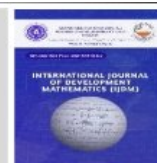




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Hyers-Ulam-Rassias Stability of Nonlinear Second Order Differential Equations with Forcing Term

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Abstract

This paper investigates the Hyers-Ulam-Rassias stability of two classes of nonlinear second-order differential equations with forcing terms. Using the Gronwall-Bellman-Bihari integral inequality and its generalizations, we establish sufficient conditions for stability and derive explicit formulas for the corresponding stability constants. The existence and uniqueness of solutions are established under appropriate Lipschitz conditions. Several special cases are analyzed, and illustrative examples are provided to demonstrate the applicability of our theoretical results.

1 Introduction

The stability problem for functional equations originated with a question posed by Ulam (1960) in 1940: "Under what conditions does there exist an additive mapping near an approximately additive mapping?" Hyers (1941) provided the first affirmative answer for additive functions on Banach spaces in 1941. Subsequently, Aoki (1950), Bourgin (1951), Rassias (1978) generalized this result, leading to the concepts of Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

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The extension of Ulam stability to differential equations began with [Alsina and Ger \(1998\)](#), who investigated the stability of the exponential function. Subsequently, [Takahasi et al. \(2002\)](#), [Takahasi et al. \(2004\)](#), [Miura et al. \(2003\)](#), ([Miura et al., 2003](#)) extended these results to first-order linear differential equations. [Jung \(2005\)](#), [Jung \(2006\)](#), [Jung \(2006\)](#) further generalized these findings. [Rus \(2010\)](#), [Rus \(2010\)](#) demonstrated stability of differential and integral equations using the Gronwall lemma and weakly Picard operators.

[Wang et al. \(2008\)](#), [Li and Shen \(2010\)](#) established Hyers-Ulam stability for first-order and second-order linear differential equations with constant coefficients. For nonlinear equations [Qarawani \(2012\)](#), [Qarawani \(2012\)](#), [Algfary and Jung \(2014\)](#), [Fakunle and Arawomo \(2018\)](#), [Fakunle and Arawomo \(2018\)](#), [Fakunle and Arawomo \(2019\)](#), [Fakunle et al. \(2023\)](#), [Fakunle et al. \(2023\)](#), [Fakunle and Arawomo \(2022\)](#), [Fakunle et al. \(2023\)](#), [Fakunle et al. \(2025\)](#) have obtained stability results.

In this paper, we investigate the Hyers-Ulam-Rassias stability of the following nonlinear second-order differential equations:

$$\begin{aligned} K_1(t, u(t), u'(t))u''(t) + np(t)K_2(t, u(t), u'(t))u'(t) \\ + q(t)u(t) + Q(t, u(t)) = P(t, u(t), u'(t)), \end{aligned} \quad (1)$$

$$\begin{aligned} u''(t) + np(t)K_2(t, u(t), u'(t))u'(t) + q(t)u(t) + Q(t, u(t)) \\ = P(t, u(t), u'(t)), \end{aligned} \quad (2)$$

for all $t > 0$, subject to the initial conditions

$$u(t_0) = u'(t_0) = 0. \quad (3)$$

Here, $p, q \in C(\mathbf{R}_+, \mathbf{R}_+)$, $K_1, K_2 \in C(\mathbf{I} \times \mathbf{R}^2, \mathbf{R})$, $P \in C(\mathbf{R}_+ \times \mathbf{R}^2, \mathbf{R})$, $Q \in C(\mathbf{R}_+ \times \mathbf{R}, \mathbf{R})$, with $K_1(t_0, 0, 0) = 0$, $K_2(t_0, 0, 0) = 0$, $Q(t_0, 0) = 0$, and $\mathbf{I} = (t_0, \infty)$, $\mathbf{R}_+ = [0, \infty)$, $\mathbf{R} = (-\infty, \infty)$.

The paper is organized as follows. Section 2 presents Methodology definitions and lemmas. Section 3 establishes existence and uniqueness of solutions. Section 4 presents the main stability results. Section 5 provides illustrative examples, and Section 6 offers concluding remarks.

2 Methodology

Definition 2.1. Equation (1) is said to have Hyers-Ulam-Rassias stability if there exists a constant $C_\varphi > 0$ such that for every positive function $\varphi : \mathbf{I} \rightarrow \mathbf{R}_+$ and every function $u \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$\begin{aligned} |K_1(t, u, u')u'' + np(t)K_2(t, u, u')u' + q(t)u + Q(t, u) \\ - P(t, u, u')| \leq \varphi(t), \end{aligned} \quad (4)$$

there exists an exact solution $u_0 \in C^2(\mathbf{I}, \mathbf{R}_+)$ of (1) such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t), \quad \forall t \in \mathbf{I}. \quad (5)$$

Definition 2.2. If there is a positive constant $C_\varphi > 0$ with the following property, the differential equation (2) has the Hyers-Ulam-Rassias stability with starting condition (3), provided that for every $u(t) \in C^2(\mathbf{R}_+)$, it satisfies

$$|u''(t) + np(t)K_2(t, u(t), u'(t))u'(t) + q(t)u(t) + Q(t, u(t)) - P(t, u(t), u'(t))| \leq \varphi(t), \quad (6)$$

then there exists a function $u_0(t) \in C^2(\mathbf{R}_+)$ satisfies (2) with initial condition (3) such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t),$$

we call such C_φ a Hyers-Ulam-Rassias constant for the differential equation.

Lemma 2.3 (Bihari (1957)). Let $u, f \in C([t_0, T], \mathbf{R}_+)$, $K > 0$, $M \geq 0$, and let $\omega : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a continuous, nondecreasing function with $\omega(u) > 0$ for $u > 0$. If

$$u(t) \leq K + M \int_{t_0}^t f(s)\omega(u(s)) ds, \quad t_0 \leq t < T, \quad (7)$$

then for all $t \in [t_0, T']$ where $T' \leq T$ is such that the following expression is defined,

$$u(t) \leq \Omega^{-1}\left(\Omega(K) + M \int_{t_0}^t f(s) ds\right), \quad (8)$$

where $\Omega(u) = \int_{u_0}^u \frac{d\tau}{\omega(\tau)}$, $u_0 > 0$, and Ω^{-1} denotes the inverse function.

Theorem 2.4 (Mean Value Theorem for Integrals Murray (1974)). If $f, g \in C([t_0, t], \mathbf{R})$ and g does not change sign on $[t_0, t]$, then there exists $\xi \in [t_0, t]$ such that

$$\int_{t_0}^t f(s)g(s) ds = f(\xi) \int_{t_0}^t g(s) ds. \quad (9)$$

Theorem 2.5 (Generalized Gronwall Inequality Fakunle and Arawomo (2022), Fakunle et al. (2023)). Let $u, r, h, \rho, g \in C(\mathbf{I}, \mathbf{R}_+)$ and let $\beta, \varpi, \gamma \in \Psi$ be nonnegative, monotonic, nondecreasing continuous functions with γ submultiplicative. If

$$u(t) \leq \rho(t) + A \int_{t_0}^t r(s)\beta(u(s)) ds + B \int_{t_0}^t h(s)\varpi(u(s)) ds + L \int_{t_0}^t g(s)\gamma(u(s)) ds, \quad (10)$$

for constants $A, B, L > 0$, then

$$\begin{aligned} u(t) \leq & \rho(t) \Upsilon^{-1} \left[\Upsilon(1) + L \int_{t_0}^t g(s) \gamma \left(\Omega^{-1} \left(\Omega(1) \right. \right. \right. \\ & \left. \left. \left. + B \int_{t_0}^s h(\alpha) \varpi(T(\alpha)) d\alpha \right) T(s) \right) ds \right] \\ & \times \Omega^{-1} \left(\Omega(1) + B \int_{t_0}^t h(s) \varpi(T(s)) ds \right) T(t), \end{aligned} \quad (11)$$

where $T(t) = F^{-1}(F(1) + A \int_{t_0}^t r(s) ds)$, $\Upsilon(u) = \int_{u_0}^u \frac{ds}{\gamma(s)}$, and $F(u) = \int_{u_0}^u \frac{ds}{\beta(s)}$, $\Omega(u) = \int_{u_0}^u \frac{ds}{\varpi(s)}$.

3 Existence and Uniqueness

Before establishing stability results, we address the existence and uniqueness of solutions.

Theorem 3.1 (Existence and Uniqueness for Equation (1)). *Assume the following conditions hold:*

1. K_1, K_2, P, Q, p, q are continuous in their respective domains.
2. There exist Lipschitz constants $L_{K_1}, L_{K_2}, L_P, L_Q > 0$ such that for all arguments in a neighborhood,

$$\begin{aligned} |K_1(t, u_1, u'_1) - K_1(t, u_2, u'_2)| &\leq L_{K_1}(|u_1 - u_2| + |u'_1 - u'_2|), \\ |K_2(t, u_1, u'_1) - K_2(t, u_2, u'_2)| &\leq L_{K_2}(|u_1 - u_2| + |u'_1 - u'_2|), \\ |P(t, u_1, u'_1) - P(t, u_2, u'_2)| &\leq L_P(|u_1 - u_2| + |u'_1 - u'_2|), \\ |Q(t, u_1) - Q(t, u_2)| &\leq L_Q|u_1 - u_2|. \end{aligned}$$

3. The functions p, q are bounded on $\mathbf{I}$.
4. $K_1(t_0, 0, 0) \neq 0$ (non-singular initial condition).
5. The initial conditions (3) are given.

Then there exists a unique solution $u \in C^2(\mathbf{I}, \mathbf{R}_+)$ to the initial value problem (1), (3) on some interval $[t_0, t_0 + \delta]$.

Proof. Define $\mathbf{z}(t) = (z_1(t), z_2(t))^T = (u(t), u'(t))^T$. The second-order equation (??) can be written as the first-order system

$$\mathbf{z}'(t) = \mathbf{F}(t, \mathbf{z}(t)), \quad \mathbf{z}(t_0) = \mathbf{0},$$

where

$$\mathbf{F}(t, \mathbf{z}) = \left(z_2, \frac{P(t, z_1, z_2) - np(t)K_2(t, z_1, z_2)z_2 - q(t)z_1 - Q(t, z_1)}{K_1(t, z_1, z_2)} \right)^T.$$

Under the Lipschitz assumptions and the condition $K_1(t_0, 0, 0) \neq 0$, $\mathbf{F}$ is Lipschitz in $\mathbf{z}$ uniformly in t on a bounded domain. By the Picard-Lindelöf theorem, there exists a unique local solution. $\square$

4 Results

For convenience, we state the following assumptions that will be used throughout this section:

(H1) $P(t, u, u') = \phi(t)\kappa(u)(h(u'))^4$, with $\phi \in C(\mathbf{I}, \mathbf{R}_+)$, $\kappa, h \in C(\mathbf{R}_+, \mathbf{R}_+)$.

(H2) $K_1(t, u, u') = \alpha(t)\gamma(u)\mu(u')(u')^n$, with $\alpha \in C(\mathbf{I}, \mathbf{R}_+)$, $\gamma, \mu \in C(\mathbf{R}_+, \mathbf{R}_+)$, $n \in \mathbf{N}$.

(H3) $K_2(t, u, u') = a(t)u + b(t)g(u)u'$, with $a, b \in C(\mathbf{I}, \mathbf{R}_+)$, $g \in C(\mathbf{R}_+, \mathbf{R}_+)$.

(H4) $Q(t, u) = v(t)m(t)\varpi(u)$, with $v, m \in C(\mathbf{I}, \mathbf{R}_+)$, $\varpi \in C(\mathbf{R}_+, \mathbf{R}_+)$.

(H5) There exists $\varrho > 0$ such that $\int_{t_0}^t \varphi(s) ds \leq \varrho\varphi(t)$ for all $t \in \mathbf{I}$.

(H6) The following integrals are bounded as $t_0 \rightarrow \infty$:

$$\begin{aligned} \int_{t_0}^t \phi(s) ds &\leq n_1, & \int_{t_0}^t p(s)b(s) ds &\leq n_2, & \int_{t_0}^t v(s)m(s) ds &\leq n_3, \\ \int_{t_0}^t \alpha(s) ds &\leq n_4, & \int_{t_0}^t q(s) ds &\leq n_5. \end{aligned}$$

(H7) There exist constants $\lambda, \eta > 0$ such that $|u'(t)| \leq \lambda$, $|u''(t)| \leq \eta$ for all $t \in \mathbf{I}$.

(H8) For $F(u) = \int_0^u s ds$, we have $|F(u(t))| \geq |u(t)|$.

(H9) The functions $p(t), a(t)$ are positive and there exists $\rho > 0$ such that $p(\rho)a(\rho) > 0$.

4.1 Stability for Equation (1)

Theorem 4.1. *Under assumptions (H1)-(H9), equation (1) is Hyers-Ulam-Rassias stable. Moreover, a Hyers-Ulam-Rassias constant is given by*

$$\begin{aligned} C_\varphi &= \frac{\varrho}{np(\rho)a(\rho)} (1 + \varpi(\chi)n_3 + \tau n_5) \\ &\times \Upsilon^{-1} \left[\Upsilon(1) + \frac{h^4(\delta)}{np(\rho)a(\rho)} n_1 \kappa \left(\Omega^{-1} \left(\Omega(1) + \frac{n\lambda^2}{np(\rho)a(\rho)} n_2 g(T^*) \right) T^* \right) \right] \\ &\times \Omega^{-1} \left(\Omega(1) + \frac{n\lambda^2}{np(\rho)a(\rho)} n_2 g(T^*) \right) T^*, \end{aligned} \quad (12)$$

where $T^* = F^{-1}(F(1) + \frac{\mu(\lambda)\lambda^n\eta}{np(\rho)a(\rho)} n_4)$, and χ, τ, δ, π are points obtained from the mean value theorem.

Proof. From inequality (4) and assumptions (H1)-(H4), we have

$$\begin{aligned} \alpha(t)\gamma(u)\mu(u')(u')^n u'' + np(t)a(t)uu' + np(t)b(t)g(u)u' \\ + q(t)u + v(t)m(t)\varpi(u) - \phi(t)\kappa(u)h^4(u') \leq \varphi(t). \end{aligned} \quad (13)$$

Integrating from t_0 to t and applying the mean value theorem (Theorem 2.4), there exist $\xi, \rho, \tau, \pi, \chi, \delta \in [t_0, t]$ such that

$$\begin{aligned} & \mu(u'(\xi))(u'(\xi))^n u''(\xi) \int_{t_0}^t \alpha(s) \gamma(u(s)) ds + np(\rho) a(\rho) \int_{t_0}^t u(s) u'(s) ds \\ & + n(u'(\pi))^2 \int_{t_0}^t p(s) b(s) g(u(s)) ds + u(\tau) \int_{t_0}^t q(s) ds \\ & + \varpi(u(\chi)) \int_{t_0}^t v(s) m(s) ds - h^4(u'(\delta)) \int_{t_0}^t \phi(s) \kappa(u(s)) ds \leq \int_{t_0}^t \varphi(s) ds. \end{aligned} \quad (14)$$

Let $F(u) = \int_0^u s ds$. Then $\int_{t_0}^t u(s) u'(s) ds = F(u(t))$. Using this and taking absolute values, we obtain

$$\begin{aligned} np(\rho) a(\rho) |F(u(t))| & \leq \int_{t_0}^t \varphi(s) ds + n\lambda^2 \int_{t_0}^t p(s) b(s) g(|u(s)|) ds \\ & + |u(\tau)| \int_{t_0}^t q(s) ds + \varpi(|u(\chi)|) \int_{t_0}^t v(s) m(s) ds \\ & + \mu(\lambda) \lambda^n \eta \int_{t_0}^t \alpha(s) \gamma(|u(s)|) ds + h^4(\lambda) \int_{t_0}^t \phi(s) \kappa(|u(s)|) ds. \end{aligned} \quad (15)$$

By assumption (H8), $|F(u(t))| \geq |u(t)|$. Thus,

$$\begin{aligned} |u(t)| & \leq \frac{1}{np(\rho) a(\rho)} (1 + \varpi(|u(\chi)|) n_3 + |u(\tau)| n_5) \int_{t_0}^t \varphi(s) ds \\ & + \frac{\mu(\lambda) \lambda^n \eta}{np(\rho) a(\rho)} \int_{t_0}^t \alpha(s) \gamma(|u(s)|) ds \\ & + \frac{n\lambda^2}{np(\rho) a(\rho)} \int_{t_0}^t p(s) b(s) g(|u(s)|) ds \\ & + \frac{h^4(\lambda)}{np(\rho) a(\rho)} \int_{t_0}^t \phi(s) \kappa(|u(s)|) ds. \end{aligned} \quad (16)$$

Applying Theorem 2.5 and using assumptions (H5)-(H7) yields the desired bound $|u(t)| \leq C_\varphi \varphi(t)$, establishing Hyers-Ulam-Rassias stability. $\square$

4.2 Stability for Equation (2)

Theorem 4.2. *Under assumptions (H1), (H3)-(H9) with the additional condition that $u'(t) \geq \sigma > 0$ for some constant σ , equation (2) is Hyers-Ulam-Rassias stable with*

constant

$$C_\varphi = \frac{\varrho}{np(\rho)a(\rho)}(1 + \delta n_5) \\ \times \Upsilon^{-1} \left[\Upsilon(1) + \frac{h^4(\sigma)}{np(\rho)a(\rho)} n_1 \kappa \left(\Omega^{-1} \left(\Omega(1) + \frac{n_3}{np(\rho)a(\rho)} g(T_2^*) \right) T_2^* \right) \right] \\ \times \Omega^{-1} \left(\Omega(1) + \frac{n_3}{np(\rho)a(\rho)} g(T_2^*) \right) T_2^*, \quad (17)$$

where $T_2^* = F^{-1} \left(F(1) + \frac{n\sigma}{np(\rho)a(\rho)} n_2 \right)$.

Proof. Evaluate inequality (6) using hypotheses (H1), (H3) and (H4) to get

$$u''(t) + np(t)a(t)u(t)u'(t) + np(t)b(t)g(u(t)u'(t)) \\ + q(t)u(t) + v(t)m(t)\varpi(u(t)) - \phi(t)\kappa(u(t))(h(u'(t)))^4 \leq \varphi(t) \quad (18)$$

Apply the hypotheses of Theorem 4.2 to reduce inequality (18) to obtain

$$np(t)a(t)u(t)u'(t) + np(t)b(t)g(u(t)\sigma) \\ + q(t)u(t) + v(t)m(t)\varpi(u(t)) - \phi(t)\kappa(u(t))(h(\sigma))^4 \leq \varphi(t) \quad (19)$$

Integrating inequality (19) and use the Theorem 2.4 that is, there exists $\xi, \delta \in [t_0, t]$ such that

$$np(\xi)a(\xi) \int_{t_0}^t u(s)u'(s)ds + n\sigma \int_{t_0}^t p(s)b(s)g(u(s))ds + u(\delta) \int_{t_0}^t q(s)u(s)ds \\ + \int_{t_0}^t v(s)m(s)\varpi(u(s))ds - (h(\sigma))^4 \int_{t_0}^t \phi(s)\kappa(u(s))ds \leq \int_{t_0}^t \varphi(s)ds$$

Using equation (19) and (H8) to get

$$|u(t)| \leq \frac{1}{np(\xi)a(\xi)} \left(\int_{t_0}^t \varphi(s)ds + u(\delta) \int_{t_0}^t q(s)ds \right) \\ + \frac{n\sigma}{np(\xi)a(\xi)} \int_{t_0}^t p(s)b(s)g(|u(s)|)ds + \frac{1}{np(\xi)a(\xi)} \int_{t_0}^t v(s)m(s)\varpi(|u(s)|)ds \\ + \frac{(h(\sigma))^4}{np(\xi)a(\xi)} \int_{t_0}^t \phi(s)\kappa(|u(s)|)ds$$

Applying Theorem 2.5 and using assumptions (H5)-(H7) yields the desired bound $|u(t)| \leq C_\varphi \varphi(t)$, establishing Hyers-Ulam-Rassias stability $\square$

4.3 Homogeneous Cases

Theorem 4.3. Under assumptions (H2)-(H4), (H6)-(H9) with $P \equiv 0$, equation (13) reduces to

$$K_1(t, u, u')u'' + np(t)K_2(t, u, u')u' + q(t)u + Q(t, u) = 0, \quad (20)$$

which is Hyers-Ulam-Rassias stable with constant

$$\begin{aligned} C_\varphi &= \frac{\varrho}{np(\rho)a(\rho)}(1 + n_3\varpi(\chi) + n_5\tau) \\ &\times \Omega^{-1}\left(\Omega(1) + \frac{n\lambda^2}{np(\rho)a(\rho)}n_2g\left(F^{-1}\left(F(1) + \frac{\mu(\lambda)\lambda^n\eta}{np(\rho)a(\rho)}n_4\right)\right)\right) \\ &\times F^{-1}\left(F(1) + \frac{\mu(\lambda)\lambda^n\eta}{np(\rho)a(\rho)}n_4\right). \end{aligned} \quad (21)$$

Proof. The inequality (4) with $P \equiv 0$ and (H2)-(H4) are used to obtain

$$\begin{aligned} &\alpha(t)\gamma(u(t))\mu(u'(t))u'(t)^nu''(t) + np(t)a(t)u(t)u'(t) \\ &+ np(t)b(t)g(u(t)u'(t) + q(t)u(t) + v(t)m(t)\varpi(u(t)) \leq \varphi(t) \end{aligned} \quad (22)$$

Integrating both sides of inequality (22) from t_0 to t , applying Theorem 2.4 that is, there exists $\xi, \rho, \tau, \pi, \chi \in [t_0, t]$ such that

$$\begin{aligned} &\mu(u'(\xi))u'(\xi)^nu''(\xi) \int_{t_0}^t \alpha(s)\gamma(u(s))ds + np(\rho)a(\rho) \int_{t_0}^t u'(s)u(s)ds \\ &+ n(u'(\pi))^2 \int_{t_0}^t p(s)b(s)g(u(s))ds + u(\tau) \int_{t_0}^t q(s)ds \\ &+ \varpi(u(\chi)) \int_{t_0}^t v(s)m(s)ds \leq \int_{t_0}^t \varphi(s)ds \end{aligned} \quad (23)$$

Using (H8) to arrive at

$$\begin{aligned} |u(t)| &= \frac{1}{np(\rho)a(\rho)} \left(1 + \varpi(|u(\chi)|) \int_{t_0}^t v(s)m(s)ds + |u(\tau)| \int_{t_0}^t q(s)ds \right) \\ &\int_{t_0}^t \varphi(s)ds + \frac{\mu(|u'(\xi)|)|u'(\xi)|^n|u''(\xi)|}{np(\rho)a(\rho)} \int_{t_0}^t \alpha(s)\gamma(|u(s)|)ds \\ &+ \frac{n(|u'(\pi)|)^2}{np(\rho)a(\rho)} \int_{t_0}^t p(s)b(s)g(|u(s)|)ds \end{aligned}$$

Applying Theorem 2.5 and using assumptions (H6)-(H9) yields the desired bound $|u(t)| \leq C_\varphi\varphi(t)$, establishing Hyers-Ulam-Rassias stability $\square$

Theorem 4.4. Under assumptions (H3)-(H4), (H6)-(H9) with $P \equiv 0$, equation (2) reduces to

$$u'' + np(t)K_2(t, u, u')u' + q(t)u + Q(t, u) = 0, \quad (24)$$

which is Hyers-Ulam-Rassias stable with constant

$$\begin{aligned} C_\varphi &= \frac{\varrho}{np(\xi)a(\xi)}(1 + \delta n_5) \\ &\times \Omega^{-1} \left(\Omega(1) + \frac{n_3}{np(\xi)a(\xi)} \varpi \left(F^{-1} \left(F(1) + \frac{n\sigma}{np(\xi)a(\xi)} n_2 \right) \right) \right) \\ &\times F^{-1} \left(F(1) + \frac{n\sigma}{np(\xi)a(\xi)} n_2 \right). \end{aligned} \quad (25)$$

Proof. Let (H3),(H4) be used together with integrating from t_0 to t and the application of Theorem 2.4 that is, there exists $\xi, \sigma, \delta \in [t_0, t]$ such that

$$\begin{aligned} np(\xi)a(\xi) \int_{t_0}^t u(s)u'(s)ds + n\sigma \int_{t_0}^t p(s)b(s)g(u(s))ds + u(\delta) \int_{t_0}^t q(s)u(s)ds \\ + \int_{t_0}^t v(s)m(s)\varpi(u(s))ds \leq \int_{t_0}^t \varphi(s)ds \end{aligned}$$

The use of (H6)-(H9) lead to the following

$$\begin{aligned} |u(t)| &\leq \frac{1}{np(\xi)a(\xi)} \left(\int_{t_0}^t \varphi(s)ds + \frac{u(\delta)}{np(\xi)a(\xi)} \int_{t_0}^t q(s)ds \right) \\ &+ \frac{n\sigma}{np(\xi)a(\xi)} \int_{t_0}^t p(s)b(s)g(|u(s)|)ds + \frac{1}{np(\xi)a(\xi)} \int_{t_0}^t v(s)m(s)\varpi(|u(s)|)ds \end{aligned}$$

Applying Theorem 2.5 and using assumptions (H6)-(H9) yields the desired bound $|u(t)| \leq C_\varphi \varphi(t)$, establishing Hyers-Ulam-Rassias stability $\square$

5 ILLUSTRATIVE EXAMPLES

Example 5.1. Consider the nonlinear second-order differential equation

$$\frac{1}{t^2}u^4(t)(u'(t))^6u''(t) + \frac{1}{t^4}u^2(t)(u'(t))^6 + t^4u(t) + \frac{1}{t^4}u^2(t) = \frac{1}{t^4}u^2(t)(u'(t))^4, \quad t > 0,$$

with initial conditions $u(1) = u'(1) = 0$.

We identify the functions:

$$\begin{aligned} K_1(t, u, u') &= \frac{1}{t^2}u^4(u')^6, \quad K_2(t, u, u') = \frac{1}{t^4}u^2(u')^5, \quad q(t) = t^4, \\ Q(t, u) &= \frac{1}{t^4}u^2, \quad P(t, u, u') = \frac{1}{t^4}u^2(u')^4, \quad p(t) = \frac{1}{t^4}. \end{aligned}$$

The integrals on $[1, \infty)$ satisfy:

$$\begin{aligned} \int_1^\infty \phi(s) ds &= \int_1^\infty s^{-4} ds = \frac{1}{3}, & \int_1^\infty p(s)b(s) ds &= \int_1^\infty s^{-8} ds = \frac{1}{7}, \\ \int_1^\infty v(s)m(s) ds &= \int_1^\infty s^{-4} ds = \frac{1}{3}, & \int_1^\infty \alpha(s) ds &= \int_1^\infty s^{-2} ds = 1, \\ \int_1^\infty q(s) ds &= \int_1^\infty s^4 ds = \infty. \end{aligned}$$

Since $\int_1^\infty q(s) ds$ diverges, we consider the equation on a finite interval $[1, T]$ with T sufficiently large. Theorem 4.1 then guarantees Hyers-Ulam-Rassias stability with an explicitly computable constant.

Example 5.2. Consider the nonlinear second-order differential equation

$$u''(t) + \frac{1}{t^4}u^2(t)(u'(t))^6 + t^4u(t) + \frac{1}{t^4}u^2(t) = \frac{1}{t^4}u^2(t)(u'(t))^4, \quad t > 0,$$

with initial conditions $u(1) = u'(1) = 0$.

This equation is of the form (2) with

$$\begin{aligned} K_2(t, u, u') &= \frac{1}{t^4}u^2(u')^5, & q(t) &= t^4, & Q(t, u) &= \frac{1}{t^4}u^2, \\ P(t, u, u') &= \frac{1}{t^4}u^2(u')^4, & p(t) &= \frac{1}{t^4}. \end{aligned}$$

The integrals on $[1, \infty)$ diverge for $q(t)$, so we consider the interval $[1, 2]$. Taking $\varphi(t) = 1$, we have $\int_1^2 1 ds = 1 = \varrho \cdot 1$ with $\varrho = 1$. Then using the bounds $|u'(t)| \leq \lambda$, $|u''(t)| \leq \eta$, Theorem 4.2 guarantees Hyers-Ulam-Rassias stability.

6 Conclusion

In this paper, we have investigated the Hyers-Ulam-Rassias stability of two classes of nonlinear second-order differential equations with forcing terms. Using the Gronwall-Bellman-Bihari integral inequality and its generalizations, we have:

1. Established sufficient conditions for Hyers-Ulam-Rassias stability.
2. Derived explicit formulas for the corresponding stability constants.
3. Addressed the existence and uniqueness of solutions under Lipschitz conditions.
4. Analyzed several special cases, including the homogeneous case.
5. Provided illustrative examples demonstrating the applicability of our results.

These results extend and complement existing literature on Ulam stability for nonlinear differential equations. Future work may focus on extending these results to higher-order equations, systems of equations, or equations with delays.

Conflicts of Interest

The author declared no conflict of interest in the manuscript.

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