International Journal of Development Mathematics Vol 3 Issue 2 Page No. 099 - 114

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals

Computational Numerical Method for Direct Solutions of Second Order Oscillatory Problems

Danladi, Nathaniel[a] and Adedayo A. Emmanuel[a]

[a]Department of Mathematics, Modibbo Adama University, Yola, Adamawa State, Nigeria

ARTICLE INFO

Article history:
Received 10 February 2026
Received in revised form 20 May 2026
Accepted 30 May 2026

Keywords:
Computational Numerical Method, Second-Order Differential Equations, Oscillatory Problems, Hybrid Linear Multistep Method, Numerical Stability, Initial Value Problems

MSC 2020 Subject classification:
65D05, 41A05

ABSTRACT

This study presents a Computational Numerical Method (CNM) for directly solving second-order oscillatory initial value problems without reducing them to systems of first-order differential equations. The CNM is derived by constructing a power series polynomial as a basis function, which is differentiated and collocated at some points to form a continuous hybrid linear multistep method. The method's properties, including order, error constant, consistency, zero-stability, convergence and region of absolute stability, are rigorously analyzed to ensure accuracy and reliability. Numerical simulations are conducted on physical and engineering problems, including second-order cooling oscillatory differential equation, simple harmonic motion and Stiefel linear oscillatory differential equations. The results demonstrate near-perfect agreement between analytical and numerical solutions confirmed that, the CNM is efficient and effectiveness in modeling oscillatory differential equations in dynamic systems. The results indicate that the CNM is a robust tool for accurately simulation of second-order oscillatory differential equations.

1. Introduction

Second-order oscillatory differential equations frequently arise in many areas of science and engineering, including mechanical vibrations, electrical circuits, quantum mechanics and wave propagation phenomena. These problems are typically modeled by second-order ordinary differential equations whose solutions exhibit oscillatory behavior over time. Traditional numerical methods often transform such equations into systems of first-order equations before solving them, which can increase computational cost and reduce efficiency (Adewale & Sabo, 2024). As a result, there has been high interest in developing computational numerical methods that can directly handle second-order formulations without reduction, thereby preserving the essential structure of the problem and improving accuracy (Omole & Ogunware, 2018, Kwari et al. 2021).

Computational numerical methods for the direct solution of second-order oscillatory problems are specifically designed to exploit the oscillatory nature of the solution. These methods, such as direct integration schemes, trigonometrically-fitted methods, and block methods, aimed to achieve higher accuracy, stability and efficiency by minimizing phase-lag and numerical damping errors. By avoiding the transformation into first-order systems, these approaches reduce computational complexity and allow for better approximation of periodic solutions over long intervals (Adewale & Sabo, 2024). Consequently, they have become essential tools in solving real-life oscillatory models where precision and computational efficiency are critical (Adewale & Sabo, 2024).

* Corresponding author. Tel.: +2347036677817
E-mail address: nathanieldanladi@mau.edu.ng (Danladi, N.)
https://doi.org/10.62054/ijdm/0302.07

Mathematical modeling has become a powerful tool in recent years for advancing our understanding of processes in biology, physics and medicine, particularly in areas such as dynamic systems, body cooling and simple harmonic motion. These models have significantly contributed to developments in both mathematical theory and the biosciences, as noted by Sabo et al. (2021) and Elazzouzi et al. (2019). The application of mathematics in these fields has fostered innovative approaches and opened new opportunities for interdisciplinary collaboration. This is especially evident in dynamic problems and the study of thermal processes, where mathematical techniques provide deeper insights and more precise predictions (Kubuye & Omar, 2015a).

The numerical solution of higher-order ordinary differential equations is typically achieved by reducing them to a system of first-order differential equations, after which an appropriate numerical method for first-order equations is applied. This approach computes the solution incrementally, one point at a time. However, the primary drawbacks of this method include its high computational cost, increased complexity, potential accuracy issues due to errors, challenges in programming the method, and significant time consumption (Kubuye & Omar, 2015a).

This study focuses on direct solution initial value problems (IVPs) of second-order ordinary differential equations (ODEs) of the form:

$$\gamma''(t) = f(t,\gamma,\gamma'), y(\gamma_0) = \gamma_0, \gamma'(t_0) = \gamma_0', \ \ t \in [t_0, t_n] \tag{1.1}$$

where t_0 represents initial value, γ_0 denotes the solution at time t_0, f remains continuous over the integration interval. We operate under the assumption that equation (1.1) adheres to the existence and uniqueness theorem of differential equations. Furthermore, we presume that solutions to equations akin to (1.1) remain bounded. It's crucial to clarify that a solution $y(t)$ to equation (1.1) is deemed bounded if,

$$\sup_{t \in \Re} \|y(t)\| < \infty \tag{1.2}$$

The numerical solution of Equation (1.1) has attracted significant attention among researchers, as discussed in the works of Kyagya et al. (2021), Kubuye and Omar (2015b), Adeyeye and Omar (2017) and Skwame et al. (2019).

Research has shown that solving equations of the form (1.1) directly is often more efficient than reducing them to systems of first-order ordinary differential equations, as highlighted by Kwari et al. (2023). This understanding has motivated many researchers to develop direct methods for solving Equation (1.1) without transforming it into a system of first-order equations. Several approaches for the direct solution of Equation (1.1) have been widely reported in the literature. Notable contributions include the works of Adeniran and Ogundare (2015), Kwanamu et al. (2021), Donald et al. (2021), Ayinde et al. (2023) and Ishaq et al. (2024).

Other important studies include Ibrahim, (2017), who developed a hybrid multistep method specifically for solving second-order initial value problems without reducing them to systems of first-order equations. The study demonstrated that hybridization enhances both accuracy and stability through the inclusion of additional off-step points, thereby improving the order of the method. Similarly, Kamoh et al. (2018) introduced a block procedure with continuous coefficients based on shifted Legendre polynomials as basis functions. Their method provided a direct and efficient framework for solving second-order ODEs, with improved approximation achieved through continuous representation of the solution. In the same manner, Raymond et al. (2018) proposed an implicit two-step hybrid block method incorporating an off-grid point and third-derivative consideration. Their approach achieved better convergence and higher accuracy, particularly for stiff and oscillatory problems, by combining block techniques with higher-derivative information.

More recent contributions have focused on improving the efficiency of numerical schemes for oscillatory differential equations. Sabo et al. (2024) presented a two-step block approach tailored for oscillatory systems, demonstrating through simulations that the method achieves high accuracy and reduced computational cost. Adewale & Sabo (2024) further explored the application of numerical methods in modeling oscillatory motion, particularly mass-spring systems, highlighting the reliability of direct computational techniques in capturing dynamic behavior. Additionally, Aloko et al. (2024) extended these ideas by developing an efficient scheme for the direct simulation of third and fourth order oscillatory differential equations, emphasizing improved stability and reduced error propagation. Collectively, these studies underscore the growing advancement in direct numerical methods, particularly block and hybrid techniques, as powerful tools for solving oscillatory problems with enhanced precision and computational efficiency. These studies underscore the growing interest and advancements of computational numerical method direct solutions of second-order oscillatory ordinary differential equations efficiently.

2. Method

2.1 Derivation of Computational Numerical Method

The computational numerical method were derived in this section, where an approximate solution to a power series polynomial of the form

$$\gamma(t) = \sum_{j=0}^{\varphi} \alpha_j t^j \tag{2.1}$$

is considered as a basis function for the direct solution of the second initial value problems of the form (1.1). Where $t \in [a,b]$, the $a's$ are real unknown parameters to be determined, $\varphi = \omega + \varpi - 1$ and $\omega + \varpi$ is the sum of the number of interpolation and collocation points.

Differentiate equation (2.1) two times to obtain

$$\gamma''(t) = \sum_{j=0}^{\varphi} j(j-1) a_j t^{j-2} \tag{2.2}$$

Now, interpolating (2.1) at point $\omega = \frac{1}{2}, 1$ and collocating (2.2) at $\varpi = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1$ which lead to a system of equation in a matrix form as

$$\begin{bmatrix}
1 & t_{n+\frac{1}{2}} & t^2_{n+\frac{1}{2}} & t^3_{n+\frac{1}{2}} & t^4_{n+\frac{1}{2}} & t^5_{n+\frac{1}{2}} & t^6_{n+\frac{1}{2}} & t^7_{n+\frac{1}{2}} & t^8_{n+\frac{1}{2}} & t^9_{n+\frac{1}{2}} \\
1 & t_{n+1} & t^2_{n+1} & t^3_{n+1} & t^4_{n+1} & t^5_{n+1} & t^6_{n+1} & t^7_{n+\frac{1}{2}} & t^8_{n+1} & t^9_{n+1} \\
0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 & 56t_n^6 & 72t_n^7 \\
0 & 0 & 2 & 6t_{n+\frac{1}{6}} & 12t^2_{n+\frac{1}{6}} & 20t^3_{n+\frac{1}{6}} & 30t^4_{n+\frac{1}{6}} & 42t^5_{n+\frac{1}{6}} & 56t^6_{n+\frac{1}{6}} & 72t^7_{n+\frac{1}{6}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{4}} & 12t^2_{n+\frac{1}{4}} & 20t^3_{n+\frac{1}{4}} & 30t^4_{n+\frac{1}{4}} & 42t^5_{n+\frac{1}{4}} & 56t^6_{n+\frac{1}{4}} & 72t^7_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{3}} & 12t^2_{n+\frac{1}{3}} & 20t^3_{n+\frac{1}{3}} & 30t^4_{n+\frac{1}{3}} & 42t^5_{n+\frac{1}{3}} & 56t^6_{n+\frac{1}{3}} & 72t^7_{n+\frac{1}{3}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{2}} & 12t^2_{n+\frac{1}{2}} & 20t^3_{n+\frac{1}{2}} & 30t^4_{n+\frac{1}{2}} & 42t^5_{n+\frac{1}{2}} & 56t^6_{n+\frac{1}{2}} & 72t^7_{n+\frac{1}{2}} \\
0 & 0 & 2 & 6t_{n+\frac{2}{3}} & 12t^2_{n+\frac{2}{3}} & 20t^3_{n+\frac{2}{3}} & 30t^4_{n+\frac{2}{3}} & 42t^5_{n+\frac{2}{3}} & 56t^6_{n+\frac{2}{3}} & 72t^7_{n+\frac{2}{3}} \\
0 & 0 & 2 & 6t_{n+\frac{3}{4}} & 12t^2_{n+\frac{3}{4}} & 20t^3_{n+\frac{3}{4}} & 30t^4_{n+\frac{3}{4}} & 42t^5_{n+\frac{3}{4}} & 56t^6_{n+\frac{3}{4}} & 72t^7_{n+\frac{3}{4}} \\
0 & 0 & 2 & 6t_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} & 30t^4_{n+1} & 42t^5_{n+1} & 56t^6_{n+1} & 72t^7_{n+1}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}
=
\begin{bmatrix} \gamma_{n+\frac{1}{2}} \\ \gamma_{n+2} \\ f_n \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \tag{2.3}$$

using Gaussian elimination method, (2.3) is solved for the $a_j's$. The values of the $a_j's$ obtained are then substituted into (1.1), after some manipulations, this gives a continuous hybrid linear multistep method of the form;

$$\gamma(t)=\alpha_{\frac{1}{2}}(t)\gamma_{n+\frac{1}{2}}+\alpha_1(t)\gamma_{n+1}+h^2\left[\sum_{j=0}^{1}\beta_j(t)f_{n+j}+\beta_{v_i}(t)f_{n+v_i}\right],\ v_i=0,\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4} \qquad (2.4)$$

the coefficient $\alpha_{\frac{1}{2}},\alpha_1,\beta_0,\beta_{\frac{1}{6}},\beta_{\frac{1}{4}},\beta_{\frac{1}{3}},\ \beta_{\frac{1}{2}},\beta_{\frac{2}{3}},\beta_{\frac{3}{4}},\beta_1$ are given by;

$$\alpha_{\frac{1}{2}}=2-2t$$

$$\alpha_1=2t-1$$

$$\beta_0=\frac{41}{13440}-\frac{1867}{40320}t+\frac{1}{2}t^2-\frac{113}{36}t^3+\frac{427}{36}t^4-\frac{3367}{120}t^5+\frac{1876}{45}t^6-\frac{113}{3}t^7+\frac{132}{7}t^8-4t^9$$

$$\beta_{\frac{1}{6}}=-\frac{54}{175}t+\frac{648}{35}t^3-\frac{594}{5}t^4+\frac{9072}{25}-\frac{3132}{5}t^6+\frac{21816}{35}t^7-\frac{11664}{35}t^8+\frac{2592}{35}t^9$$

$$\beta_{\frac{1}{4}}=\frac{2}{21}+\frac{26}{175}t-\frac{512}{15}t^3+\frac{11392}{45}t^4-\frac{21248}{25}t^5+\frac{14080}{9}t^6-\frac{171008}{105}t^7+\frac{31488}{35}t^8-\frac{1024}{5}t^9$$

$$\beta_{\frac{1}{3}}=-\frac{243}{4480}-\frac{567}{3200}t+\frac{243}{10}t^3-\frac{1539}{8}t^4+\frac{138267}{200}t^5-\frac{6723}{5}t^6+\frac{51111}{35}t^7-\frac{5832}{7}t^8+\frac{972}{5}t^9$$

$$\beta_{\frac{1}{2}}=\frac{17}{105}-\frac{9}{35}t-8t^3+\frac{202}{3}t^4-\frac{1304}{5}t^5+\frac{8252}{15}t^6-\frac{4504}{7}t^7+\frac{2736}{7}t^8-96t^9$$

$$\beta_{\frac{2}{3}}=-\frac{243}{4480}+\frac{1647}{22400}t+\frac{81}{20}t^3-\frac{351}{10}t^4+\frac{28269}{200}t^5-\frac{1566}{5}t^6+\frac{13581}{35}t^7-\frac{8748}{35}t^8+\frac{324}{5}t^9$$

$$\beta_{\frac{3}{4}}=\frac{2}{21}-\frac{278}{1575}t+\frac{512}{315}t^3-\frac{128}{9}t^4+\frac{4352}{75}t^5-\frac{5888}{45}t^6+\frac{17408}{105}t^7-\frac{768}{7}t^8-\frac{1024}{35}t^9$$

$$\beta_1=\frac{41}{13440}-\frac{143}{22400}t+\frac{1}{30}t^3-\frac{107}{360}t^4+\frac{249}{200}t^5-\frac{131}{45}t^6+\frac{407}{405}t^7-\frac{96}{35}t^8+\frac{4}{5}t^9$$

evaluating (2.4) at non interpolating points to obtain the continuous form as,

$$\left.\begin{array}{l}
\gamma_n-2y_{n+\frac{1}{2}}-\gamma_{n+1}=\frac{41}{13440}h^2f_n+\frac{2}{21}h^2f_{n+\frac{1}{4}}-\frac{243}{4480}h^2f_{n+\frac{1}{3}}+\frac{17}{105}h^2f_{n+\frac{1}{2}}-\frac{243}{4480}h^2f_{n+\frac{2}{3}}+\frac{2}{21}f_{n+\frac{3}{4}}+\frac{41}{13440}f_{n+1}\\
\gamma_{n+\frac{1}{6}}-\frac{5}{3}\gamma_{n+\frac{1}{2}}-+\frac{2}{3}\gamma_{n+1}=\frac{8851}{8817984}h^2f_n-\frac{188}{8505}h^2f_{n+\frac{1}{6}}+\frac{52522}{688905}h^2f_{n+\frac{1}{4}}-\frac{4979}{90720}h^2f_{n+\frac{1}{3}}+\frac{10117}{91854}h^2f_{n+\frac{1}{2}}-\frac{20473}{544320}h^2f_{n+\frac{2}{3}}+\frac{44146}{688905}h^2f_{n+\frac{3}{4}}+\frac{22283}{11022480}h^2f_{n+1}\\
\gamma_{n+\frac{1}{4}}-\frac{3}{2}\gamma_{n+\frac{1}{2}}+\frac{1}{2}\gamma_{n+1}=\frac{1749}{2293760}h^2f_n-\frac{4941}{286720}h^2f_{n+\frac{1}{6}}+\frac{359}{6720}h^2f_{n+\frac{1}{4}}-\frac{106191}{2293760}h^2f_{n+\frac{1}{3}}+\frac{23393}{286720}h^2f_{n+\frac{1}{2}}-\frac{64233}{2293760}h^2f_{n+\frac{2}{3}}+\frac{43}{896}h^2f_{n+\frac{3}{4}}+\frac{10441}{6881280}h^2f_{n+1}\\
\gamma_{n+\frac{1}{3}}-\frac{4}{3}\gamma_{n+\frac{1}{2}}+\frac{1}{3}\gamma_{n+1}=\frac{22819}{44089920}h^2f_n-\frac{20}{1701}h^2f_{n+\frac{1}{6}}+\frac{25052}{688905}h^2f_{n+\frac{1}{4}}-\frac{6737}{181440}h^2f_{n+\frac{1}{3}}+\frac{12182}{229635}h^2f_{n+\frac{1}{2}}-\frac{10019}{544320}h^2f_{n+\frac{2}{3}}+\frac{628}{19683}h^2f_{n+\frac{3}{4}}+\frac{44659}{44089920}h^2f_{n+1}\\
\gamma_{n+\frac{2}{3}}-\frac{2}{3}\gamma_{n+\frac{1}{2}}-\frac{1}{3}\gamma_{n+1}=-\frac{45149}{88179840}h^2f_n+\frac{20}{1701}h^2f_{n+\frac{1}{6}}-\frac{5098}{137781}h^2f_{n+\frac{1}{4}}+\frac{2963}{72576}h^2f_{n+\frac{1}{3}}-\frac{7211}{229635}h^2f_{n+\frac{1}{2}}+\frac{24061}{1088640}h^2f_{n+\frac{2}{3}}-\frac{22418}{688905}h^2f_{n+\frac{3}{4}}-\frac{88829}{88179840}f_{n+1}\\
\gamma_{n+\frac{3}{4}}-\frac{1}{2}\gamma_{n+\frac{1}{2}}-\frac{1}{2}\gamma_{n+1}=-\frac{5179}{6881280}h^2f_n+\frac{4941}{286720}h^2f_{n+\frac{1}{6}}-\frac{727}{13440}h^2f_{n+\frac{1}{4}}+\frac{136323}{2293760}h^2f_{n+\frac{1}{3}}-\frac{37883}{860160}h^2f_{n+\frac{1}{2}}+\frac{18873}{458752}h^2f_{n+\frac{2}{3}}-\frac{109}{2240}h^2f_{n+\frac{3}{4}}-\frac{10373}{6881280}f_{n+1}
\end{array}\right\} \qquad (2.5)$$

differentiating (2.4) once, yields

$$\gamma'(t)=\sigma_{\frac{1}{2}}(t)\gamma_{n+\frac{1}{2}}+\sigma_1(t)\gamma_{n+1}+h^2\left[\sum_{j=0}^{1}\beta'_j(t)f_{n+j}+\beta'_{v_i}(t)f_{n+v_i}\right],\ v_i=0,\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4} \qquad (2.6)$$

the coefficient $\sigma_{\frac{1}{2}},\sigma_1,\beta'_0,\beta'_{\frac{1}{6}},\beta'_{\frac{1}{4}},\beta'_{\frac{1}{3}},\ \beta'_{\frac{1}{2}},\beta'_{\frac{2}{3}},\beta'_{\frac{3}{4}},\beta'_1$ are given by;

$$\sigma_{\frac{1}{2}} = -2$$

$$\sigma_1 = 2$$

$$\beta'_0 = -\frac{1867}{40320} + t - \frac{113}{12}t^2 + \frac{427}{9}t^3 - \frac{3367}{24}t^4 + \frac{3752}{15}t^5 - \frac{791}{3}t^6 + \frac{1056}{7}t^7 - 36t^8$$

$$\beta'_{\frac{1}{6}} = -\frac{54}{175} + \frac{1944}{35}t^2 - \frac{2376}{5}t^3 + \frac{9072}{5}t^4 - \frac{18792}{5}t^5 + \frac{21816}{5}t^6 - \frac{93312}{35}t^7 + \frac{23328}{35}t^8$$

$$\beta'_{\frac{1}{4}} = \frac{26}{175} - \frac{512}{5}t^2 + \frac{45568}{45}t^3 - \frac{21248}{5}t^4 + \frac{28160}{3}t^5 - \frac{171008}{15}t^6 + \frac{251904}{35}t^7 - \frac{9216}{5}t^8$$

$$\beta'_{\frac{1}{3}} = -\frac{567}{3200} + \frac{729}{10}t^2 - \frac{1539}{2}t^3 + \frac{138267}{40}t^4 - \frac{40338}{5}t^5 + \frac{51111}{5}t^6 - \frac{46656}{7}t^7 + \frac{8748}{5}t^8$$

$$\beta'_{\frac{1}{2}} = -\frac{143}{22400} + \frac{1}{10}t^2 - \frac{107}{90}t^3 + \frac{249}{40}t^4 - \frac{262}{15}t^5 + \frac{407}{15}t^6 - \frac{768}{35}t^7 + \frac{36}{5}t^8$$

$$\beta'_{\frac{2}{3}} = -\frac{1647}{22400} + \frac{243}{20}t^2 - \frac{702}{5}t^3 + \frac{28269}{40}t^4 - \frac{9396}{5}t^5 + \frac{13581}{5}t^6 - \frac{69984}{35}t^7 + \frac{2916}{5}t^8$$

$$\beta'_{\frac{3}{4}} = -\frac{278}{1575} - \frac{512}{105}t^2 + \frac{512}{9}t^3 - \frac{4352}{15}t^4 + \frac{11776}{15}t^5 - \frac{17408}{15}t^6 + \frac{6144}{7}t^7 - \frac{9216}{35}t^8$$

$$\beta'_1 = -\frac{143}{22400} + \frac{1}{10}t^2 - \frac{107}{90}t^3 + \frac{249}{40}t^4 - \frac{626}{15}t^5 + \frac{407}{15}t^6 - \frac{768}{35}t^7 + \frac{36}{5}t^8$$

Evaluate equation (2.6) at all point, to yields the following discrete computational numerical method (CNM) as

$$\left.\begin{aligned}
&h\gamma'_n + 2\gamma_{n+\frac{1}{2}} - 2\gamma_{n+1} = -\frac{1867}{40320}hf_n - \frac{54}{175}hf_{n+\frac{1}{6}} + \frac{26}{175}hf_{n+\frac{1}{4}} - \frac{567}{3200}hf_{n+\frac{1}{3}} - \frac{9}{35}hf_{n+\frac{1}{2}} + \frac{1647}{22400}hf_{n+\frac{2}{3}} - \frac{278}{1575}hf_{n+\frac{3}{4}} - \frac{143}{22400}hf_{n+1} \\
&h\gamma'_{n+\frac{1}{6}} + 2\gamma_{n+\frac{1}{2}} - 2\gamma_{n+1} = -\frac{13609}{4898880}hf_n + \frac{167}{4725}hf_{n+\frac{1}{6}} - \frac{114598}{382725}hf_{n+\frac{1}{4}} + \frac{5611}{50400}hf_{n+\frac{1}{3}} - \frac{8789}{25515}hf_{n+\frac{1}{2}} - \frac{35107}{302400}hf_{n+\frac{2}{3}} - \frac{74014}{382725}hf_{n+\frac{3}{4}} - \frac{18511}{3061800}hf_{n+1} \\
&h\gamma'_{n+\frac{1}{4}} + 2\gamma_{n+\frac{1}{2}} - 2\gamma_{n+1} = -\frac{1711}{573440}hf_n + \frac{24273}{358400}hf_{n+\frac{1}{6}} - \frac{23659}{100800}hf_{n+\frac{1}{4}} + \frac{273213}{2867200}hf_{n+\frac{1}{3}} - \frac{73343}{215040}hf_{n+\frac{1}{2}} + \frac{46953}{409600}hf_{n+\frac{2}{3}} - \frac{6479}{33600}hf_{n+\frac{3}{4}} - \frac{156281}{25804800}hf_{n+1} \\
&h\gamma'_{n+\frac{1}{3}} + 2\gamma_{n+\frac{1}{2}} - 2\gamma_{n+1} = -\frac{811}{279936}hf_n + \frac{302}{4725}hf_{n+\frac{1}{6}} - \frac{72418}{382725}hf_{n+\frac{1}{4}} + \frac{28039}{201600}hf_{n+\frac{1}{3}} - \frac{8777}{25515}hf_{n+\frac{1}{2}} + \frac{69989}{604800}hf_{n+\frac{2}{3}} - \frac{73954}{382725}hf_{n+\frac{3}{4}} - \frac{296341}{48988800}hf_{n+1} \\
&h\gamma'_{n+\frac{1}{2}} + 2\gamma_{n+\frac{1}{2}} - 2\gamma_{n+1} = -\frac{67}{20160}hf_n + \frac{27}{350}hf_{n+\frac{1}{6}} - \frac{386}{1575}hf_{n+\frac{1}{4}} + \frac{1539}{5600}hf_{n+\frac{1}{3}} - \frac{11}{42}hf_{n+\frac{1}{2}} + \frac{1161}{11200}hf_{n+\frac{2}{3}} - \frac{298}{1575}hf_{n+\frac{3}{4}} - \frac{11}{1800}hf_{n+1} \\
&h\gamma'_{n+\frac{2}{3}} + 2\gamma_{n+\frac{1}{2}} - 2\gamma_{n+1} = -\frac{27569}{9797760}hf_n + \frac{302}{4725}hf_{n+\frac{1}{6}} - \frac{10894}{54675}hf_{n+\frac{1}{4}} + \frac{436339}{201600}hf_{n+\frac{1}{3}} - \frac{3713}{25515}hf_{n+\frac{1}{2}} + \frac{116789}{604800}hf_{n+\frac{2}{3}} - \frac{77794}{382725}hf_{n+\frac{3}{4}} - \frac{292261}{48988800}hf_{n+1} \\
&h\gamma'_{n+\frac{3}{4}} + 2y_{n+\frac{1}{2}} - 2\gamma_{n+1} = -\frac{3067}{1032192}hf_n + \frac{24273}{358400}hf_{n+\frac{1}{6}} - \frac{2371}{11200}hf_{n+\frac{1}{4}} + \frac{662013}{2867200}hf_{n+\frac{1}{3}} - \frac{11349}{71680}hf_{n+\frac{1}{2}} + \frac{717471}{2867200}hf_{n+\frac{2}{3}} - \frac{17117}{100800}hf_{n+\frac{3}{4}} - \frac{17329}{2867200}hf_{n+1} \\
&h\gamma'_{n+1} + 2\gamma_{n+\frac{1}{2}} - 2\gamma_{n+1} = -\frac{61}{4480}hf_n - \frac{54}{175}hf_{n+\frac{1}{6}} + \frac{1514}{1575}hf_{n+\frac{1}{4}} - \frac{23409}{22400}hf_{n+\frac{1}{3}} + \frac{11}{15}hf_{n+\frac{1}{2}} - \frac{17793}{22400}hf_{n+\frac{2}{3}} + \frac{334}{525}hf_{n+\frac{3}{4}} + \frac{10793}{201600}hf_{n+1}
\end{aligned}\right\} \quad (2.7)$$

Now, combining equation (2.5) and (2.7) together to obtained the CNM, which can be written as

$$\left.\begin{aligned}
\gamma_{n+\frac{1}{6}} &= \gamma_n + \frac{1}{6}h\gamma'_n + \frac{1000061}{176359680}h^2 f_n + \frac{1247}{42525}h^2 f_{n+\frac{1}{6}} - \frac{150733}{3444525}h^2 f_{n+\frac{1}{4}} + \frac{104833}{3628800}h^2 f_{n+\frac{1}{3}} - \frac{409}{45927}h^2 f_{n+\frac{1}{2}} + \frac{47623}{10886400}h^2 f_{n+\frac{2}{3}} - \frac{5989}{3444525}h^2 f_{n+\frac{3}{4}} + \frac{30853}{881798400}h^2 f_{n+1} \\
\gamma_{n+\frac{1}{4}} &= \gamma_n + \frac{1}{4}h\gamma'_n + \frac{191741}{20643840}h^2 f_n + \frac{85887}{1433600}h^2 f_{n+\frac{1}{6}} - \frac{379}{4800}h^2 f_{n+\frac{1}{4}} + \frac{599157}{11468800}h^2 f_{n+\frac{1}{3}} - \frac{13789}{860160}h^2 f_{n+\frac{1}{2}} + \frac{90099}{11468800}h^2 f_{n+\frac{2}{3}} - \frac{629}{201600}h^2 f_{n+\frac{3}{4}} + \frac{719}{11468800}h^2 f_{n+1} \\
y_{n+\frac{1}{3}} &= \gamma_n + \frac{1}{3}h\gamma'_n + \frac{14221}{1102248}h^2 f_n + \frac{3874}{42525}h^2 f_{n+\frac{1}{6}} - \frac{373376}{3444525}h^2 f_{n+\frac{1}{4}} + \frac{617}{8100}h^2 f_{n+\frac{1}{3}} + \frac{5314}{229635}h^2 f_{n+\frac{1}{2}} + \frac{3853}{340200}h^2 f_{n+\frac{2}{3}} - \frac{15488}{3444525}h^2 f_{n+\frac{3}{4}} + \frac{311}{3444525}h^2 f_{n+1} \\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}h\gamma'_n + \frac{1621}{80640}h^2 f_n + \frac{27}{175}h^2 f_{n+\frac{1}{6}} - \frac{89}{525}h^2 f_{n+\frac{1}{4}} + \frac{6399}{44800}h^2 f_{n+\frac{1}{3}} - \frac{1}{30}h^2 f_{n+\frac{1}{2}} + \frac{783}{44800}h^2 f_{n+\frac{2}{3}} - \frac{11}{1575}h^2 f_{n+\frac{3}{4}} + \frac{19}{134400}h^2 f_{n+1} \\
\gamma_{n+\frac{2}{3}} &= \gamma_n + \frac{2}{3}h\gamma'_n + \frac{18812}{688905}h^2 f_n + \frac{9248}{42525}h^2 f_{n+\frac{1}{6}} - \frac{796672}{3444525}h^2 f_{n+\frac{1}{4}} + \frac{3022}{14175}h^2 f_{n+\frac{1}{3}} - \frac{5024}{229635}h^2 f_{n+\frac{1}{2}} + \frac{166}{6075}h^2 f_{n+\frac{2}{3}} - \frac{34816}{3444525}h^2 f_{n+\frac{3}{4}} + \frac{682}{3444525}h^2 f_{n+1} \\
\gamma_{n+\frac{3}{4}} &= \gamma_n + \frac{3}{4}h\gamma'_n + \frac{14187}{458752}h^2 f_n + \frac{356481}{1433600}h^2 f_{n+\frac{1}{6}} - \frac{5841}{22400}h^2 f_{n+\frac{1}{4}} + \frac{2827791}{11468800}h^2 f_{n+\frac{1}{3}} - \frac{3753}{286720}h^2 f_{n+\frac{1}{2}} + \frac{461457}{11468800}h^2 f_{n+\frac{2}{3}} - \frac{129}{11200}h^2 f_{n+\frac{3}{4}} + \frac{2637}{11468800}h^2 f_{n+1} \\
\gamma_{n+1} &= \gamma_n + h\gamma'_n + \frac{109}{2520}h^2 f_n + \frac{54}{175}h^2 f_{n+\frac{1}{6}} - \frac{128}{525}h^2 f_{n+\frac{1}{4}} + \frac{81}{350}h^2 f_{n+\frac{1}{3}} - \frac{2}{21}h^2 f_{n+\frac{1}{2}} + \frac{27}{1400}h^2 f_{n+\frac{2}{3}} - \frac{128}{1575}h^2 f_{n+\frac{3}{4}} + \frac{1}{300}h^2 f_{n+1} \\
\gamma'_{n+\frac{1}{6}} &= \gamma'_n + \frac{426463}{9797760}hf_n + \frac{65}{189}hf_{n+\frac{1}{6}} - \frac{34292}{76545}hf_{n+\frac{1}{4}} + \frac{11633}{40320}hf_{n+\frac{1}{3}} - \frac{2228}{25515}hf_{n+\frac{1}{2}} + \frac{5149}{120960}hf_{n+\frac{2}{3}} - \frac{1292}{76545}hf_{n+\frac{3}{4}} + \frac{3313}{9797760}hf_{n+1} \\
\gamma'_{n+\frac{1}{4}} &= \gamma'_n + \frac{223577}{5160960}hf_n + \frac{26973}{71680}hf_{n+\frac{1}{6}} - \frac{7727}{20160}hf_{n+\frac{1}{4}} + \frac{156249}{573440}hf_{n+\frac{1}{3}} - \frac{18047}{215040}hf_{n+\frac{1}{2}} + \frac{23571}{573440}hf_{n+\frac{2}{3}} - \frac{47}{2880}hf_{n+\frac{3}{4}} + \frac{1691}{5160960}hf_{n+1} \\
\gamma'_{n+\frac{1}{3}} &= \gamma'_n + \frac{26581}{612360}hf_n + \frac{352}{945}hf_{n+\frac{1}{6}} - \frac{25856}{76545}hf_{n+\frac{1}{4}} + \frac{797}{2520}hf_{n+\frac{1}{3}} - \frac{2216}{25515}hf_{n+\frac{1}{2}} + \frac{319}{7560}hf_{n+\frac{2}{3}} - \frac{256}{15309}hf_{n+\frac{3}{4}} + \frac{41}{122472}hf_{n+1} \\
\gamma'_{n+\frac{1}{2}} &= \gamma'_n + \frac{1733}{40320}hf_n + \frac{27}{70}hf_{n+\frac{1}{6}} - \frac{124}{315}hf_{n+\frac{1}{4}} + \frac{405}{896}hf_{n+\frac{1}{3}} - \frac{1}{210}hf_{n+\frac{1}{2}} + \frac{27}{896}hf_{n+\frac{2}{3}} - \frac{4}{315}hf_{n+\frac{3}{4}} + \frac{11}{40320}hf_{n+1} \\
\gamma'_{n+\frac{2}{3}} &= \gamma'_n + \frac{3329}{76545}hf_n + \frac{352}{945}hf_{n+\frac{1}{6}} - \frac{26624}{76545}hf_{n+\frac{1}{4}} + \frac{124}{315}hf_{n+\frac{1}{3}} - \frac{2848}{25515}hf_{n+\frac{1}{2}} + \frac{113}{945}hf_{n+\frac{2}{3}} - \frac{2048}{76545}hf_{n+\frac{3}{4}} + \frac{32}{76545}hf_{n+1} \\
\gamma'_{n+\frac{3}{4}} &= y'_n + \frac{24849}{573440}hf_n + \frac{26973}{71680}hf_{n+\frac{1}{6}} - \frac{807}{2240}hf_{n+\frac{1}{4}} + \frac{234009}{573440}hf_{n+\frac{1}{3}} - \frac{7083}{71680}hf_{n+\frac{1}{2}} + \frac{101331}{573440}hf_{n+\frac{2}{3}} - \frac{3}{448}hf_{n+\frac{3}{4}} + \frac{39}{114688}hf_{n+1} \\
\gamma'_{n+1} &= \gamma'_n + \frac{151}{2520}hf_n + \frac{256}{315}hf_{n+\frac{1}{4}} - \frac{243}{280}hf_{n+\frac{1}{3}} + \frac{104}{105}hf_{n+\frac{1}{2}} - \frac{243}{280}hf_{n+\frac{2}{3}} + \frac{256}{315}hf_{n+\frac{3}{4}} + \frac{151}{2520}hf_{n+1}
\end{aligned}\right\} \quad (2.8)$$

3. Analysis of Computational Numerical Method (CNM)

In this section, the analysis of the basic properties of the CNM are analyzed. These properties are order, error constant, consistency, zero-stability and region of absolute stability.

3.1 Order and Error Constant of CNM

Let the linear operator defined on the method be $\ell\left[\gamma(t);h\right]$, where,

$$\Lambda\{\gamma(t):h\} = \mathrm{A}^{(0)}\mathrm{Y}_m^{(i)} - \sum_{i=0}^{k}\frac{jh^{(i)}}{i!}\gamma_n^{(i)} - h^{(3-1)}\left[\delta_i f(\gamma_n) + b_i \mathrm{Z}(\mathrm{Y}_m)\right] \quad (3.1)$$

Expanding Y_m and $\mathrm{Z}(\mathrm{Y}_m)$ in Taylor series and comparing the coefficients of h gives

$$\Lambda\{\gamma(t):h\} = C_0\gamma(t) + C_1\gamma'(t) + \cdots + C_p h^p \gamma^p(t) + C_{p+1}h^{p+1}\gamma^{p+1}(t) + C_{p+2}h^{p+2}\gamma^{p+2}(t) + \cdots \quad (3.2)$$

Definition 3.1: The linear operator Λ and the CNM are said to be of order p if $C_0 = C_1 = \cdots = C_p = C_{p+1} = 0, \quad C_{p+2} \neq 0.$ C_{p+2} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2}h^{p+2}\gamma^{p+3}(t) + 0h^{p+3}$

$$\Lambda\{\gamma(t):h\} = C_0\gamma(t) + C_1\gamma'(t) + \cdots + C_p h^p \gamma^p(t) + C_{p+1}h^{p+1}\gamma^{p+1}(t) + C_{p+2}h^{p+2}\gamma^{p+2}(t) + \cdots \quad (3.2)$$

$$
\begin{bmatrix}
y\sum_{j=0}^{\infty}\frac{\left(\frac{1}{6}\right)^{j}}{j!}-\gamma_{n}-\frac{1}{6}h\gamma'_{n}-\frac{1000061}{176359680}h\gamma''_{n}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\gamma_{n}^{j+2}\left[\frac{1247}{42525}\left(\frac{1}{6}\right)-\frac{150733}{3444525}\left(\frac{1}{4}\right)+\frac{104833}{3628800}\left(\frac{1}{3}\right)-\frac{409}{45927}\left(\frac{1}{2}\right)+\frac{47623}{10886400}\left(\frac{2}{3}\right)-\frac{5989}{3444525}\left(\frac{3}{4}\right)+\frac{30853}{881798400}(1)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)^{j}}{j!}-\gamma_{n}-\frac{1}{4}h\gamma'_{n}-\frac{191741}{20643840}h\gamma''_{n}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\gamma_{n}^{j+2}\left[\frac{85887}{1433600}\left(\frac{1}{6}\right)-\frac{379}{4800}\left(\frac{1}{4}\right)+\frac{599157}{11468800}\left(\frac{1}{3}\right)-\frac{13789}{860160}\left(\frac{1}{2}\right)+\frac{90099}{11468800}\left(\frac{2}{3}\right)-\frac{629}{201600}\left(\frac{3}{4}\right)+\frac{719}{11468800}(1)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^{j}}{j!}-\gamma_{n}-\frac{1}{3}h\gamma'_{n}-\frac{14221}{1102248}h\gamma''_{n}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\gamma_{n}^{j+2}\left[\frac{3874}{42525}\left(\frac{1}{6}\right)-\frac{373376}{3444525}\left(\frac{1}{4}\right)+\frac{617}{8100}\left(\frac{1}{3}\right)=\frac{5314}{229635}\left(\frac{1}{2}\right)+\frac{3853}{340200}\left(\frac{2}{3}\right)-\frac{15488}{3444525}\left(\frac{3}{4}\right)+\frac{311}{3444525}(1)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{j}}{j!}-\gamma_{n}-\frac{1}{2}h\gamma'_{n}-\frac{1621}{80640}h\gamma''_{n}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\gamma_{n}^{j+2}\left[\frac{27}{175}\left(\frac{1}{6}\right)-\frac{89}{525}\left(\frac{1}{4}\right)+\frac{6399}{44800}\left(\frac{1}{3}\right)-\frac{1}{30}\left(\frac{1}{2}\right)+\frac{783}{44800}\left(\frac{2}{3}\right)-\frac{11}{1575}\left(\frac{3}{4}\right)+\frac{19}{134400}h^{2}f_{n+1}\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^{j}}{j!}-\gamma_{n}-\frac{2}{3}h\gamma'_{n}-\frac{18812}{688905}h\gamma''_{n}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\gamma_{n}^{j+2}\left[\frac{9248}{42525}\left(\frac{1}{6}\right)-\frac{796672}{3444525}\left(\frac{1}{4}\right)+\frac{3022}{14175}\left(\frac{1}{3}\right)-\frac{5024}{229635}\left(\frac{1}{2}\right)+\frac{166}{6075}\left(\frac{2}{3}\right)-\frac{34816}{3444525}\left(\frac{3}{4}\right)+\frac{682}{3444525}(1)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)^{j}}{j!}-\gamma_{n}-\frac{3}{4}h\gamma'_{n}-\frac{14187}{458752}h\gamma''_{n}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\gamma_{n}^{j+2}\left[\frac{356481}{1433600}\left(\frac{1}{6}\right)-\frac{5841}{22400}\left(\frac{1}{4}\right)+\frac{2827791}{11468800}\left(\frac{1}{3}\right)-\frac{3753}{286720}\left(\frac{1}{2}\right)+\frac{461457}{11468800}\left(\frac{2}{3}\right)-\frac{129}{11200}\left(\frac{3}{4}\right)+\frac{2637}{11468800}(1)\right] \\
\sum_{j=0}^{\infty}\frac{(1)^{j}}{j!}-\gamma_{n}-h\gamma'_{n}-\frac{109}{2520}h\gamma''_{n}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\gamma_{n}^{j+2}\left[\frac{54}{175}\left(\frac{1}{6}\right)-\frac{128}{525}\left(\frac{1}{4}\right)+\frac{81}{350}\left(\frac{1}{3}\right)-\frac{2}{21}\left(\frac{1}{2}\right)+\frac{27}{1400}\left(\frac{2}{3}\right)-\frac{128}{1575}\left(\frac{3}{4}\right)+\frac{1}{300}(1)\right]
\end{bmatrix}
=
\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}
$$

Comparing the coefficient of h, according to Skwame et al. (2019), the CNM is of uniform order $p = \begin{bmatrix} 7 & 7 & 7 & 7 & 7 & 7 & 7 \end{bmatrix}^{T}$ with its error constant are given respectively by $C_{p+2} = \begin{bmatrix} -1.3640\times10^{-10} & -1.3281\times10^{-10} & -1.3497\times10^{-10} & -1.1961\times10^{-10} & -1.4526\times10^{-10} & -1.3561\times10^{-10} & -1.4353\times10^{-09} \end{bmatrix}$

3.2 Consistency of the CNM

A CNM is said to be consistent if the following conditions are satisfied.

i. The order of the method must be greater than or equal to zero to one i.e. $p \geq 1$.

ii. $\sum_{j=0}^{k} \alpha_j = 0$

iii. $\rho(r) = \rho'(r) = 0$

iv. $\rho'''(r) = 3!\sigma(r)$

Where $\rho(r)$ and $\sigma(r)$ are first and second characteristics polynomials of the CNM. According to Skwame et al. (2019), the first condition is a sufficient condition for the associated CNM to be consistent. Hence the CNM is consistent.

3.3 Zero Stability of the CNM

Definition 3.2: the CNM is said to be zero-stable, if the roots $z_s,\ s = 1, 2, \cdots, k$ of the first characteristics polynomial $\rho(z)$ defined by $\rho(z) = \det\left(zA^{(0)} - E\right)$ satisfies $|z_s| \leq 1$ and every root satisfies $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation, Sunday (2018). The first characteristic polynomial is given by,

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{bmatrix}\right|=\left|\begin{bmatrix}z&0&0&0&0&0&-1\\0&z&0&0&0&0&-1\\0&0&z&0&0&0&-1\\0&0&0&z&0&0&-1\\0&0&0&0&z&0&-1\\0&0&0&0&0&z&-1\\0&0&0&0&0&0&z-1\end{bmatrix}\right|=z^6(z-1)$$

Thus, solving for Z in

$$z^6(z-1) \tag{3.3}$$

gives $z = 0, 0, 0, 0, 0, 0, 1$. Hence the CNM is said to be zero-stable.

3.4 Convergence of the CNM

Theorem 3.1: the necessary and sufficient conditions for linear multistep method to be convergent are that it must be consistent and zero-stable. Hence the CNM formulated is consistent Skwame et al. (2019).

3.5 Region of Absolute Stability of CNM

Definition 3.3: the region of absolute stability is the region of the complex z plane, where $z = \lambda h$ for which the CNM is absolute stable. To determine the region of absolute stability of the CNM, the method that compare neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. Thus, the method according to Sunday, (2018) is called the boundary locus method. Applying the boundary locus method on the CNM, we obtain the stability polynomial of the CNM as

$$\left.\begin{aligned}\bar{h}(w)=&\left(-\frac{1}{11612160}w^6-\frac{163}{146313216000}w^7\right)h^{14}+\left(-\frac{810757}{87787929600}w^6+\frac{113}{34836480}w^7\right)h^{12}+\left(-\frac{4473503}{376233984000}w^6-\frac{61}{829440}w^7\right)h^{10}\\&+\left(-\frac{1235057}{522547200}w^6+\frac{481}{414720}w^7\right)h^8+\left(-\frac{128809}{6967296}w^6-\frac{67}{5184}w^7\right)h^6+\left(-\frac{817}{5040}w^6+\frac{113}{1152}w^7\right)h^4+\left(-\frac{17}{30}w^6-\frac{11}{24}w^7\right)h^2-2w^6+w^7\end{aligned}\right\} \tag{3.4}$$

Using the stability polynomial in equation (3.4), we obtain the region of absolute stability in figure below as

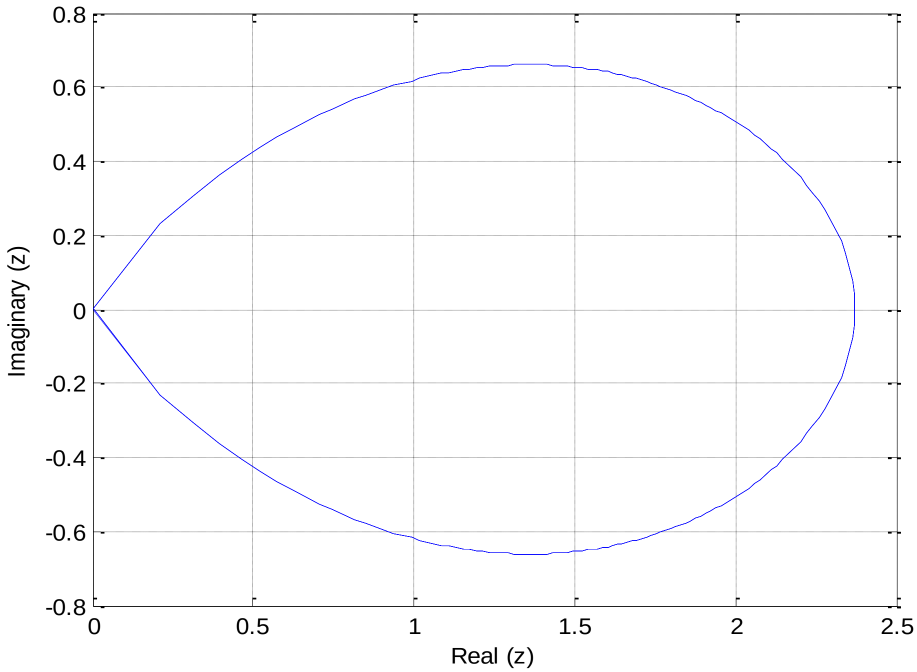

Figure 3.1: Stability region of the CNM.

The stability region of the CNM obtained in Figure 3.1 is A-stable. The stability region is the exterior of the blue contour.

4. Numerical Simulation of CNM

The computational numerical method were simulated on some physical problems from sciences and engineering problems to shows the effectiveness of the CNM. The CNM were directly applied without reducing equation (1.1) to an equivalent system of first order initial value problem. The approximate solutions are computed and compared with results from existing methods in literature.

Problem 4.1:

The Second order oscillatory cooling of a body is modeled as the temperature t degrees of a body, x minutes after being placed in a certain room, satisfies the differential equation

$$3\gamma''(t)+\gamma'(t)=0 \tag{4.1}$$

By using the substitution $z=\gamma'(t)$ or otherwise, find γ in terms of x given that $\gamma=60$ when $t=0$ and $\gamma=35$ when $t=6\ln 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

The problem is modeled as

$$\gamma''(t)=\frac{-\gamma'}{3},\ \gamma(0)=60,\ \gamma'(0)=-\frac{80}{9},\ h=0.1 \tag{4.2}$$

With analytic solution

$$\gamma(t)=26.667\exp(-0.33333t)+33.333 \tag{4.3}$$

Source [Omole & Ogunware (2018), Olanegan, *et al.* (2018)].

Problem 4.2:
The Simple Harmonic Oscillatory differential equation were modeled as Simple Harmonic Motion in which an object stretches a spring of 6 inches in equilibrium,

i. Set up the equation of motion and find its general solution.
ii. Find the displacement of the object for $t > 0$, if it's initially displaced 18 inches above equilibrium and given a downward velocity of $3\frac{ft}{s}$.

From Newton's second law of motion, we have

$$m\gamma''(t) + c\gamma'(t) + \kappa\gamma(t) = \mathrm{M} \tag{4.4}$$

By setting $c = 0$ and $\mathrm{M} = 0$, we get

$$m\gamma''(t) + \kappa\gamma = 0 \Rightarrow y''(t) + \frac{\kappa}{m}\gamma(t) = 0 \tag{4.5}$$

The equation of the weight of the object is given as follow:

$$mg = \kappa\Delta l \Rightarrow \frac{\kappa}{m} = \frac{t}{\Delta l} \tag{4.6}$$

Substituting $t = 32\frac{ft}{s^2}, \Delta l = 0.5ft$ into (4.6) we obtain

$$\frac{k}{m} = 64 \tag{4.7}$$

Substituting equation (4.7) into the equation (4.5) we get

$$\gamma''(t) + 64\gamma = 0 \tag{4.8}$$

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus, $\gamma(0) = 1.5, \gamma'(0) = -3$. We make use of (4.8) as

$$\gamma''(t) + 64\gamma(t) = 0, \gamma(0) = 1.5, \gamma'(0) = -3 \tag{4.9}$$

We obtain the analytical solution (4.9) as

$$y(t) = -0.375\sin(8t) + 1.5\cos(8t) \tag{4.10}$$

Source [Areo & Rufai (2016)].

Problem 4.3:
The second order Stiefel linear oscillatory differential equation which is modeled as

$$\gamma''(t) + \gamma'(t) = 0.001\sin(t), \gamma(0) = 0, \gamma'(t) = 0.9995 \tag{4.11}$$

is considered whose analytical solution of (4.11) is given as

$$\gamma(t) = \sin(t) - 0.0005t\cos(t) \tag{4.12}$$

Source [Olabode & Momoh, (2016) and Lydia, et al. (2021)].

The following notations were used in subsequent tables and figures.

Notations Meaning

AS	Analytical Solution
NS	Numerical Solution
ECNM	Error in Computational Numerical Method
EOO18	Error in Omole and Ogunware (2018)
EOe18	Error in Olanegan, *et al.* (2018)
EAR16	Error in Areo and Rufai (2016)
ELe21	Error in Lydia, et al., (2021)
EOM16	Error in Olabode and Momoh, (2016)

Table 4.1: Showing the results for problem 4.1

t	**AS**	**NS**	**ECNM**	**EOO18**	**EOe18**
0.1	59.12576267952015738700	59.12576267952015738700	0.0000E00	3.5500E-11	7.4764E-06
0.2	58.28018626750980633900	58.28018626750980633900	0.0000E00	4.5800E-11	2.9394E-05
0.3	57.46233114762558861700	57.46233114762558861700	0.0000E00	7.0000E-11	6.4802E-05
0.4	56.67128850781193210600	56.67128850781193210600	0.0000E00	6.5000E-12	1.1279E-05
0.5	55.90617933041637530700	55.90617933041637530700	0.0000E00	3.3300E-11	1.7250E-04
0.6	55.16615341541284956400	55.16615341541284956400	0.0000E00	4.2000E-11	2.4310E-04
0.7	54.45038843564751105000	54.45038843564751105000	0.0000E00	4.3800E-11	3.2383E-04
0.8	53.75808902305729847200	53.75808902305729847200	0.0000E00	1.0700E-10	4.1393E-04
0.9	53.08848588484580976200	53.08848588484580976200	0.0000E00	6.5800E-11	5.1271E-04
1.0	52.44083494863438001100	52.44083494863438001100	0.0000E00	1.6900E-10	6.1951E-04

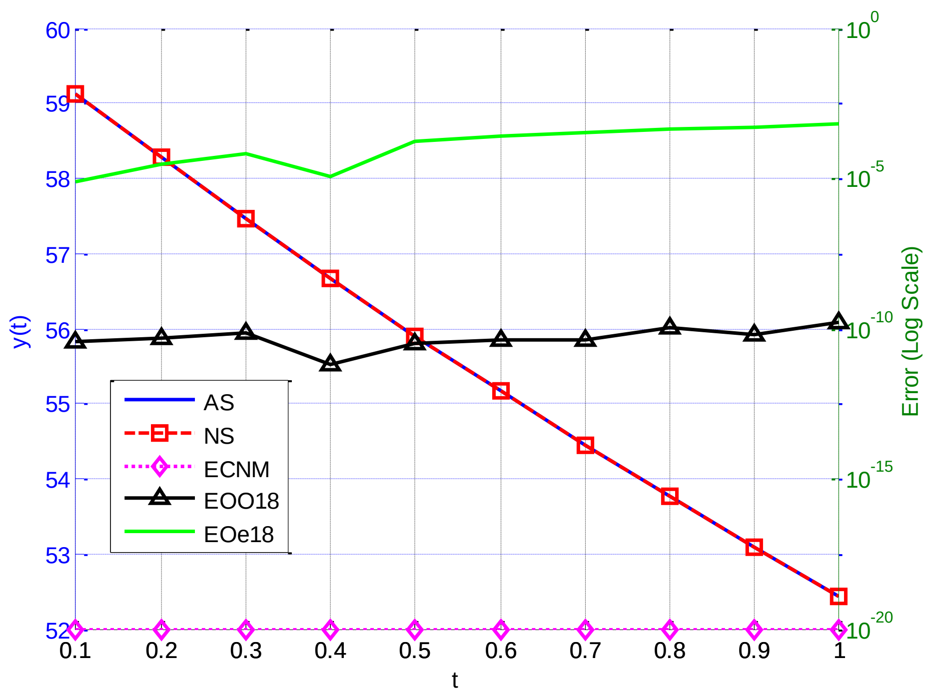

Figure 4.1: Graphical Curve of table 4.1

Table 4.2: Showing the results for problem 4.2

t	**AS**	**NS**	**ECNM**	**EAR16**
0.1	0.776051529933427 09579	0.77605152993274408426	6.8301E-13	3.3496E-07
0.2	- 0.41863938459249752594	-0.41863938459387367324	1.3762E-12	1.6371E-06
0.3	- 1.3593892660185498469	-1.35938926601955541960	1.0056E-12	3.2716E-06
0.4	- 1.4755518599067871611	-1.47555185990606872960	1.0056E-13	3.5979E-06
0.5	- 0.69666449555494477770	-0.69666449555213113975	2.8136E-12	1.3589E-06
0.6	0.50481020347261010590	0.50481020347619324768	3.5831E-12	2.9143E-06
0.7	1.4000738069674951883	1.40007380696939826270	1.9031E-12	6.7226E-06
0.8	1.4460714263183540043	1.44607142631665691830	1.6971E-12	7.0589E-06
0.9	0.61490152285494961183	0.61490152284989092499	5.0587E-12	2.6543E-06
1.0	- 0.58925939319668845548	-0.58925939320237650700	5.6881E-12	4.6056E-06

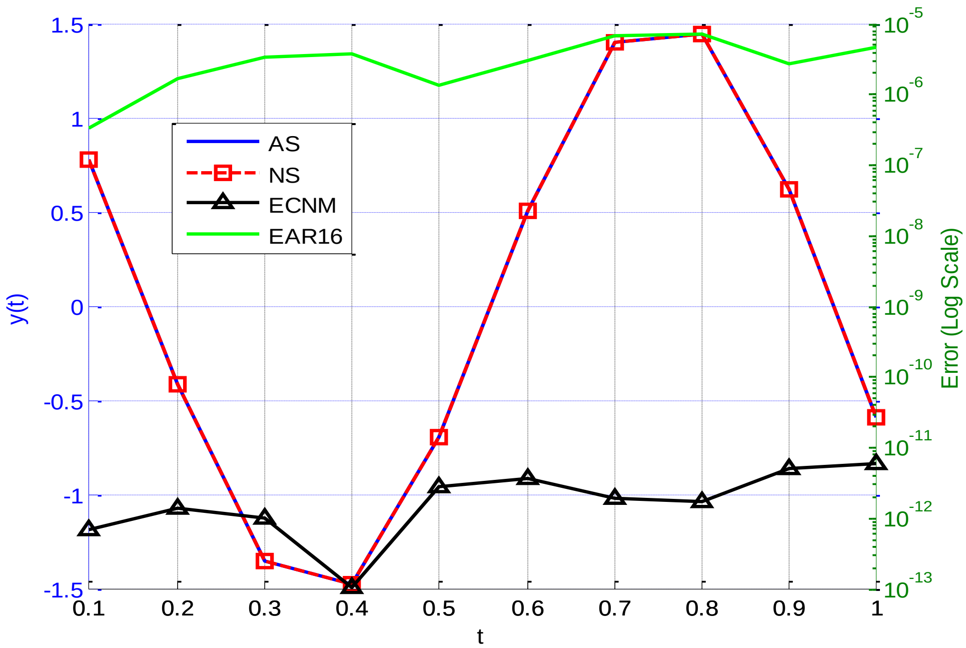

Figure 4.2: Graphical Curve of table 4.2

Table 4.3: Showing the results for problem 4.3

t	**AS**	**NS**	**ECNM**	**ELe21**	**EOM16**
0.1	0.99500915694885810751	0.99500915694885810750	1.0000E-20	2.8269E-12	1.0169E-11
0.2	0.98008644477432113724	0.98008644477432113723	1.0000E-20	5.8994E-12	2.0390E-11
0.3	0.95538081715660522058	0.95538081715660522057	1.0000E-20	6.8309E-12	1.5451E-13
0.4	0.92113887767134681290	0.92113887767134681288	2.0000E-20	1.4991E-12	8.1063E-11
0.5	0.87770241827502376687	0.87770241827502376685	2.0000E-20	1.8395E-12	2.5377E-10
0.6	0.82550500765169680785	0.82550500765169680783	2.0000E-20	1.6559E-11	5.4848E-10
0.7	0.76506766347502161813	0.76506766347502161811	2.0000E-20	1.2970E-11	9.9571E-10
0.8	0.69699365178352523002	0.69699365178352523001	1.0000E-20	8.4312E-11	1.6260E-10

0.9	0.62196246537999682400	0.62196246537999682400	0.0000E00	5.3240E-11	2.4697E-10
1.0	0.54072304136054366565	0.54072304136054366565	0.0000E00	3.2126E-11	3.5575E-10

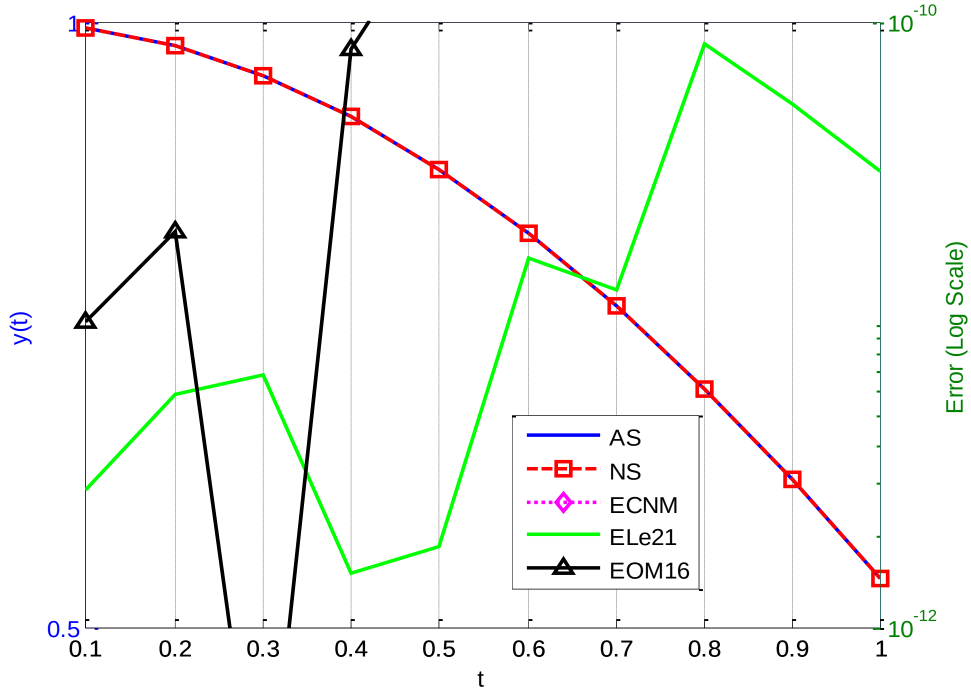

Figure 4.3: Graphical Curve of table 4.3

5. Discussion of Results

The derivation of the Computational Numerical Method (CNM) is based on constructing an approximate solution in the form of a power series polynomial, which serves as a basis for the direct solutions of second-order oscillatory differential equations. By differentiating the polynomial twice and applying interpolation at selected points along with collocation at others, a system of linear equations is obtained and solved using Gaussian elimination. The resulting coefficients are substituted back to form a continuous hybrid linear multistep method. This method is then evaluated at non-interpolating points to produce a continuous form, which, when differentiated and evaluated at all points, gives the discrete CNM. Finally, combining the continuous and discrete forms results in a fully formulated CNM capable of approximating solutions to second-order differential equations without first reducing them to first-order systems.

The analysis of the CNM examines its essential numerical properties to ensure reliability and accuracy. The method's order, error constant, consistency, and zero-stability are evaluated using Taylor series expansion and characteristic polynomials. The CNM is found to be of uniform order seven with a well-defined truncation error and it satisfies the conditions for consistency. Zero-stability is confirmed by examining the roots of the first characteristic polynomial, ensuring no root exceeds the allowed multiplicity. Convergence follows from the combination of consistency and zero-stability which is the necessary and sufficient conditions for linear multistep method to be convergent. Finally, the region of absolute stability is determined using the boundary locus method and the resulting stability region is illustrated in Figure 3.1, confirming that the CNM is stable over the specified domain and suitable for practical numerical computations in engineering and scientific applications.

The results obtained for Problem 4.1, which models the second-order oscillatory cooling of a body, show a very strong agreement between the analytical solution (AS) and the numerical solution (NS) across all selected time values. From Table 4.1, the values of AS and NS are identical to the displayed significant figures, indicating that the Computational Numerical Method (CNM) reproduces the exact behaviour of the analytical model. The graphical curve in Figure 4.1 further confirms this observation, as the numerical curve coincides perfectly with the analytical curve. This demonstrates that the CNM accurately captures the exponential decay behaviour of the cooling process and preserves the physical characteristics of the model without requiring reduction to a system of first-order equations.

For Problem 4.2, which represents a simple harmonic motion system, Table 4.2 shows that the numerical results closely match the analytical solution at all grid points. The displacement values oscillate between positive and negative magnitudes, reflecting the periodic motion of the spring-mass system. The CNM effectively captures both amplitude and phase of the oscillation. Figure 4.2 illustrates this periodic behaviour clearly, as the numerical and analytical curves overlap throughout the interval considered. This confirms that the method is stable and reliable for oscillatory systems and accurately models the dynamic response of the physical system.

In Problem 4.3, involving the second-order Stiefel linear oscillatory differential equation, Table 4.3 again demonstrates near-perfect agreement between AS and NS. The computed solutions decrease smoothly over the interval, reflecting the expected behaviour of the model. The graphical representation in Figure 4.3 shows that the numerical curve follows the analytical curve precisely, indicating that the CNM maintains high consistency even for more structured linear oscillatory equations. The absence of visible deviation in the figure suggests strong convergence properties of the method.

6. Conclusion

The study developed and analyzed a Computational Numerical Method (CNM) for the direct solution of second-order oscillatory initial value problems. The method was derived using a power series polynomial as a basis function, differentiated, and collocated to form a continuous hybrid linear multistep approach. Key properties of the CNM, including order, error constant, consistency, zero-stability, convergence, and absolute stability, were evaluated, demonstrating the method's reliability and accuracy. Numerical simulations on problems such as body cooling, simple harmonic motion, and Stiefel linear oscillatory equations showed that the CNM closely reproduces analytical solutions with negligible computational error, outperforming existing methods in the literature. The graphical analyses further confirmed that the method effectively captures oscillatory behaviors and dynamic responses in physical and engineering systems.

In conclusion, the CNM provides a robust, efficient, and accurate framework for solving second-order oscillatory differential equations without the need to reduce them to first-order systems. Its strong stability properties, high precision, and ease of implementation make it suitable for a wide range of applications in science and engineering. The results validate the CNM as a powerful computational tool for modeling dynamic processes, and its application can be extended to more complex oscillatory systems, higher-order differential equations, and real-world physical problems where direct and precise numerical solutions are essential. Further research could extend the study by developing a numerical methods for solving Volterra integro-differential equation of second kind and applying CNM on boundary value problems.

REFERENCES

Adeniran, O., & Ogundare, B. (2015). An efficient hybrid numerical scheme for solving general second-order initial value problems (IVPs). *International Journal of Applied Mathematical Research, 4*, 411–419.

Adewale, A. J., & Sabo, J. (2024). Simulating the dynamics of oscillating differential equations of mass in motion. *International Journal of Development Mathematics, 1*(1), 54–69. https://doi.org/10.62054/ijdm/0101.07

Adeyeye, O., & Omar, Z. (2017). A 4-step 5-point hybrid block method for the direct solution of second-order initial value problems. *Journal of Mathematics and Computer Science, 17*, 527–534.

Aloko, M. D., Ayinde, M. A., Usman, A. D., & Sabo, J. (2024). An efficient scheme for direct simulation of third and fourth oscillatory differential equations. *International Journal of Development Mathematics, 1*(1), 72–89. https://doi.org/10.62054/ijdm/0101.06

Areo, E. A., & Rufai, M. A. (2016). An efficient one-eighth step hybrid block method for solving second-order initial value problems of ODEs. *International Journal of Differential Equations and Applications, 15*(2), 117–139.

Ayinde, A. M., Ibrahim, S., Sabo, J., & Silas, D. (2023). The physical application of motion using single-step block method. *Journal of Material Science Research and Review, 6*(4), 708–719.

Donald, J. Z., Skwame, Y., Sabo, J., & Ayinde, A. M. (2021). The use of linear multistep method on implicit one-step second derivative block method for direct solution of higher-order initial value problems. *Abacus (Mathematics Science Series), 48*(2), 224–237.

Elazzouzi, A., Lamrani, A. A., Tilioua, M., & Torres, D. F. M. (2019). Analysis of a SIRI epidemic model with distributed delay and relapse. *Statistics, Optimization & Information Computing, 7*, 545–557.

Ibrahim, I. H. (2017). Hybrid multistep method for solving second-order initial value problems. *Journal of the Egyptian Mathematical Society, 25*, 355–362.

Ishaq, A. A., Ayinde, M. M., Sabo, J., Ibrahim, S., & Jaiyeola, B. S. (2024). Three-step collocation method for solution of third-derivative initial value problem. *International Journal of Development Mathematics, 1*(1), 25–36. https://doi.org/10.62054/ijdm/0101.03

Kamoh, N. M., Abada, A. A., & Soomiyol, M. C. (2018). A block procedure with continuous coefficients for direct solution of second-order initial value problems of ODEs using shifted Legendre polynomial as basic function. *International Journal of Multidisciplinary Research and Development, 5*(4), 236–241.

Kuboye, J. O., & Omar, Z. (2015a). Derivation of a six-step block method for direct solution of second-order ordinary differential equations. *Mathematics and Computers in Applications, 20*, 151–159.

Kuboye, J. O., & Omar, Z. (2015b). Multistep collocation block method for direct solution of second-order ordinary differential equations. *American Journal of Applied Sciences, 12*(9), 663–668.

Kwanamu, J. A., Skwame, Y., & Sabo, J. (2021). Block hybrid method for solving higher-order ordinary differential equations using power series on implicit one-step second derivative. *FUW Trends in Science and Technology Journal, 6*(2), 576–582.

Kwari, L. J., Sunday, J., Ndam, J. N., Shokri, A., & Wang, Y. (2023). On the simulations of second-order oscillatory problems with applications to physical systems. *Axioms, 12*(3), 282. https://doi.org/10.3390/axioms12030282

Kwari, L. J., Sunday, J., Ndam, J. N. & James, A. A. (2021). On the numerical approximations and simulations of damped and undamped Duffing oscillators. *Science Forum (Journal of Pure and Applied Science), 21*, 503–515.

Olabode, B. T., & Momoh, A. L. (2016). Continuous hybrid multistep methods with Legendre basic function for treatment of second-order stiff ODEs. *American Journal of Computational and Applied Mathematics, 6*(2), 38–49.

Olanegan, O. O., Ogunware, B. G., & Alakofa, C. O. (2018). Implicit hybrid points approach for solving general second-order ordinary differential equations with initial values. *Journal of Advances in Mathematics and Computer Science, 27*(3), 1–14.

Omole, E. A., & Ogunware, B. G. (2018). 3-point single hybrid block method (3PSHBM) for direct solution of general second-order initial value problems of ordinary differential equations. *Journal of Scientific Research & Reports, 20*(3), 1–11.

Raymond, D., Skwame, Y., & Sunday, J. (2018). An implicit two-step one off-grid point third-derivative hybrid block method for the direct solution of second-order ordinary differential equations. *Academic Journal of Applied Mathematical Sciences, 4*(2), 8–14.

Sabo, J., Ayinde, A. M., Oyedepo, T., & Ishaq, A. A. (2024). Simulation of two-step block approach for solving oscillatory differential equations. *Cumhuriyet Science Journal, 45*(2), 366–378. https://doi.org/10.17776/csj.1345303

Sabo, J., Kyagya, T. Y., & Vashawa, W. J. (2021). Numerical simulation of one-step block method for treatment of second-order forced motions in mass-spring systems. *Asian Journal of Research and Reviews in Physics, 5*(2), 1–11.

Skwame, Y., Sabo, J., & Mathew, M. (2019). The treatment of second-order ordinary differential equations using equidistant one-step block hybrid method. *Asian Journal of Probability and Statistics, 5*(3), 1–9.

Sunday, J. (2018). On the oscillation criteria and computation of third-order oscillatory differential equations. *Communications in Mathematics and Applications, 6*(4), 615–625.