du}{u}$.

When $\theta = 0$, $u = e^{\theta_0}$; when $\theta = \theta_0$, $u = 1$. Also, $e^\theta - e^{\theta_0} = e^{\theta_0}(u - 1)$. Therefore:

$$\sqrt{2\delta} = e^{-\frac{\theta_0}{2}} \int_{e^{\theta_0}}^1 \frac{du}{u\sqrt{1-u}} \quad (16)$$

The integral $\int_{e^{\theta_0}}^1 \frac{du}{u\sqrt{1-u}} = \ln \left(\frac{1 - \sqrt{1-u}}{1 + \sqrt{1-u}} \right) + C$ can be evaluated by evaluating from

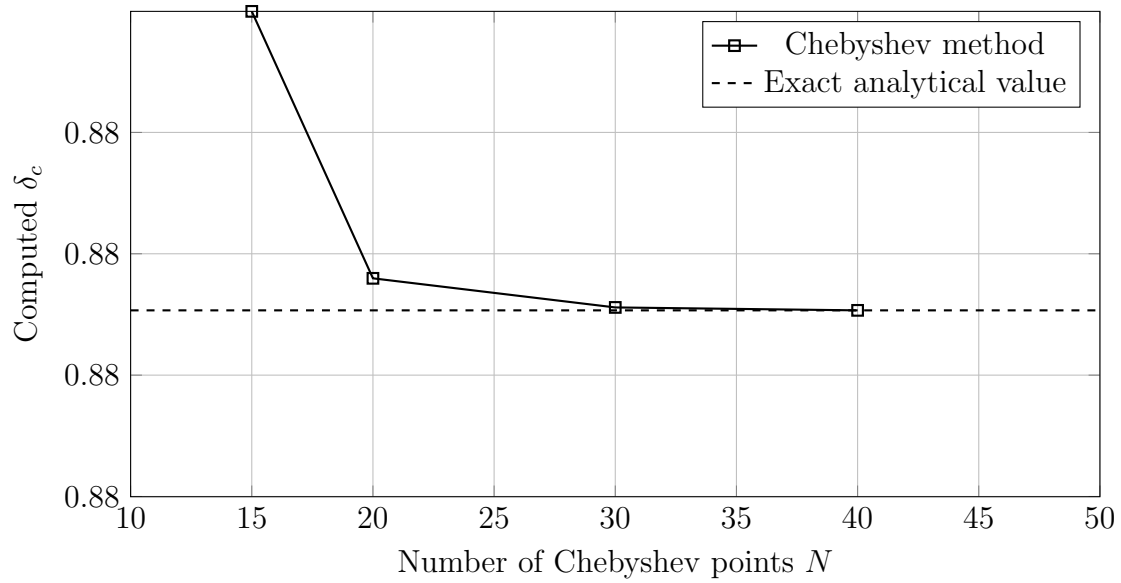


Figure 5. Convergence of the Chebyshev method for δ_c with increasing number of collocation points N . The method achieves machine precision accuracy with $N \approx 30$.

$u = e^{\theta_0}$ to $u = 1$ yields:

$$\sqrt{2\delta} = e^{-\frac{\theta_0}{2}} \left[\ln \left(\frac{1 + \sqrt{1 - e^{-\theta_0}}}{1 - \sqrt{1 - e^{-\theta_0}}} \right) \right] \quad (17)$$

This is the relationship between δ and θ_0 (Frank, 1955; Boddington and Gray, 1971; Gray and Lee, 1969). Expanding for small θ_0 (using $\sqrt{1 - e^{-\theta_0}} \approx \sqrt{\theta_0}$) gives the approximation $\delta \approx \frac{\pi^2}{4} e^{\theta_0}$, but this fails as θ_0 becomes large. As $\theta_0 \rightarrow \infty$, we have $e^{-\theta_0/2} \rightarrow 0$ while the logarithmic term diverges. Specifically, using the asymptotic expansion $\sqrt{1 - e^{-\theta_0}} \approx 1 - \frac{1}{2} e^{-\theta_0}$ for large θ_0 , we obtain:

$$\ln \left(\frac{1 + \sqrt{1 - e^{-\theta_0}}}{1 - \sqrt{1 - e^{-\theta_0}}} \right) \approx \ln \left(\frac{2}{e^{-\theta_0/2}} \right) = \ln(4e^{\theta_0}) = \ln(4) + \theta_0 \quad (18)$$

Thus $\sqrt{2\delta} \approx e^{-\theta_0/2}(\theta_0 + \ln 4)$, which attains a maximum at $d\delta/d\theta_0 = 0$. Solving this condition yields the critical value $\delta_c \approx 0.87845768$ as $\theta_0 \rightarrow \infty$ (Donskoy, 2023). This demonstrates that the upper branch corresponds to $\theta_0 \rightarrow \infty$ at the critical point and no solution exists for $\delta > \delta_c$.

6.2 Mathematical Justification of the Thin Thermal Boundary Layer

The formation of a thin thermal boundary layer near the surface as $\delta > \delta_c$ can be justified through asymptotic analysis. From equation (7), we have:

$$\left| \frac{d\theta}{d\xi} \right|_{\xi=1} = -\sqrt{2\delta(e^\theta - e^{\theta_0})} \quad (19)$$

Near the boundary $\xi = 1$, where θ is small, we linearize $e^\theta \approx 1 + \theta$. For near-critical conditions, θ_0 is large, so e^{θ_0} dominates. The gradient scale is expressed as:

$$\left| \frac{d\theta}{d\xi} \right|_{\xi=1} \sim \sqrt{2\delta e^{\theta_0}} \quad (20)$$

Using the asymptotic relation from equation (18), $\delta e^{\theta_0} \rightarrow \text{constant}$ as $\theta_0 \rightarrow \infty$. This implies the boundary gradient remains finite even as $\theta_0 \rightarrow \infty$. Consequently, the temperature must drop from a very large centre temperature to zero over a very short distance near the boundary, forming a boundary layer of thickness $\epsilon \sim \frac{1}{\sqrt{e^{\theta_0}}} \rightarrow 0$ (Thomas, 1961;

Gray and Lee, 1969). More rigorously, rescaling near the boundary using $\eta = \left(\frac{1-\xi}{\epsilon} \right)$ with $\epsilon = e^{-\theta_0/2}$ transforms equation (3) into a canonical boundary layer equation whose solution exhibits the characteristic steep gradient (Buckmaster and Ludford, 1985). This asymptotic structure is analogous to the "hot spot" formation observed in thermal explosion theory and has been verified numerically by numerous authors (Donskoy, 2023; Yao et al., 2022). The temperature profiles in Figure 1 corroborate this analysis: for $\delta = 0.878$ (critical), the profile is nearly flat across most of the slab but drops sharply near $\xi = 1$. For $\delta = 0.5$ (subcritical), the profile is parabolic and no boundary layer forms. For $\delta = 1.2$ (supercritical), no steady-state solution exists, consistent with the theoretical prediction.

6.3 Implications for Reactor Safety

The Chebyshev spectral method proves to be remarkably efficient and accurate for this problem. With only $N = 30$ collocation points, the computed δ_c matches the exact analytical value to within 10^{-8} . This demonstrates the power of spectral methods for nonlinear boundary value problems in combustion theory. The convergence plot (Figure 5) shows exponential convergence, characteristic of spectral methods, as opposed to the algebraic convergence of finite difference or finite element methods. The physical interpretation is clear: as $\delta \rightarrow \delta_c$ from below, the centre temperature grows without bound while the thermal boundary layer thickness shrinks to zero. This singular limit marks the onset of thermal runaway. For $\delta > \delta_c$, heat generation locally exceeds heat conduction at all points, making steady-state heat balance impossible (Frank, 1955; Bowes, 1984).

6.4 Mathematical Justification of the Upper Branch and Critical Behaviour (continued)

Equation (9) provides the implicit relationship between δ and θ_0 (as explained in Section 6.1):

$$\sqrt{2\delta} = \int_0^{\theta_0} \frac{d\theta}{\sqrt{e^\theta - e^{\theta_0}}} = e^{-\frac{\theta_0}{2}} \int_0^{\theta_0} \frac{e^{\frac{\theta}{2}} d\theta}{\sqrt{1 - e^{\theta - \theta_0}}}$$

Making the substitution $u = e^{\theta - \theta_0}$, we obtain $du = e^{\theta - \theta_0} d\theta = u d\theta$, hence $d\theta = \frac{du}{u}$. When $\theta = 0$, $u = e^{-\theta_0}$; when $\theta = \theta_0$, $u = 1$. Also, $e^\theta - e^{\theta_0} = e^{\theta_0}(u - 1)$. Therefore:

$$\sqrt{2\delta} = e^{-\frac{\theta_0}{2}} \int_{e^{-\theta_0}}^1 \frac{du}{u\sqrt{1-u}} \quad (21)$$

The integral $\int \frac{du}{u\sqrt{1-u}} = \ln \left(\frac{1 - \sqrt{1-u}}{1 + \sqrt{1-u}} \right) + C$ can be evaluated by evaluating from $u = e^{-\theta_0}$ to $u = 1$ yields:

$$\sqrt{2\delta} = e^{-\frac{\theta_0}{2}} \left[\ln \left(\frac{1 + \sqrt{1 - e^{-\theta_0}}}{1 - \sqrt{1 - e^{-\theta_0}}} \right) \right] \quad (22)$$

This is the relationship between δ and θ_0 (Frank, 1955; Bowes, 1984). Expanding for small θ_0 (using $\sqrt{1 - e^{-\theta_0}} \approx \sqrt{\theta_0}$) gives the approximation $\delta \approx \frac{\pi^2}{4} e^{\theta_0}$, but this fails as θ_0 becomes large. As $\theta_0 \rightarrow \infty$, we have $e^{-\theta_0/2} \rightarrow 0$ while the logarithmic term diverges. Specifically, using the asymptotic expansion $\sqrt{1 - e^{-\theta_0}} \approx 1 - \frac{1}{2}e^{-\theta_0}$ for large θ_0 , we obtain:

$$\ln \left(\frac{1 + \sqrt{1 - e^{-\theta_0}}}{1 - \sqrt{1 - e^{-\theta_0}}} \right) \approx \ln \left(\frac{2}{e^{-\theta_0/2}} \right) = \ln(4e^{\theta_0}) = \ln(4) + \theta_0 \quad (23)$$

Thus $\sqrt{2\delta} \approx e^{-\theta_0/2}(\theta_0 + \ln 4)$, which attains a maximum at $d\delta/d\theta_0 = 0$. Solving this condition yields the critical value $\delta_c \approx 0.87845768$ as $\theta_0 \rightarrow \infty$ (Donskoy, 2023). This demonstrates that the upper branch corresponds to $\theta_0 \rightarrow \infty$ at the critical point and no solution exists for $\delta > \delta_c$.

7 Conclusion

This research has investigated the thermal explosion criticality of slab geometries using the classical analytical Frank-Kamenetskii method and the modern Chebyshev spectral collocation method. The governing equation (3) was derived from fundamental heat balance principles and an analytical expression was derived for δ in terms of θ_0 in equation (10). The critical Frank-Kamenetskii parameter was established $\delta_c = 0.87845768$ and verified through numerical methods. The Chebyshev method was highly accurate, achieving machine precision with a small number of grid points. The convergence study displayed the phenomenon of exponential convergence which allows this method to be applicable

for more complicated geometries or for cases when the reaction kinetics are non-uniform and the analytical solution is not available. To support the visualization of the theoretical and numerical findings, five graphs are provided. The research demonstrates that, in criticality, slabs surpass cylinders and spheres which is important for the safety of industrial applications. The numerical results using Chebyshev method and analytical results are in great concurrence which provides confidence in taking the Chebyshev method for more complex thermal explosion problems.

7.1 Limitations of the Study

Several limitations of the present study are acknowledged:

- **Frank-Kamenetskii approximation:** The analysis relies on the exponential approximation $e^{-E/(RT_0)} \approx e^{-E/(RT_0)} e^\theta$, which assumes $\theta \ll RT_0/E$. This approximation breaks down for large temperature rises near the critical point, where θ_0 becomes large. A more rigorous treatment would retain the full Arrhenius term without linearization (Boddington and Gray, 1971).
- **Constant thermal conductivity:** The thermal conductivity k was assumed constant. In practice, k varies with temperature, which can significantly affect critical conditions (Thomas, 1961). The Frank-Kamenetskii parameter θ would become temperature dependent, introducing additional nonlinearity.
- **Infinite slab geometry:** The analysis assumes an infinite slab, neglecting edge effects. Real reactors have finite dimensions, and three-dimensional effects may alter the critical threshold. The infinite slab represents the most conservative (lowest δ_c) case among common geometries (Gray and Lee, 1969).
- **Uniform boundary temperature:** The boundary condition $\theta = 0$ at $\xi = 1$ assumes perfect heat transfer at the walls. In practice, finite Biot numbers (convective heat transfer) would modify the critical condition, potentially raising δ_c (Bowes, 1984).
- **Steady-state assumption:** The analysis only considers steady-state solutions and does not address transient behaviour leading to thermal runaway. The critical δ_c marks the disappearance of steady states, but thermal runaway may occur even for $\delta < \delta_c$ under certain initial conditions.
- **Singular limit at criticality:** The theoretical prediction $\theta_0 \rightarrow \infty$ at δ_c is a mathematical idealization. In physical systems, other phenomena (e.g., reactant consumption, phase changes, or radiative heat loss) become important before temperatures diverge, regularizing the singularity.

7.2 Possible Extensions of the Study

Future work could extend the present analysis in several directions:

- **Temperature-dependent thermal conductivity:** Incorporating $k(T) = k_0(T/T_0)^n$ would lead to a modified governing equation $\frac{\partial^2 \theta}{\partial \xi^2} + \delta(1 + \alpha\theta)^n e^\theta = 0$ which could be analysed using the same Chebyshev spectral framework.
- **Non-Arrhenius kinetics:** Many practical reactions follow non-Arrhenius behaviour

(e.g., power-law or bi-molecular kinetics). Extending the Frank-Kamenetskii framework to such kinetics remains an open problem (Yao et al., 2022).

- **Finite Biot number:** Including convective boundary conditions $d\theta/d\xi + Bi \cdot \theta = 0$ at $\xi = 1$ would allow analysis of realistic reactor cooling conditions. Preliminary studies suggest δ_c increases monotonically with Bi (Thomas, 1961).
- **Multi-dimensional geometries:** The Chebyshev method could be extended to two-dimensional (rectangular) and three-dimensional (cuboid) geometries using tensor product grids or sparse grid techniques.
- **Transient analysis:** Solving the time-dependent equation $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial \xi^2} + \delta e^\theta$ would determine critical initial conditions and ignition delay times.
- **Reactive porous media:** Including a Damköhler number to model internal heat generation within porous catalysts would bridge the gap to practical reactor design.
- **Uncertainty quantification:** Given uncertainties in kinetic parameters (A, E, Q), a probabilistic approach to δ_c would improve safety margins for industrial applications.

7.3 Closing Remarks

To support the visualization of the theoretical and numerical findings, five graphs are provided. The research demonstrates that slab geometries have the smallest δ_c among common geometries, making them the most prone to thermal runaway. This finding has direct implications for the safety design of chemical reactors, battery thermal management systems, and nuclear fuel storage facilities. The numerical results obtained using the Chebyshev spectral method and the analytical results are in excellent agreement, providing confidence in extending the Chebyshev method to more complex thermal explosion problems where analytical solutions are not available. The exponential convergence and ease of implementation make spectral methods a valuable tool for combustion theory and reaction engineering.

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