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Combinatorial Properties of Order Decreasing and Order Reversing (IDR_n) Partial One– to–One Transformation Semigroup

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ABSTRACT

We study the semigroup of partial one-to-one transformations on a finite chain that are simultaneously order-decreasing and order-reversing. This class arises naturally as the intersection of two well-studied transformation semigroups. For a finite chain $X_n = \{1,2,3, \dots, n\}$,, we give a detailed combinatorial analysis with respect to fixed points, height, and image changes. Explicit counting formulas, generating polynomials, and recurrence relations are obtained and justified combinatorially. Our results unify and extend earlier enumerations for order-decreasing and order-reversing partial one-to-one transformation semigroups and provide refined distributions via univariate and bivariate generating functions.

1. Introduction

Transformation semigroups defined by order-theoretic constraints possess rich algebraic and combinatorial structures and have been studied extensively. Prominent examples include semigroups of order-preserving, order-reversing, order-decreasing, and order-increasing (partial) transformations on a finite chain, Fernandes et al (2004), Umar (1992), Cornelius et al (2023). In this paper, we focus on transformations that are *simultaneously* order-decreasing and order-reversing.

The semigroup of order-decreasing transformations was initiated by Umar (1992b), while order-reversing partial one-to-one transformations have been studied more recently, Cornelius et al (2023). Despite these developments, a systematic combinatorial treatment of the *intersection* of these classes, especially with respect to fixed points, height, and image changes has received limited attention. The aim of this paper is to provide such a treatment, including explicit enumeration formulas, generating functions, and refinements that extend known results.

2. Preliminaries and Definitions

Let $X_n = \{1,2,3, \dots, n\}$ be a finite chain with the natural order. A partial one-to-one transformation α on X_n is an injective partial map $\alpha: Dom(\alpha) \longrightarrow X_n$, Umar (2010).

We say that α is:

i) order-decreasing if $x\alpha \leq x$ for all$x \in Dom(\alpha)$, Umar (2010)

ii) order-reversing if for all $x, y \in Dom(\alpha), x \leq y$ implies $x\alpha \geq y\alpha$, Cornelius et al (2023)

Denote by IDR_n the set of all order-decreasing and order-reversing partial one-to-one transformations on X_n under composition, $\alpha \in IDR_n$ forms a semigroup and the following parameters ware used:

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iii) fixed points: $\{x \in Dom(\alpha): x\alpha = x\}$
iv) height: $|Im(\alpha)|$
v) image changes: the number of indices i such that successive values in the ordered domain satisfy $x_i(\alpha) \neq x_{i+1}(\alpha)$.

3. Combinatorial Results on Order-Decreasing (ID_n)

We briefly recall known results for the semigroup (ID_n) of order-decreasing partial one-to-one transformations on X_n, following Umar (1992) and subsequent refinements.

Proposition 3.1. Let $n, k \geq 0$, The number of elements of (ID_n) with exactly k fixed points is given by $\binom{n}{k}_{a_{n-k}}$. where a_n denotes the number of nilpotent elements in the corresponding symmetric inverse semigroup on n points.
Proof. Let $(\alpha \in ID_n)$with fixed-point set F. Since α is injective and order-decreasing, the restriction of α to $X_n \backslash F$ is nilpotent after relabeling. The claim follows by counting choices of F and nilpotent restrictions.
Proposition 3.2 let $I_n = ID_n$, then,

$$F(n; m) = \binom{n}{m} B_{n-m}, where\ B_n\ is\ the\ n^{th}\ Bell'number.$$

Proof. Let $\alpha \in ID_n$and let $x_1, x_2, x_3, \ldots, x_n$ be fixed points of α. Since α is one to one and order-decreasing, if follows that for $x \in (X_n \backslash \{x_1, x_2, \ldots, x_m\}) \cap Dom\alpha$, then we have $x\alpha \in (X_n \backslash \{x_1, x_2, \ldots, x_m\})$ and $\alpha x < x$. Therefore the restriction of α to $(X_n \backslash \{x_1, x_2, \ldots, x_m\})$ is well defined and is a nilpotent element of $(X_n \backslash \{x_1, x_2, \ldots, x_m\})$. Hence the number of nilpotent that can be formed by these $n - m$ elements (after relabeling) is B_{n-m}, Umar (1993).

The following corollaries ware deduced: Corollary 3.3 Let $I_n = ID_n$. Then,

$i.$ $F(n; k) = \binom{n}{k} B_k$.

$ii.$ Borwein et al (1989), $F(n; p) = S(n; n - p)$

$iii.$ Borwein et al (1989), $|ID_n| = B_{n+1}$.

$iv.$ $|E(ID_n)| 2^n\ and\ |N(ID_n)| B_n. where; S(n, r)$ the Stirling number of the second kind $S(n, r) = S(n - 1, r - 1) + rS(n - 1, r), S(n, 1) = 1 = S(n, n)$

$$v.\ B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

$vi.$ Cornelius (2024), $f(n, i) = \begin{cases} \frac{(n-1)(n-2)(n^2+9n+12)}{24}, n \geq 3 \\ \sum_{i=0}^{n} \frac{\binom{n-1}{i}}{((i+1)\binom{n+i+1}{n-i})}, \ n \geq 1 \end{cases}$

$vii.$ Cornelius (2024),$f(n; j) = \{a_n = n + (n - 1)(2^n - 2),\ \ n \geq 0$

$viii$. Cornelius (2024) $f(n;l) = \begin{cases} a_n = \frac{(2^n+1)(2^n+2)}{6}, n \geq 0 \\ a_n = \frac{(5(4^n)+(-2)^n)}{6}, n \geq 0 \end{cases}$

4 Combinatorial results on order-reversing (IOR_n)

First, note that $k = w^+(\alpha)$ is underfined when $p = 0$. Due to the presence of the empty map, it seems reasonale to define $k = 0$ if $p = 0\ 0r\ r = 0; and\ F(n;r) = F(n;p) = F(n;k) = F(n;r,k) = F(n;r,p) = F(n;p,k) = 1, if\ any\ of\ r.p\ or\ k\ is\ 0.$

This and other observations were record in the following lemma, which will be used implicitly whenever needed, Umar 2010), Ganyushkin and Manzorchuk (2001), Ganyushkin and Kormysheva (1993), and Limscomp (1986)

Lemma 4.1 let $X_n = \{1,2,3, \ldots, n\}$ and $P = \{r, q, p, m, k\}$. Where for a given $\alpha \in P_n$ we set $r = b(\alpha), q = c(\alpha), p = h(\alpha), m = f(\alpha) and\ k = w^+(\alpha)$. We also define $F(n;r) = F(n;p) = F(n;k) = F(n;r,k) = F(n;r,p) = F(n;p,k) = 1$ if any $r.p\ or\ k\ is\ o.$ Then we have the following:

1. $n \geq r \geq p \geq m \geq 0$
2. $n \geq k \geq p \geq m \geq 0$
3. $n \geq r \geq q \geq r - p \geq 0$
4. $r = 1 \Rightarrow p = 1$
5. $k = 1 \Rightarrow p = 1 \Rightarrow m \leq 1$
6. $r = 0 \Leftrightarrow p = 0 \Leftrightarrow k = 0.$

We now recall some known key enumeration results recorded including a characterization by image set and corresponding binomial enumerations Michael et al (2023).

Theorem 4.2 Let $I_n = IOR_n$, then $|IOR_n| = \binom{2n}{n}, \ n \geq 0.$

Proof,

For all the elements $\alpha = \begin{pmatrix} 1 & 2 & \ldots n \\ x_1 & x_2 & \ldots n \end{pmatrix} \in IOR_n$ can be uniquely determined by $x_1, x_2, \ldots, x_n$ for $i = 1,2,3, \ldots, n$. Set $y_i = x_i + i$, then the mapping between $(x_1, x_2, \ldots, x_n) \rightarrow (y_1, y_2, \ldots, y_n)$ is a bijection between the set of all $(x_1, x_2, \ldots, x_n)$ such that $1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq n$ and the set of all $(y_1, y_2, \ldots, y_n)$ such that $1 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq n + n$. If follows that $(y_1, y_2, \ldots, y_n)$ is uniquely determined by the $n - elements$ subset $(y_1, y_2, \ldots, y_n)$ of $\{1,2,3, \ldots, 2n\}$. Hence $|IOR_n| = \binom{2n}{n}, \ n \geq 0.$

The following corollaries ware deduced:

Corollary 4.3 let $I_n = IOR_n$ then,

$$f(n;p)=\begin{cases}\binom{n+p}{n}, & n\geq 0, \quad p=0\\ \binom{n}{p}, & n\geq 0, \quad p\geq 0\\ \binom{n^2}{p}=n^2p, & n\geq 1, \quad p=1\\ \binom{n^2}{p-i}, & n\geq 1, \quad p\geq 1, \quad i\geq 0\\ \left(\frac{(n-1)((n-1)+1)}{p}\right)^2, & n\geq 2, p=2\\ \left(\frac{(n-1)((n-1)+1)}{n-p}\right), & n\geq 2, \quad p\geq 0\end{cases}$$

Corollary 4.4 let $I_n = IOR_n$ then,

$$f(n;k)=\begin{cases}\binom{n+k-1}{k}, & n\geq 1, \quad k\geq 0\\ n+k-1, \quad n\geq 1, & k=1\\ \binom{n+k}{n}, \quad n\geq 0, & k=0\\ \binom{n+1}{k}, n\geq 2, & k=2\\ \binom{2k+1}{n-1}, n\geq 2, & k\geq 0\\ \binom{2k-1}{n}, n\geq 1, & k\geq 1\end{cases}$$

Corollary 4.5 let $I_n = IOR_n$ then,

$$f(n;\dot{a})=\begin{cases}\dfrac{7(3^n)+2n+5}{4}, n\geq 2\\ a_n=(n-\dot{a})^2, n=\dot{a}\geq 2\end{cases}$$

Corollary 4.6 let $I_n = IOR_n$ then,

$$f(n;\dot{b})=\begin{cases}\binom{n+1}{p+k}=\frac{n(n+1)}{p+k} & n\geq 1, \quad p=k=1\\ a_n=a_{n-1}+a_{n-3}+a_{n-4}, & a_0=1, a_1=2, a_3=3\\ 4^n+n & n\geq 2\\ a_n=4a_{n-1}+a_{n-2}, & a_0=2, a_1=5\end{cases}$$

Corollary 4.7 let $I_n = IOR_n$ then,

$f(n;\dot{c}) = \{a_n = 4(3^{n-3}), \quad a_1 = 1, a_2 = 2, \quad n \geq 3$

Corollary 4.8 let $I_n = IOR_n$ then,

$$f(n;i)=\begin{cases}a_n=2a_{n-1}+(n-1) \quad a_0=1, a_2=1, n\geq 2\\ a_n=a_{n-1}+a_{n-2}+a_{n-3}+4n-8, n\geq 3\end{cases}$$

Corollary 4.9 let $I_n = IOR_n$ then,

$f(n;j) = \{a_n = 2a_{n-1} + a_{n-2} - a_{n-3}, \ a_0 = 1, a_1 = 3, a_2 = 6, \quad n \geq 3$

Corollary 4.10 let $I_n = IOR_n$ then,

$$f(n;l) = \begin{cases} a_n = \frac{2^n(n^3 - 3n^2 + 2n + 48}{48}, n \geq 0 \end{cases}$$

5. Combinatorial Results for Order-Decreasing and Order-Reversing (IDR_n)

Let $\alpha \in I_n$, then a transformation α is said to be $(IDR_n), if\ \forall\ x, y \in Dom\alpha, x\alpha \geq x\ and\ x \leq \Longrightarrow x\alpha \geq \alpha$.
The following are the elements of IDR_n for $n = 1,2,3$.

For $n = 0$, IDR_0 have,

$$\begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}$$

For $n = 1$, IDR_1 have,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ \emptyset \end{pmatrix}$$

For $n = 2$, IDR_2 have,

$$\begin{pmatrix} 1 & 2 \\ 1 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & \emptyset \end{pmatrix}.$$

For $n = 3$, IDR_3 have,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 2 \end{pmatrix},$$

$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{pmatrix}$. It follows for $= 4,5, \ldots$. We now present some results on (IDR_n)

Lemma 5.1 An element $\alpha \in IDR_n$, is uniquely determined by its image set.

Proof. Since α is order-decreasing and order-reversing, its values form a non-increasing sequence along the ordered domain. Once the image set is fixed, monotonicity and injectively forces a unique assignment.

The following corollaries ware deduced:

Corollaries 5.2 Let $I_n = IDR_n$ then,

i. $|IDR_n| = 2^n, n \geq 1$
ii. $f(n;m) = \{a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5}\ a_{0,3} = a_4 = 1, n \geq 5$

iii. $f(n:m) = \binom{n}{m} - n,\ n = m \geq 2$

iv. $f(n;m) = 2^n,\ n \geq 1$

v. $$f(n;p) = \begin{cases} \frac{n(n+1)}{2}, & n \geq 1 \\ \binom{n}{p}, & n = p \geq 0 \\ \binom{n}{p} - p, & n = p \geq 2 \\ \frac{n(n-1)(n-1)(n-3)}{24}, & n \geq 2 \end{cases}$$

vi. $$f(n;k) = \begin{cases} nk + 1, & n \geq 0, k = 0 \\ \binom{n}{k} - k, & n = k \geq 0 \\ (n-k), & n \geq 2, k = 1 \\ \frac{n(n+1)}{2}, & n \geq 1 \\ \frac{n(n+1)(n+2)}{6}, & n \geq 0 \end{cases}$$

vii. $$f(n;\dot{a}) = \begin{cases} \binom{n}{\dot{a}} - \dot{a}, & n = \dot{a} \geq 2 \\ a_n = 2a_{n-1} - a_{n-2} + a_{n-3} & a_0 = 0, a_1 = 1, a_2 = 2, n \geq 3 \\ 2n^2 & n \geq 0 \end{cases}$$

viii. $$f(n,\dot{b}) = \begin{cases} n\dot{b} + 1, & n \geq 0, \dot{b} = 0 \\ \binom{n}{\dot{b}} - \dot{b}, & n = \dot{b} \geq 2 \\ \frac{-6+5n+n^3}{6}, & n \geq 1 \end{cases}$$

ix. $$f(n;\dot{c}) = \begin{cases} \binom{n}{\dot{c}} - \dot{c}, & n \geq 2, \dot{c} \geq 0 \\ a_n = a_{n-1} - a_{n-2} + a_{n-3}, & a_0 = 0, a_1 = 1, a_2 = 2, n \geq 3 \\ a_n = a_{n-1} + a_{n-2} + a_{n-3}, & a_0 = a_1 = 1, a_2 = 0, n \geq 3 \end{cases}$$

x. $$f(n,i) = \begin{cases} a_n = \frac{n(n+1)(n+2)}{6}, & a_0 = 0,\ n \geq 1 \\ a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3}, & a_{0,1,2} = 0, a_3 = 1, n \geq 3 \\ \frac{n(n+1)}{2}, & n \geq 1 \\ a_n = 3a_{n-1} - a_{n-2}, & a_0 = 0, a_1 = 1, n \geq 2 \end{cases}$$

xi. $$f(n;j) = \begin{cases} a_n = 2a_{n-1} - a_{n-2} + a_{n-3}, & a_0 = a_1, a_2 = 1,\ n \geq 3 \\ \binom{n+1}{2}, & n \geq 1 \\ a_n = a_{n-1} + a_{n-2} + a_{n-3}, & a_0 = a_1 = 0, a_2 = 1,\ n \geq 3 \end{cases}$$

xii. $$f(n;l) = \begin{cases} a_n = 3a_{n-1} - a_{n-2}, & a_0 = a_1 = 1, \quad n \geq 2 \\ \binom{n+1}{2}, & n \geq 1 \\ a_n = \frac{n(n+1)(n+2)}{6}, & a_0 = 0,\ n \geq 1 \end{cases}$$

6. Generating Functions and Refinements

Theorem 6.1 Let X_n be a finite chain and let $IDR_n^{(k)}$ be the set of order-decreasing and order-reversing partial one to one transformation with at least k image changes, then

$$|IDR_n| = \sum_{i=1}^{k} \binom{n-1}{i}.$$

Proof. Let $\alpha \in IDR_n$, suppose α is order-reversing, then the sequence $\alpha(x_n) \geq \alpha(x_2\alpha) \geq \alpha(x_3) \geq \cdots \geq \alpha(x_n)$ is non-increasing. Now, let define a change point at position i if $\alpha(i) > \alpha(i+1)$, then there are exactly $(n-1)$ possible position for the change points. Choosing a subset $\mathcal{S}$, that is, $\mathcal{S} \subseteq \{1,2,3,\ldots,n\}$ it sizes i uniquely determines α and restricting to at most k changes, which implies $|\mathcal{S}| \geq k$, and therefore the number of admissible transformations is

$$\sum_{i=1}^{k} \binom{n-1}{i}.$$

The following lemmas ware deduced:

Lemma 6.2 Let $m \geq 1$ be fixed, define

$$IDR_n^m = \sum_{i=0}^{m} \binom{n-1}{i}$$

Then for $m = n-1$, we have

$$IDR_n^{(n-1)} = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{(n-1)}, n \geq 0.$$

Lemma 6.3 The $q - analogue\ refinement.$

We refine the enumeration of IDR_n by assigning a weight q to each change point in the IDR_n. If a transformation has exactly i change point, it is assigned a q^i, the total weight enumeration is therefore,

$$F_n(q) = \sum_{i=1}^{n-1} q^i \binom{n-1}{i} = (1+q)^{n-1}$$

This polynomial encodes the distribution of transformation according to the number of image change and reduces to $2^{(n-1)}\ when\ q = 1.$

Lemma 6.4 The fixed-point refinement.

Let $f(n;m)$ denote the number of elements of $IDR_n$with exactly m fixed points, then introducing a weight u for fixed point led to the generating polynomial

$$F_n(u) = \sum_{m=0}^{n} f(n;m)\, u^m$$

This recurrence relations established earlier imply that $F_n(u)$ satisfies a linear recurrence in n and the initial values suggest that $F_n(u)$ is a polnimial of degree n with nonnegative integer coefficient this provide a refined combinatorial classification of elements of IDR_n.

Lemma 6.5 A bivariate generating function

A bivariate generating both refinements yield a bivariate generating function

$$G_n(u,q) = \sum_{\alpha \in IDR_n} u^{|fix\alpha|} q^{c(\alpha)}$$

Where $c(\alpha)$ denotes the number of changes of α. The specialization $u = 1$ recovers $(1+q)^{(n-1)}$ while $q = 1$ yield the fixed-point generating polynomial $F_n(u)$. This bivariate framework unifies the enumeration of IDR_n with respect to multiple structural parameters.

Lemma 6.6 The $q - indeterminate$.

Let q be an indeterminate, then

$$F_n(q) = \sum_{i=1}^{n-1} q^i \binom{n-1}{i}.$$

7. Conclusion

We have presented a complete combinatorial enumeration of the semigroup IDR_n of order-decreasing and order-reversing partial one-to-one transformations on a finite chain. The total number of such transformations is 2^n, and refined distributions with respect to image changes and fixed points are encoded by natural generating polynomials. Further directions include the study of regular and inverse Subsemigroups determined by additional constraints, as well as extensions to transformations on more general finite posets.

REFERENCES

Borwein D., Rankin S. and Renner L. (1989):. *Enumeration of injective partial transformation*. Discrete Math. 73, 291-296.

Fernandes V. H., Gomes G. M. S. and Jesus M. M. (2004):. *Presentation for some monoids of injective partial transformations on a finite chain.* Southeast Asian Bull, math. 28, 903-918.

Ganyusahkin O. and Manzochuk V. (2001):. *The full finite inverse symmetric semigroup* IS_n, Preprint 2001:37, chalmers University of Technology and Goteborg University, Goteborg.

Ganyusahkin O. and Kormysheva T. V. (1993):. *Chain decomposition of partial permutations and conjugacy classes in* IS_n, Visnyk Kyiv. University 2, 10-18.

Limpscomb S. (1986):. *Cyclic subsemigroups of symmetric inverse semigroups*, semigroup Forum 34(2) 2344-248.

Michael Cornelius, MI Bello, NH Manjak and Ishiaku Zubairu. (2023):. *On the Order-Reversing Partial One-to-One Transformation Semigroup* (IOR_n). Int. J Res. Acad. Word. 2(5):42-45.

Michael Cornelius (2024):. *Combinatorial results on three variable functions of subsemigroups of symmetric inverse semigroup*, Ph.D Thesis, Abubakar Tafawa Balewa University, Nigeria.

Umar. A. (1992):. *Semigroups of order-decreasing transformation*, Ph. D Thesis, University of St. Andrews.

Umar. A. (1992b):. *On the semigroups of order-decreasing finite full transformations*. Proc. Roy. Soc. Edindurgh 120A, 129-142.

Umar. A. (2010):. *Some combinatorial problems in the theory of symmetric inverse semigroups*. Algebra Discrete Math. 9, 115-126.