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Analytical Shooting Technique Based on Adomian Decomposition for solving Fractional Order Boundary Value Problems

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A B S T R A C T

In order to tackle the boundary value problem of fractional order, this study devised a semi-analytical shooting technique. In order to express the answer as a series, the Adomian decomposition approach was incorporated into the shooting methodology. Additionally, new shooting slopes with a better rate of convergence were presented for boundary value problems of higher order, along with the technique for deriving the extra equations that correspond to them. The accuracy of the developed approach was demonstrated through four numerical illustrations; absolute errors, relative errors, and mean deviations of the errors were calculated for all the illustrations. For every set of challenges, the tolerance value—the difference between two consecutive shooting slopes—was also computed. For each of the four problems under consideration, the computed tolerance values likewise showed a decrease as shooting slopes increased. This is an additional measure that supports the effectiveness of the shooting strategy.

1 Introduction

A differential equation that is subject to boundary conditions (BCs), which act as constraints, is known as a boundary value problem (BVP) (Fayode *et al*., 2025). A BVP solution needs to meet BCs. BVPs take into consideration phenomena like heat conduction and vibration in structures and involve variables that depend on several dimensions. BVPs can be represent in real-world issues as in Bashiri (2018) and Ijadi (2024), they have a wide range of applications in real-world scenarios. They are also useful in the engineering industry. BVP can be classified as either factional or classical order.

Fractional-order boundary value problems (BVPs) play a significant role in both pure and applied mathematics because of their ability to accurately model systems with memory and hereditary properties. They have found extensive applications in fluid mechanics, electrical circuit analysis, diffusion processes, damping phenomena, mathematical biology, and many other areas of science and engineering. Consequently, the development of reliable and efficient methods for solving fractional-order BVPs has become an important area of scientific research (Khalouta & Kadem, 2019). Numerous numerical and analytical techniques have been proposed to solve such

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problems, as demonstrated in the works of Khalouta and Kadem (2019), El-Ajou *et al*. (2013), El-Borai et al. (2015), Khodabakhsi et al. (2014), Sadiq and Rehman (2022), Bekri *et al*. (2025), and Oderinu *et al*. (2026).

Several researchers have also applied fractional calculus to model real-world phenomena using different fractional derivatives. For example, Aydogan *et al*. (2020) developed a new mathematical model for rabies disease based on the Caputo–Fabrizio fractional derivative. The model was analysed using the Adomian Decomposition Method and the Laplace transform to investigate its dynamic behaviour and to compare it with the corresponding integer-order rabies model. Similarly, George *et al*. (2023) investigated fractional integro-differential equations using the Hilfer fractional derivative, which provides an interpolation between the Riemann–Liouville and Caputo fractional derivatives, and illustrated the effectiveness of the approach through several numerical examples. Furthermore, Aydogan (2021) employed the Adams–Bashforth numerical scheme to obtain approximate solutions of a COVID-19 transmission model formulated with the Atangana–Baleanu–Caputo fractional derivative, demonstrating the applicability of fractional-order models in epidemiological studies.

Fractional boundary value problems were solved with -shifted Chebyshev operational matrices by introducing -shifted Chebyshev polynomials and later formulated -shifted Chebyshev operational matrices from Chebyshev polynomial in Sadiq and Rehman (2022). Several examples were presented using the method and the results were compared with other techniques, which showed the effectiveness of the method. Murad (2022) worked on the existence and uniqueness of solutions for two point boundary conditions of nonlinear fractional differential equation with Banach space using the contraction mapping principle and Brouwer fixed point theorem with Bielecki norm. The stability of the solution was also established using Ulam-Hyers and Ulam-Hyers-Rassias stability theorems. The example presented was solved by using the successive approximation method, which showed the accuracy of the suggested method.

An integral boundary value problem of fractional differential equations with a sign-changed parameter in Banach spaces' existence and uniqueness of solution was investigated by Yang *et al.* (2021). The approach used is a recent fixed point theorem of increasing $\psi-(h,r)$ -concave operators defined on ordered sets. Also, a monotone iterative scheme to approximate the unique solution was presented and two examples were given to illustrate the results. A new existence and uniqueness procedure was established in Derbazi and Hammouche (2020) for boundary value problems of Caputo fractional differential equation with nonlocal and fractional integral boundary conditions by using the Banach contraction principle. An example was used to validate the procedure and the result was found to be favourable. The authors in Murad (2022), Ford and Morgado (2011) and Zhao and Lui (2013) also work on existence and uniqueness of solution of fractional boundary value problems.

Shooting technique is an approach used in solving BVP's by reducing them to equivalent Initial Value Problems (IVPs), it involves finding solutions to the IVPs with different initial conditions, until a solution is found that satisfies the boundary conditions of the BVP's. This method was developed due to difficulty encountered in solving BVP's as a result of the complexity of BC's. The number of IVP's to be derived from BVP depends on Shooting Gradient (SG) procedure employed; a few of the procedure been used is Secant Method in Edun and Akinlabi (2021) and Manyonge *et al.* (2017), Newton's Method, Modified Newton's method and Cubic Newton's Method in Oderinu *et al.* (2022). The resulting IVP generated with the shooting technique will be solved with the Adomian decomposition method (ADM), to produce an analytical result in the form of a polynomial. Many researchers have made use of ADM in solving IVP's such as Johansyah *et al.* (2022) and Othman and Hasan (2020).

The research gaps that this article aims to address are as follows: addressing the discretization error commonly encountered in the methods used to solve the resulting IVPs; broadening the scope of the shooting method to encompass fractional-order BVPs, as most authors have focused on second-order problems; and enhancing the shooting slope to achieve improved convergence of solutions. The purpose of this research is to provide a numerical strategy that eliminates discretization error and provides an approximate analytical solution using the Adomian decomposition method in the shooting scheme. Additionally, to enhance the accuracy of the solution.

2 Basic Definition

Definition 2.1 (Alkresheh and Ismail, 2016). The Riemann Liouville fractional integral of order α for a function *f(x)* is defined as

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{m}(x-r)^{\alpha-1}f(r)dr,$$

$$\alpha > 0, m > a, a \geqslant 0.$$

The following properties can be deduced from definition 2.1

1. If $\alpha = 0$, $I^{\alpha}f(x) = f(x)$ is the identity operator.
2. $(I^{\alpha}I^{\beta})f(x) = (I^{\alpha+\beta})f(x), \quad \alpha, \beta > 0,$
3. if $f(x) = x^{n}$ for some $n > -1$ and $\alpha > 0$ then $I^{\alpha}f(x) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}x^{n+\alpha}$

Definition 2.2 (Khalouta and Kadem, 2019). The Riemann Liouville fractional derivative of order α for a function *f(x)* is defined as

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{0}^{x}(x-r)^{n-\alpha-1}f(r)dr, \;\; x > 0$$

where $n-1 < \alpha \leq n, n \in N$.

Definition 2.3(Khalouta and Kadem, 2019 and Albadarneh *et al.*, 2016). The Caputo fractional derivative of order α for a function $f(x)$ is defined as

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{m}\frac{f^{n}(s)}{(x-r)^{\alpha+1-n}}dr, \;\; x > 0,$$

where $n-1 < \alpha \leq n, n \in N$. From this definition the following properties are deduced:

1. $I^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{(m-a)^{k}}{k!}$
2. If f is continuous and $\alpha \leq 0$ then $D^{\alpha}I^{\alpha}f = f$.

3. If $f(x) = x^n$ for some $n \leq 0$ then

$$D^\alpha f(x) = \begin{cases} 0 & if \alpha = n \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} & if n \in N,\ \alpha \neq n \end{cases}$$

3 Method

One of the main focuses of this study is to utilize a shooting strategy that relies on a semi-analytical method for solving the resulting IVPs; specifically, the method of Adomian Decomposition Method (ADM) will be employed. In the next subsection, there will be a concise presentation on ADM to provide necessary context.

3.1 Concept of ADM for IVP

The procedure of ADM as described in Liu and Yang (2018) and Okeke *et al.* (2019) entails the following steps: Let us examine a nonlinear differential equation in operator form, denoted by

$$G(y(x)) = f(x), \tag{1}$$

with

$$G = L + R + N, \tag{2}$$

in such a way that (1) is transformed into

$$L(y(x)) + R(y(x)) + N(y(x)) = f(x). \tag{3}$$

L represents the linear operator that characterizes the largest derivative, R represents the linear operator with a lower order than L and N represents the nonlinear operator that will be decomposed using Adomian polynomial. It is worthy to note the existence of L^{-1} which leads (3) to transform into

$$y(x) = L^{-1}f(x) - L^{-1}R(y(x)) - L^{-1}N(y(x)) + \psi, \tag{4}$$

where the term ψ fulfills the condition

$$L\psi = 0. \tag{5}$$

Breaking down the true solution y of the equation into an infinite sum of different degrees, That is $y(x) = \sum_{j=0}^{\infty} y_j$ and the nonlinear term $N(y(x))$ is simplified by the use of the polynomial (Rahman *et al*., 2015)

$$B_j = \frac{1}{j!}\frac{d^j}{ds^j}\left[N\left(\sum_{i=0}^{j} s^i y_i\right)\right]_{j=0}, \tag{6}$$

Where B_j is adomial polynomial, then

$$\begin{aligned} y_0 &= \psi_0 + L^{-1}f(x), \\ y_1 &= -L^{-1}R(y_0) - L^{-1}B_0, \\ y_2 &= -L^{-1}R(y_1) - L^{-1}B_1, \\ &\vdots \\ y_{n+1} &= -L^{-1}R(y_n) - L^{-1}B_n, \qquad n \geq 0, \end{aligned} \tag{7}$$

thus the solution to the given problem can be written in an infinite series as

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{8}$$

Comprehensive information regarding this procedure can be acquired in Liu and Yang (2018) and Okeke *et al.* (2019)

3.2 Analytical Shooting Method

3.2.1 Linear shooting method

Considering a higher-order BVP given as

$$D^{\alpha}y = h(x, y, y', \dots, y^{\gamma}), \qquad c \leq x \leq d, \qquad \alpha > 1, \tag{9}$$

where $\gamma = \alpha - \mu$ and $\mu \leq 1$ such that γ is an integer and the boundary conditions is given as $y(c) = \rho, y(d) = \beta$. The shooting method for classical order BVP in Rahman *et al.* (2015) will be adjusted to formulate the method of solution to solve linear FBVP. If (9) is a linear FBVP, it requires that (9) is reduced to an initial value problem (IVP) of the form

$$D^{\alpha}y = h(x, y, y', \dots, y^{\gamma}), \quad c \leq x \leq d, \tag{10}$$
$$y(c) = \rho, y'(c) = \rho_1, \dots, y^{\gamma}(c) = 0,$$

and a supplementary IVP of the form

$$D^{\alpha}z(x, Q) = \frac{\partial h}{\partial y} z(x, Q) + \frac{\partial h}{\partial y'} z'(x, Q) + \dots + \frac{\partial h}{\partial y^{\gamma}} z^{\gamma}(x, Q), \tag{11}$$
$$z(0) = 0, z'(0) = 0, \cdots, z^{\gamma} = 0.$$

Eq. (10) and (11) will be solved with ADM, the result obtained from the two equations will be combined with

$$Y(x) = y(x) + \frac{\beta - y(d)}{z(d)} \cdot z(x). \tag{12}$$

where Y(x) is the solution to (9).

3.2.2 Nonlinear shooting method

The shooting Strategy in Oderinu and Aregbesola (2014) and Burden and Faires (2010) is used and adapted to provide a novel shooting technique.

Considering a higher-order BVP of the form (9), the approximation solution of (9) will be obtained by using solutions to a set of IVP's that include a parameter Q, which is called the shooting angle.

$$D^{\alpha}y = h(x, y, y', \dots, y^{\gamma}), \quad c < x < d, \tag{13}$$
$$y(c) = \rho, y'(c) = \rho_1, \dots, y^{\gamma}(c) = Q.$$

In order to guarantee this, $Q = Q_t$ will be selected in a manner that

$$\lim_{x \to \infty} y(d, Q_t) = y(d) = \beta. \tag{14}$$

The initial shooting angle will be Q_0,

$$D^{\alpha} y = h(x, y, y', \ldots, y^{\gamma}), \quad c \le x \le d, \tag{15}$$
$$y(c) = \rho, y'(c) = \rho_1, \ldots, y^{\gamma}(c) = Q_0.$$

Solving (15) with ADM, the result obtained will be in the form of a polynomial, which is a continuous form. If the solution obtained at Q_0, does not satisfy (14), then the shooting angle will be adjusted to Q_1, this can be repeated severely unit the solution satisfies (14).

This article's procedure for this $Q_t, t = 1,2, \ldots$ is noteworthy since it makes use of Newton's, modified Newton's, and cubic procedures for root finding. These calculations require additional differential equations that will aid in figuring out the trajectory's course.

The case of second order ODE has been provided in Okeke *et al.* (2019) and Lui and Yang (2018) while the derivation of the fractional order will be presented in the next section.

The formulas

$$Q_t = Q_{t-1} - \frac{y(d, Q_{t-1}) - \beta}{\frac{\partial y}{\partial Q}(d, Q_{t-1})}, \tag{16}$$

$$Q_t = Q_{t-1} - \left(\frac{2\left(y(d, Q_{t-1}) - \beta\right)\frac{\partial y}{\partial Q}(d, Q_{t-1})}{2\left(\frac{\partial y}{\partial Q}(d, Q_{t-1})\right)^2 - \left(y(d, Q_{t-1}) - \beta\right)\left(\frac{\partial^2 y}{\partial Q^2}(d, Q_{t-1})\right)} \right), \tag{17}$$

which represent Newton's Method(NM) and Halley's Method (HM) respectively, where

$$Q_0 = \frac{\beta - \rho}{d - c}. \tag{18}$$

The parameters $\frac{\partial y}{\partial Q}$ and $\frac{\partial^2 y}{\partial Q^2}$ in (16) and (17) requires to form differential equation to satisfies them.

Differentiating both sides of (13) partially with respect to Q, in order to form a differential equation for $\frac{\partial y}{\partial Q}$

$$\frac{\partial}{\partial Q} D^{\alpha} y(x, Q) = \frac{\partial h}{\partial Q}(x, y(x, Q), y'(x, Q), \ldots, y^{\gamma}(x, Q)), \tag{19}$$

$$\begin{aligned} \frac{\partial}{\partial Q} D^{\alpha} y(x, Q) = {} & \frac{\partial h}{\partial x}\left(x, y(x, Q), y'(x, Q), \ldots, y^{\gamma}(x, Q)\right) \frac{\partial x}{\partial Q}(x, Q) + \\ & \frac{\partial h}{\partial y}\left(x, y(x, Q), y'(x, Q), \ldots, y^{\gamma}(x, Q)\right) \frac{\partial y}{\partial Q}(x, Q) + \\ & \frac{\partial h}{\partial y'}\left(x, y(x, Q), y'(x, Q), \ldots, y^{\gamma}(x, Q)\right) \frac{\partial y'}{\partial Q}(x, Q) + \cdots + \\ & \frac{\partial h}{\partial y^{\gamma}}(x, y(x, Q), y'(x, Q), \ldots, y^{\gamma}(x, Q)) \frac{\partial y^{\gamma}}{\partial Q}(x, Q), \end{aligned} \tag{20}$$

since x and Q are independent, then, $\frac{\partial x}{\partial Q} = 0$, (20) result to

$$\begin{aligned} \frac{\partial}{\partial Q} D^{\alpha} y(x, Q) = {} & \frac{\partial h}{\partial y}\left(x, y(x, Q), y'(x, Q), \ldots, y^{\gamma}(x, Q)\right) \frac{\partial y}{\partial Q}(x, Q) + \\ & \frac{\partial h}{\partial y'}\left(x, y(x, Q), y'(x, Q), \ldots, y^{\gamma}(x, Q)\right) \frac{\partial y'}{\partial Q}(x, Q) + \cdots + \\ & \frac{\partial h}{\partial y^{\gamma}}(x, y(x, Q), y'(x, Q), \ldots, y^{\gamma}(x, Q)) \frac{\partial y^{\gamma}}{\partial Q}(x, Q), \end{aligned} \tag{21}$$

Let $z(x,Q)=\frac{\partial h}{\partial q}(x,Q)$ in (21) to have

$$D^{\alpha}z(x,Q)=\frac{\partial h}{\partial y}\big(x,y(x,Q),y'(x,Q),\dots,y^{\gamma}(x,Q)\big)z(x,Q)+$$

$$\frac{\partial h}{\partial y'}\big(x,y(x,Q),y'(x,Q),\dots,y^{\gamma}(x,Q)\big)z'(x,Q)+\cdots+$$

$$\frac{\partial h}{\partial y^{\gamma}}(x,y(x,Q),y'(x,Q),\dots,y^{\gamma}(x,Q))z^{\gamma}(x,Q), \tag{22}$$

(22) is the required supplementary ODE for NM, subject to the initial value condition

$$\frac{\partial}{\partial Q}y(c)=0,\quad \frac{\partial}{\partial Q}y'(c)=0,\dots,\frac{\partial}{\partial Q}y^{\gamma}(c)=1. \tag{23}$$

Eq (22) and the initial value condition in (23) can be rewritten as

$$D^{\alpha}z(x,Q)=h_y z+h_{y'}z'+h_{y''}z''+\cdots+h_{y^{\gamma}}z^{\gamma} \tag{24}$$

$$z(c,Q)=0,\quad z'(c,Q)=0,\dots,z^{\gamma}=1.$$

Theorem 3.1 *Let* $D^{\alpha}y=h(x,y,y',\cdots,D^{\gamma}y)$ *be a differential equation, and there exists a solution on* $[c,d]$, *with* $z(x,Q)=\frac{\partial y}{\partial Q}$ *then for every* $\alpha>1$,

$$D^{\alpha}z(x,Q)=h_y z+h_{y'}z'+h_{y''}z''+\cdots+h_{y^{\gamma}}z^{\gamma}=\sum_{i=0}^{\gamma}h_{D^i y}D^i z. \tag{25}$$

Proof:

Since

$$D^{\alpha}z(x,Q)=\sum_{i=0}^{\gamma}h_{D^i y}D^i z. \tag{26}$$

Employing the principle of mathematical induction, the order of FBVP that can be solved with the shooting technique must be greater than 1, for this reason, when $\alpha=2$ will be the starting point.

$$D^{2}z(x,Q)=\sum_{i=0}^{1}h_{D^i y}D^i z. \tag{27}$$

Recall that $\gamma=\alpha-\mu$ and $\mu\le 1$ such that γ is an integer. Hence (27) can be written as

$$z''(x,Q)=\sum_{i=0}^{1}h_{y^i}z^i. \tag{28}$$

that is

$$z''(x,Q)=h_y z+h_{y'}z'. \tag{29}$$

to establish that (29) is true.

Considering

$$y''(x,Q)=h(x,y,y'),$$

$$\frac{\partial y''}{\partial Q}=\frac{\partial h}{\partial Q}(x,Q)=h(x,y,y'),$$

$$z''(x,Q)=\frac{\partial h}{\partial x}\cdot\frac{\partial x}{\partial Q}+\frac{\partial h}{\partial y}\cdot\frac{\partial y}{\partial Q}+\frac{\partial h}{\partial y'}\cdot\frac{\partial y'}{\partial Q}$$

$$z''(x,Q)=h_y z+h_{y'}z'. \tag{30}$$

Since (29) and (30) are the same, then (25) is true.

Now, let us assume that when $\alpha = k$ it is true and then

$$D^{(k)}z(x,Q) = \sum_{i=0}^{k-\mu} h_{y^{(i)}} z^{(i)}. \tag{31}$$

Simplifying (31) to become

$$D^{(k)}z(x,Q) = h_y z + h_{y'} z' + h_{y''} z'' + \cdots + f_{y^{(k-\mu)}} z^{(k-\mu)}, \tag{32}$$

assuming (32) is true. Then let $\alpha = k + \mu$; then (25) becomes

$$D^{(k+1)}z(x,Q) = \sum_{i=0}^{k} h_{y^{(i)}} z^{(i)}, \tag{33}$$

$$D^{(k+1)}Z(x,Q) = h_y z + f_{y'} z' + h_{y''} + \cdots + h_{y^{(k)}} z^{(k)}, \tag{34}$$

Let consider

$$D^{(k+1)}y(x,Q) = h\big(x, y, y', \cdots, y^{(k)}\big), \tag{35}$$

to show that (34) holds, taking partial derivative of (35) w.r.t. Q to have

$$\frac{\partial}{\partial Q} D^{(k+1)}y(x,Q) = \frac{\partial h}{\partial x}\cdot\frac{\partial x}{\partial Q} + \frac{\partial h}{\partial y}\cdot\frac{\partial y}{\partial Q} + \frac{\partial h}{\partial y'}\cdot\frac{\partial y'}{\partial Q} + \cdots + \frac{\partial h}{\partial y^{(k)}}\cdot\frac{\partial y^{(k)}}{\partial Q}, \tag{36}$$

let $z^{(k+1)} = \frac{\partial y^{(k+1)}}{\partial Q}$ then (36) can be written as

$$D^{(k+1)}z(x,Q) = \sum_{i=0}^{k} h_{y^{(i)}} z^{(i)}. \tag{37}$$

Since (33) and (37) are the same, then (25) is true for $\alpha = k + 1$.

To obtain $\frac{\partial^2 y}{\partial Q^2}$ in (17) it requires taking the second partial derivative of (13) w.r.t. Q. That is

$$\frac{\partial^2}{\partial Q^2} D^\alpha y(x,Q) = \frac{\partial^2 h}{\partial Q^2}\left(x, y(x,Q), y'(x,Q)\cdots, y^\gamma(x,Q)\right) \tag{38}$$

$$\frac{\partial}{\partial Q}\left(\frac{\partial h}{\partial Q}\right) = \frac{\partial}{\partial Q}\left(\frac{\partial h}{\partial x}\cdot\frac{\partial x}{\partial Q} + \frac{\partial h}{\partial y}\cdot\frac{\partial y}{\partial Q} + \frac{\partial h}{\partial y'}\cdot\frac{\partial y'}{\partial Q} + \cdots + \frac{\partial h}{\partial y^\mu}\cdot\frac{\partial y^\gamma}{\partial Q}\right) \tag{39}$$

Since x and Q are independent, then $\frac{\partial x}{\partial Q} = 0$ and (39) becomes

$$\frac{\partial^2}{\partial Q^2} D^\alpha y = \frac{\partial h}{\partial y}\cdot\frac{\partial^2 y}{\partial Q^2} + \frac{\partial y}{\partial Q}\cdot\frac{\partial^2 f}{\partial y \partial Q} + \frac{\partial h}{\partial y'}\cdot\frac{\partial^2 y'}{\partial Q^2} + \frac{\partial y'}{\partial Q}\cdot\frac{\partial^2 h}{\partial y' \partial Q} + \cdots + \frac{\partial h}{\partial y^\mu}\cdot\frac{\partial^2 y^\mu}{\partial Q^2} + \frac{\partial y^\mu}{\partial Q}\cdot\frac{\partial^2 h}{\partial y^\mu \partial Q}. \tag{40}$$

In (40) the terms $\frac{\partial y}{\partial Q}\cdot\frac{\partial^2 f}{\partial y \partial Q}$, $\frac{\partial y'}{\partial Q}\cdot\frac{\partial^2 h}{\partial y' \partial Q}$ and $\frac{\partial y^\mu}{\partial Q}\cdot\frac{\partial^2 h}{\partial y^\mu \partial Q}$ are further simplified also let $w = \frac{\partial^2 y}{\partial Q^2}$ such that (40) becomes

$$\begin{aligned} D^\alpha w(x,Q) = {} & h_y \cdot w + z\big(h_{yy}\cdot z + h_{yy'}\cdot z' + \cdots + h_{yy^\gamma}\cdot z^\mu\big) \\ & + h_y' \cdot w' + z'\big(h_{y'y}\cdot z + h_{y'y'}\cdot z' + \cdots + h_{y'y^\gamma}\cdot z^\mu\big) \\ & + \cdots + h_y^\gamma \cdot w^\gamma + z^\gamma\big(h_{y^\gamma y}\cdot z + h_{y^\mu y'}\cdot z' + \cdots + h_{y^\mu y^\gamma}\cdot z^\gamma\big), \end{aligned} \tag{41}$$

and (41) is the second supplementary ODE required to be solved when using (17) to adjust the shooting angle. The initial value condition is derived by taking the second derivative of the initial value condition in (13) to have

$$w(c,Q) = 0,\ w'(c,Q) = 0, \cdots, w^\mu = 0. \tag{42}$$

Theorem 3.2 *Let $D^\alpha y = h(x, y, y', \cdots, D^\gamma y)$ be a differential equation, and there exist a solution on $[c,d]$, with $w(x,Q) = \frac{\partial^2 y}{\partial Q^2}$ then for every $\alpha > 1$,*

$$D^{\alpha}w(x,Q)=\sum_{i=0}^{\gamma}\left[h_{y^{(i)}}\cdot w^{(i)}+z^{(i)}\left(\sum_{k=0}^{\gamma}\left(h_{y^{(i)}y^{(k)}}\cdot z^{(k)}\right)\right)\right]. \tag{43}$$

Theorem (3.2) was proved with the same procedure used in proofing theorem (3.1) using mathematical induction.

3.3 Derivation of Modified Fourth Order Method (MFOM)

Let

$$f(x_n+h)=f(x_n)+f'(x_n)(x_{n+1}-x_n)+\frac{1}{2!}f''(x_n)(x_{n+1}-x_n)^2+\frac{1}{3!}f'''(x_n)(x_{n+1}-x_n)^3+\cdots \tag{44}$$

setting $f(x_n+h)=0$, then (44) will be equivalent to

$$f'(x_n)(x_{n+1}-x_n)+\frac{1}{2}f''(x_n)(x_{n+1}-x_n)^2+\frac{1}{6}f'''(x_n)(x_{n+1}-x_n)^3=-f(x_n), \tag{45}$$

$$x_{n+1}-x_n=\frac{-6f(x_n)}{6f'(x_n)+3f''(x_n)(x_{n+1}-x_n)+f'''(x_n)(x_{n+1}-x_n)^2} \tag{46}$$

from the first order of Taylor series $x_{n+1}-x_n=-\frac{f(x_n)}{f'(x_n)}$, then (48) becomes

$$x_{n+1}=x_n-\frac{6f(x_n)}{6f'(x_n)+3f''(x_n)\left(-\frac{f(x_n)}{f'(x_n)}\right)+f'''(x_n)\left(-\frac{f(x_n)}{f'(x_n)}\right)^2} \tag{47}$$

$$x_{n+1}=x_n-\frac{6f(x_n)f'(x_n)^2}{6f'(x_n)^3-3f(x_n)f'(x_n)f''(x_n)+f(x_n)^2f'''(x_n)} \tag{48}$$

so if $Q_t=x_{n+1}$ then (48) becomes

$$Q_t=Q_{t-1}-\frac{6f(Q_{t-1})f'(Q_{t-1})^2}{6f'(Q_{t-1})^3-3f(Q_{t-1})f'(Q_{t-1})f''(Q_{t-1})+f(Q_{t-1})^2f'''(Q_{t-1})} \tag{49}$$

where $t=1,2,3,\cdots$

$$Q_t=Q_{t-1}-\left(\frac{6(y(d,Q_{t-1})-\beta)\left(\frac{\partial y}{\partial Q}(d,Q_{t-1})\right)^2}{6\left(\frac{\partial y}{\partial Q}(d,Q_{t-1})\right)^2-3\left((y(d,Q_{t-1})-\beta)\left(\frac{\partial y}{\partial Q}(d,Q_{t-1})\right)\right)\left(\frac{\partial^2 y}{\partial Q^2}(d,Q_{t-1})\right)+(y(d,Q_{t-1})-\beta)^2\left(\frac{\partial^3 y}{\partial Q^3}(d,Q_{t-1})\right)}\right). \tag{50}$$

To obtain $\frac{\partial^3 y}{\partial Q^3}$ in (50), it requires taking the third derivative of (13) partially w.r.t. Q, that is

$$\frac{\partial^3}{\partial Q^3}D^{\alpha}y(x,Q)=\frac{\partial}{\partial Q}\left[\frac{\partial}{\partial Q}\left(\frac{\partial h}{\partial Q}\right)\right] \tag{51}$$

$$\begin{aligned}&\frac{\partial}{\partial Q}\left[\frac{\partial}{\partial Q}\left(\frac{\partial h}{\partial Q}\right)\right]=\\&\frac{\partial}{\partial Q}\left(\frac{\partial h}{\partial y}\cdot\frac{\partial^2 y}{\partial Q^2}+\frac{\partial^2 h}{\partial y\partial Q}\cdot\frac{\partial y}{\partial Q}+\frac{\partial h}{\partial y'}\cdot\frac{\partial^2 y'}{\partial Q^2}+\frac{\partial^2 h}{\partial y'\partial Q}\cdot\frac{\partial y'}{\partial Q}+\frac{\partial h}{\partial y''}\cdot\frac{\partial^2 y''}{\partial Q^2}+\frac{\partial^2 h}{\partial y''\partial Q}\cdot\frac{\partial y''}{\partial Q}+\cdots+\right)\\&\frac{\partial}{\partial Q}\left(\frac{\partial h}{\partial y^{(\mu)}}\cdot\frac{\partial^2 y^{(\mu)}}{\partial Q^2}+\frac{\partial^2 h}{\partial y^{(\mu)}\partial Q}\cdot\frac{\partial y^{(\mu)}}{\partial Q}\right).\end{aligned} \tag{52}$$

In (48) terms like, $\frac{\partial^2 h}{\partial y\partial Q}\cdot\frac{\partial y}{\partial Q}$, $\frac{\partial^2 h}{\partial y'\partial Q}\cdot\frac{\partial y'}{\partial Q}$, $\frac{\partial^2 h}{\partial y''\partial Q}\cdot\frac{\partial y''}{\partial Q}$ and $\frac{\partial^2 h}{\partial y^{(\mu)}\partial Q}\cdot\frac{\partial y^{(\mu)}}{\partial Q}$ are further simplified and represent $\frac{\partial^3}{\partial Q^3}$ with g. So (52) becomes

$$D^{\alpha}g(x,Q)=f_y\cdot g+2w\left(f_{yy}\cdot z+f_{yy'}\cdot z'+\cdots+f_{yy^{\gamma}}\cdot z^{\gamma}\right)$$

$$+f_{y'}\cdot g' + 2w'\left(f_{y'y}\cdot z + f_{y'y'}\cdot z' + \cdots + f_{y'y^{\gamma}}\cdot z^{\gamma}\right)$$
$$+f_{y''}\cdot g'' + 2w''\left(f_{y''y}\cdot z + f_{y''y'}\cdot z' + \cdots + f_{y''y^{\gamma}}\cdot z^{\gamma}\right)$$
$$+\cdots + f_{y^{\gamma}}\cdot g^{\gamma} + 2w^{\gamma}\left(f_{y^{(\gamma)}y}\cdot z + f_{y^{(\gamma)}y'}\cdot z' + \cdots + f_{y^{(\gamma)}y^{\gamma}}\cdot z^{\gamma}\right). \quad (53)$$

Taking the third derivative of the initial value conditions of (13) to derive the initial value conditions for (53) to have

$$g(c,Q) = 0,\ g'(c,Q) = 0, \cdots, g^{\gamma}(c,Q) = 0. \quad (54)$$

Hence (13) is the third required supplementary equation to be solved when using (50) to adjust the shooting angle.

Theorem 3.3 *Let $D^{\alpha}y = h(x,y,y',\cdots,D^{\gamma}y)$ be a differential equation, and there exists a solution on $[c,d]$, with $g(x,Q) = \frac{\partial^3 y}{\partial Q^{3,}}$ then for every $\alpha > 1$,*

$$D^{\alpha}g(x,Q) = \sum_{i=0}^{\gamma}\left[h_{y^{(i)}}\cdot g^{(i)} + 2w^{(i)}\left(\sum_{k=0}^{\gamma}\left(h_{y^{(i)}y^{(k)}}\cdot z^{(k)}\right)\right)\right] \quad (55)$$

Theorem (3.3) was proved with the same procedure used in proving Theorem (3.1) using mathematical induction.

4 Numerical Applications

Problem 1: Consider the linear FBVP

$$D^{\frac{3}{2}}y(x) + y(x) = x^5 - x^4 + \frac{128x^{3.5}}{2\sqrt{\pi}} - \frac{64x^{2.5}}{5\sqrt{\pi}} - y(x),$$
$$0 \le x \le 1,\ \ y(0) = 0, y(1) = 0. \quad (56)$$

The exact solution of eq. (56) is $x^5 - x^4$ (Murad, 2022).

The reduced IVP of eq. (56) is

$$D^{\frac{3}{2}}y(x) + y(x) = x^5 - x^4 + \frac{128x^{3.5}}{2\sqrt{\pi}} - \frac{64x^{2.5}}{5\sqrt{\pi}} - y(x),$$
$$y(0) = 0,\ \ y'(0) = 0. \quad (57)$$

Using (24) to generate the supplementary equation involved when solving linear FBVP to have

$$D^{\frac{3}{2}}z = -z,\ \ 0 \le x \le 1,\ \ z(0) = 0, z'(0) = 1. \quad (58)$$

Using the ADM scheme in section (3.1) to solve (57) and (58), while (12) was used to generate a solution to eq. (56) to be

$$Y = -0.000004892889x + 0.000014722756x^{\frac{5}{2}} - 1.000000204x^4 + x^5 + 1.703x^{\frac{11}{2}} \times 10^{-8} - 4.0x^{\frac{13}{2}} \times 10^{-11} + 2x^7 \times 10^{-l2} + \cdots \quad (59)$$

Problem 2: Considering the linear FBVP

$$D^{\alpha}y - 2y'' = r(x),\ \ 0 \le x \le 1,\ \ 3 < \alpha \le 4$$
$$y(0) = 0,\ \ y''(0) = 0,\ \ y(1) = 0,\ \ y''(1) = -12, \quad (60)$$

where $r(x) = -24 + x + 24x^2 - x^4$ and the exact solution at $\alpha = 4$ is $y = x(1 - x^3)$ Sekson and Supaporn (2017).

The reduced IVP of (60) is

$$D^{\alpha}y - 2y'' = r(x),\ 0 \le x \le 1,\ 3 < \alpha \le 4$$
$$y(0) = 0,\ y' = v,\ y''(0) = 0,\ y'''(0) = 0, \tag{61}$$

and the supplementary equation is

$$D^{\alpha}z = 2z'' - z,\ 0 \le x \le 1,\ 3 < \alpha \le 4,$$
$$z(0) = 0,\ z' = 0,\ z''(0) = 0,\ z'''(0) = 1. \tag{62}$$

ADM scheme in section (3.1) was used to solve eq. (61) and (62) at different values of α. The result obtained at $\alpha = 4$ is

$$Y|_{\alpha=4} = 1.00001580201x - 0.000015091304x^3 - x^4$$
$$-0.000001640814x^5 - 0.000000060168x^7$$
$$-0.000000001129x^9 + 0.000000200443x^{11}$$
$$+0.000000801667x^{12} + \cdots. \tag{63}$$

Problem 3: Considering the nonlinear FBVP (Ruman and Islam 2024)

$$D^2y(x) + \Gamma\left(\frac{4}{5}\right)x^{\frac{6}{5}}D^{\frac{6}{5}}y(x) + \frac{11}{9}\Gamma\left(\frac{5}{6}\right)x^{\frac{1}{6}}D^{\frac{1}{6}}y(x) - (Dy(x))^2 = 2 + \frac{1}{10}x^2. \tag{64}$$

The boundary condition is given as $y(0) = 1$ and $y(1) = 2$, and the exact solution is $y(x) = 1 + x^2$. The reduced IVP of (64) is

$$D^2y(x) + \Gamma\left(\frac{4}{5}\right)x^{\frac{6}{5}}D^{\frac{6}{5}}y(x) + \frac{11}{9}\Gamma\left(\frac{5}{6}\right)x^{\frac{1}{6}}D^{\frac{1}{6}}y(x) - (Dy(x))^2 = 2 + \frac{1}{10}x^2,$$
$$y(0) = 1,\ y'(0) = Q_t. \tag{65}$$

Where Q_t is SG, which will be adjusted till the solution satisfies BC. The initial SG, $Q_0 = 1$ was determined with (18) and the result obtained at this point is

$$y(x) = 1.0 + 1.5\,x^2 + 0.755555555556\,x^3 + x + 0.623611111111\,x^4 + 0.000000000372\,x^{16} + 0.000000084746\,x^{14} + 0.000000122563\,x^{13} + 0.000005388141\,x^{12} + 0.000102376703\,x^{10} + 0.000028380031\,x^{11} + 0.001869007866\,x^9 + 0.004631851512\,x^8 + 0.199798094922\,x^6 + 0.0352385051255\,x^7 + 0.508037037037\,x^5. \tag{66}$$

From (66), it was observed that $y(d, Q_0) = 5.62887751569$, which is not close to β, it requires adjusting the shooting (Q_t). Newton's Method with (16) was used to adjust the shooting angle, and the supplementary equation associated with NM is generated with Theorem (3.1) is

$$D^2z(x) += 2Dz(x)Dy(x) - \Gamma\left(\frac{4}{5}\right)x^{\frac{6}{5}}D^{\frac{6}{5}}z(x) - \frac{11}{9}\Gamma\left(\frac{5}{6}\right)x^{\frac{1}{6}}Dx^{\frac{1}{6}}z(x),$$
$$z(0) = 1,\ z'(0) = 1, \tag{67}$$

which was adjusted six times, and the result obtained at Q_6 is

$$y(x) = 1 + 1.00000000135\,x^2 - 0.000021956424\,x^3 - 0.000052002056\,x + 0.000000001010\,x^4 + 0.000005388141\,x^{12} + 0.000076979877\,x^{10} - 6.372 \times 10^{-12}\,x^{13} + 1.2521 \times 10^{-11}\,x^8 +$$

$0.000000000540\ x^6 - 0.000001659134\ x^7 - \quad 0.000006739761\ x^5 + 0.000000084746\ x^{14} + 0.000000000372\ x^{16}.$ (68)

When using HM with (17) to adjust the shooting angle, the second supplementary equation it required was generated with Theorem (3.2)

$$D^2 w(x) = \frac{-11}{9}\Gamma\left(\frac{5}{6}\right) x^{\frac{1}{6}} D^{\frac{1}{6}} w(x) + 2Dy(x)Dw(x) + 2(Dz(x))^2 - \Gamma\left(\frac{4}{5}\right) x^{\frac{6}{5}} D^{\frac{6}{5}} w(x),$$
$$w(0) = 0, \quad w'(0) = 0. \tag{69}$$

The result obtained at Q_3 HM is

$y(x) = 1 + 1.00000000135\ x^2 - 0.000021956424\ x^3 - 0.000052002057\ x + \quad 0.000000001010\ x^4 + 0.000005388141\ x^{12} + 0.000076979877\ x^{10} - \quad 0.000000001476\ x^{11} - 0.000000097192\ x^9 + 1.2514 \times 10^{-11}\ x^8 + \quad 0.000000000540\ x^6 - 0.000001659135\ x^7 - 0.00000673976\ x^5 + 0.000000084746\ x^{14} + 0.00000000037\ x^{16} - 6.37350844072 \times 10^{-12}\ x^{13}.$ (70)

Modified Fourth Order Method was also used to adjust the shooting angle with eq. (50). An additional supplementary equation required by MFOM is

$$D^2 g(x) = \frac{-11}{9}\Gamma\left(\frac{5}{6}\right) x^{\frac{1}{6}} D^{\frac{1}{6}} g(x) + 2Dy(x)Dg(x) + 4Dw(x)Dz(x) - \Gamma\left(\frac{4}{5}\right) x^{\frac{6}{5}} D^{\frac{6}{5}} g(x)$$
$$g(0) = 0, \quad g'(0) = 0, \tag{71}$$

which was generated with Theorem (3.3) and takes four iterations before the result satisfies the boundary condition. The result obtained at Q_4 is

$$\begin{aligned} y(x) = {} & 1 + 1.00000000135\ x^2 - 0.000021956424\ x^3 - 0.000052002058\ x + 0.000000001010\ x^4 + 1.25 \\ & \times 10^{-11}\ x^8 + 0.000000000540\ x^6 - 0.000001659135\ x^7 - 0.000006739761\ x^5 \\ & - 0.000001659135\ x^7 - 0.000006739761\ x^5 + 0.000000000372\ x^{16} \\ & + 0.000000084746\ x^{14} - 6.37 \times 10^{-12}\ x^{13} + 0.000005388141\ x^{12} + 0.000076979877\ x^{10} \\ & - 0.000000001476\ x^{11}. \end{aligned} \tag{72}$$

Problem 4: Considering the nonlinear FBVP (Ruman and Islam 2024)

$$D^{1.5} y(x) = \frac{\Gamma 2.9}{\Gamma 1.4} x^{0.4} - (x^{1.9} - 1)^3 + (y(x))^3, \quad y(0) = -1, \quad y(1) = 0, \tag{73}$$

the exact solution is $x^{1.9} - 1$. The reduced VP of (73) is

$$D^{1.5} y(x) = \frac{\Gamma 2.9}{\Gamma 1.4} x^{0.4} - (x^{1.9} - 1)^3 + (y(x))^3, \quad y(0) = -1, \quad y(1) = 0, \tag{74}$$

The shooting angle was adjust severally with NM, while the supplementary equation is given as

$$D^{1.5} z(x) = 3(y(x))^2 z(x), \quad z(0) = 0, \quad z'(0) = 1. \tag{75}$$
$$D^{1.5} w(x) = 3(y(x))^2 w(x) + 6y(x)(z(x))^2, \quad w(0) = 0, \quad w'(0) = 0. \tag{76}$$

Equation (76) is an additional supplementary equation to (75) needed when using HM to adjust the shooting angle while MFOM uses

$$D^{1.5} g(x) = 3(y(x))^2 g(x) + 12w(x)y(x)z(x), \quad g(0) = 0, \quad g(0) = 0, \tag{77}$$

with other supplementary equations to determine the shooting angle.

5 Results and Discussion

A shooting method for fractional order boundary value problems was developed and four numerical examples were carried out, comprising two linear and two nonlinear FBVP. Table 5 shows the approximate solution for **Problem** 1 which was a linear fractional order BVP. The table also shows the absolute error, obtained by taking the absolute difference between the exact and approximate solutions; it was observed that the error was minimal and lower than that reported in Murad (2022). The relative error and the mean deviation of the error from mean error was computed, and it was discovered that the error was at minimal.

Table 1 shows the comparison of the exact solution at the classical order and approximate solutions at different orders for Problem 2. The result obtained at the classical order is an approximate solution to the exact solution. The absolute error was only computed for the classical order due to the non-availability of the fractional-order exact solution, as shown in Table 2. It was observed that the absolute error was low, and the Mean Absolute Error (MAE) obtained was low, and likewise Mean Relative Error (MRE) and Mean Deviation Error (MDE) computed were low. The absolute error obtained was smaller than that reported by Sekson and Supaporn (2017).

Problem 3 was a nonlinear FBVP, which required adjusting the shooting angle. The result obtained at the initial shooting angle is displayed in Table 3. Column 3 is the approximate solution obtained from the proposed method, while column 2 is the exact solution. The absolute difference between the exact and approximate solutions was computed in column 4, which shows a wide difference between the exact and approximate solutions. The absolute relative error and deviation error were also shown in the table. Due to the large absolute error obtained, the shooting angle was adjusted with all the proposed shooting angle formulas introduced, where NM was updated six times before it reached minimum absolute differences, and HM was updated three times, and MFOM was adjusted four times. The result obtained at each of the last iterations of the shooting angle is shown in Table 4 in comparison with the exact solution.

Table 1: Comparison of the exact solution and the approximate solution for **Problem** 1

x	Exact Solution	Approximate Solution	Absolute Error	Relative Error	Mean Deviation	Murad (2022). Absolute error
0.0	0.000000	0.000000000000	0.000000000000			0
0.2	-0.001280	-0.001280952564	0.000000952564	0.000744190625	000001330850	0.000822
0.4	-0.015360	-0.015361812730	0.000001812730	0.000118016276	0.000000378286	0.00168

0.6	-0.051840	-0.051842521480	0.000002521480	0.000048639660	0.000000481880	0.00221
0.8	-0.081920	-0.081922698220	0.000002698220	0.000032937256	0.000002521480	0.00185
1.0	0.000000	-0.000000000109	0.000000000109		0.000001367370	0
MEAN			0.000001330850	0.000235945954	0.000001215973	

Table 2: Comparison of the exact solution and the approximate solution for **Problem** 2

x	Exact Solution at $\alpha = 4$	Approximate Solution at $\alpha = 4$	Approximate Solution at $\alpha = 3.8$	Approximate Solution at $\alpha = 3.6$	Approximate Solution at $\alpha = 3.4$	Approximate Solution at $\alpha = 3.2$
0.0	0	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.2	0.1984	0.198403039146	0.199763869999	0.200848029013	0.201027348051	0.199034939997
0.4	0.3744	0.374405338081	0.380623183760	0.384552701933	0.383080517219	0.370316010697
0.6	0.4704	0.470406094661	0.481440209847	0.486789578481	0.480264439663	0.450084435334
0.8	0.3904	0.390404436245	0.400824825132	0.404182206769	0.393634075605	0.355754332513
1.0	0	0.000000000000	0.000000000004	-0.000000000007	0.000000000002	-0.000000000010

Table 3: Absolute error classical order for **Problem** 2

x	Exact Solution at $\alpha = 4$	Approximate Solution at $\alpha = 4$	Absolute Error	Relative Error	Mean Deviation	Sekson and Supaporn (2017) Absolute error
	0	0.000000000000	0.000000000000		0.000003151355	
0.2	0.1984	0.198403039146	0.000003039146	0.000015318276	0.000000112209	0.0208592
0.4	0.3744	0.374405338081	0.000005338081	0.000014257695	0.000002186726	0.000945886
0.6	0.4704	0.470406094661	0.000006094661	0.000012956337	0.000002943305	0.0000428772
0.8	0.3904	0.390404436245	0.000004436245	0.000011363332	0.000001284889	0.00000194362
1.0	0	0.000000000000	0.000000000000		0.000003151355	
MEAN			0.000003151355	0.000013473910	0.000002138307	

The absolute difference between the exact solution and the approximate solution obtained from the proposed method is shown in Table 5. Table 6 and Table 7 show the Relative Error and Mean deviation of the error, respectively. From the three error tables, it was observed that HM converges faster than the other two methods. Also the error obtained is less than erorr computed in Ruman and Islam 2024. Table 8 shows the tolerance value (TV), which is the absolute difference between two successive shooting angles. The TV was reduced as the number of shooting angles increased.

Table 4: Comparison of exact solution and the approximate solution for **Problem** 3 at Q_0

x	Exact Solution	Q_0	Absolute Error	Relative Error	Mean Deviation
0.0	1.00	1.00000000000	0.00000000000	0.000000000000	1.244997479842
0.2	1.04	1.26721804503	0.22721804503	0.218478889452	1.017779434812
0.4	1.16	1.71040194452	0.55040194452	0.474484434931	0.694595535322
0.6	1.36	2.43393055910	1.07393055910	0.789654822868	0.171066920742
0.8	1.64	3.62955681471	1.98955681471	1.213144399213	0.744559334868
1.0	2.00	5.62887751569	3.62887751569	1.814438757845	2.383880035848
MEAN			1.24499747984	0.751700217385	1.042813123572

Table 5: Comparison of exact solution and the approximate solution for **Problem** 3 with different Shooting formula

x	Exact Solution	NM at Q_6	HM at Q_3	MFOM at Q_4
0.0	1.00	1.00000000000	1.00000000000	1.00000000000
0.2	1.04	1.03998942182	1.03998942182	1.03998942182
0.4	1.16	1.15997773061	1.15997773061	1.15997773061
0.6	1.36	1.35996396258	1.35996396258	1.35996396258
0.8	1.64	1.63995322814	1.63995322813	1.63995322813
1.0	2.00	2.00000000000	2.00000000000	2.00000000000

Table 6: Abolute Error for **Problem** 3 with different Shooting formula

x	NM at Q_6	HM at Q_3	MFOM at Q_4	Ruman and Islam 2024
0.0	0.00000000000	0.00000000000	0.00000000000	0.000000
0.2	0.00001057818	0.00001057818	0.00001057818	0.000249
0.4	0.00002226939	0.00002226939	0.00002226939	0.000474
0.6	0.00003603742	0.00003603742	0.00003603742	0.000606
0.8	0.00004677186	0.00004677187	0.00004677187	0.000514
1.0	0.00000000000	0.00000000000	0.00000000000	0.000000
MEAN	0.00001927614	0.00001927614	0.00001927614	

Table 7: Relative Error for **Problem** 3 with different Shooting formula

x	NM at Q_6	HM at Q_3	MFOM at Q_4

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0.0	0.00000000000	0.00000000000	0.00000000000
0.2	0.00001017133	0.00001017133	0.00001017133
0.4	0.00001919775	0.00001919775	0.00001919775
0.6	0.00002649810	0.00002649810	0.00002649810
0.8	0.00002851943	0.00002851943	0.00002851943
1.0	0.00000000000	0.00000000000	0.00000000000
MEAN	0.00001406443	0.00001406444	0.00001406444

Table 8: Mean Deviation Error for **Problem** 3 with different Shooting formula

x	**NM at Q_6**	**HM at Q_3**	**MFOM at Q_4**
0.0	0.00001927614	0.00001927614	0.00001927614
0.2	0.00000869796	0.00000869796	0.00000869796
0.4	0.00000299325	0.00000299325	0.00000299325
0.6	0.00001676128	0.00001676128	0.00001676128
0.8	0.00002749572	0.00002749573	0.00002749573
1.0	0.00001927614	0.00001927614	0.00001927614
MEAN	0.00001575008	0.00001575008	0.00001575008

Problem 4 was also a nonlinear FBVP, in which the results obtained are shown in Table 9 - 14. The proposed method was used and had similar behaviour to Problem 3. There are much differences between the exact solution and the approximate solution at the initial shooting angle, as shown in Table 9. Hence, the shooting angle was adjusted with NM, HM, and MFOM. NM was adjusted four times, while HM and MFOM were adjusted three times to reach the minimal differences and also satisfy the boundary conditions given with the problem. The result obtained at the last iteration of each of the methods used in adjusting the shooting angle was displayed in Table 10. The Absolute difference, Relative Error, and Mean Deviation from the mean error were computed as shown in Tables 11, 12, and 13, respectively, to showcase the accuracy of the method.

Table 9: Tolerance Value for **Problem** 3

Q_t	NM	$\lvert Q_t - Q_{t-1}\rvert$	HM	$\lvert Q_t - Q_{t-1}\rvert$	MFOM	$\lvert Q_t - Q_{t-1}\rvert$
Q_0	1.00000000000		1.000000000000		1.000000000000	

Q_1	-0.90047211370	1.90047211370	0.174928706740	0.825071293260	0.230470431286	0.769529568714
Q_2	-0.35303907321	0.54743304049	0.000393208493	0.174535498247	0.004112022862	0.226358408424
Q_3	-0.04838540967	0.30465366354	-0.000052002057	0.000445210550	-0.000051979472	0.004164002334
Q_4	-0.00154060080	0.04684480888	-0.000052002057	0.000000000000	-0.000052002058	0.000000022586
Q_5	-0.00005350571	0.00148709509			-0.000052002058	0.000000000000
Q_6	-0.00005200206	0.00000150365				
Q_7	-0.00005200206	0.00000000000				

Table 10: Comparison of exact solution and the approximate solution for **Problem** 4 at Q_0

x	Exact Solution	Q_0	Absolute Error	Relative Error	Mean Deviation
0.0	-1.000000000000	-1.000000000000	0.000000000000		0.6165472632810
0.2	-0.953015242276	-0.738787270759	0.214227971517	0.224789659193	0.4023192917640
0.4	-0.824646683778	-0.359257545069	0.465389138709	0.564349736516	0.1511581245720
0.6	-0.621132479506	0.111594838169	0.732727317675	1.179663504729	0.1161800543940
0.8	-0.345558283159	0.647978516846	0.993536800005	2.875164186262	0.3769895367240
1.0	0.000000000000	1.293402351780	1.293402351780		0.6768550884990
	MEAN		0.616547263281	1.210991771675	0.3900082265390

Table 11: Comparison of exact solution and the approximate solution for **Problem** 4 with different Shooting formula

x	Exact Sol	NM at Q_4	HM at Q_3	MFOM at Q_3
0.0	-1.00000000000	-1.00000000000	-1.00000000000	-1.00000000000
0.2	-0.95301524228	-0.95554529218	-0.95554529219	-0.95554529218
0.4	-0.82464668378	-0.82976199340	-0.82976199341	-0.82976199342
0.6	-0.62113247951	-0.62678878434	-0.62678878438	-0.62678878438
0.8	-0.34555828316	-0.34857739986	-0.34857739989	-0.34857739986
1.0	000000000000	-0.00000000004	-0.00000000007	-0.00000000006

Table 12: Abolute Error for **Problem** 4 with different Shooting formula

x	NM at Q_6	HM at Q_3	MFOM at Q_3	Ruman and Islam 2024
0.0	0.00000000000	0.00000000000	0.00000000000	0.000000
0.2	0.00253004990	0.00253004991	0.00253004991	0.00219

0.4	0.00511530962	0.00511530963	0.00511530964	0.000816
0.6	0.00565630484	0.00565630487	0.00565630487	0.000297
0.8	0.00301911670	0.00301911674	0.00301911670	0.000460
1.0	0.00000000004	0.00000000007	0.00000000006	0.000000
MAE	0.00272013018	0.00272013020	0.00272013020	

Table 13: Relative Error for **Problem** 4 with different Shooting formula

x	NM at Q_6	HM at Q_3	MFOM at Q_4
0.2	-0.00265478430	-0.00265478431	-0.00265478431
0.4	-0.00620303182	-0.00620303184	-0.00620303185
0.6	-0.00910643868	-0.00910643873	-0.00910643874
0.8	-0.00873692470	-0.00873692480	-0.00873692470
MRE	-0.00667529486	-0.00667529492	-0.00667529490

Table 14: Mean Deviation Error for **Problem** 4 with different Shooting formula

x	NM at Q_4	HM at Q_3	MFOM at Q_3
0.0	0.002720130183	0.002720130203	0.002720130197
0.2	0.000190080280	0.000190080291	0.000190080289
0.4	0.002395179440	0.002395179430	0.002395179446
0.6	0.002936174652	0.002936174667	0.002936174674
0.8	0.000298986516	0.000298986532	0.000298986501
1.0	0.002720130145	0.002720130133	0.002720130137
MDE	0.001876780203	0.001876780209	0.001876780207

Q_t	NM	$\lvert Q_t - Q_{t-1}\rvert$	HM	$\lvert Q_t - Q_{t-1}\rvert$	MFOM	$\lvert Q_t - Q_{t-1}\rvert$
Q_0	1.00000000000		1.000000000000		1.000000000000	

Q_1	-0.90047211370	1.90047211370	0.174928706740	0.825071293260	0.230470431286	0.769529568714
Q_2	-0.35303907321	0.54743304049	0.000393208493	0.174535498247	0.004112022862	0.226358408424
Q_3	-0.04838540967	0.30465366354	-0.000052002057	0.000445210550	-0.000051979472	0.004164002334
Q_4	-0.00154060080	0.04684480888	-0.000052002057	0.000000000000	-0.000052002058	0.000000022586
Q_5	-0.00005350571	0.00148709509			-0.000052002058	0.000000000000
Q_6	-0.00005200206	0.00000150365				
Q_7	-0.00005200206	0.00000000000				

Table 15: Tolerance Value for **Problem** 4

Table 9 and 15 are tolerance table for problem 3 and 4 respectively, tolerance value is the absolute difference between two successive shooting angles. These tables show case the rate of convergence of the method. HM, MFOM demonstrate high order of rate of convergence. These methods reach values extremely close to zero at Q_3, after which no significant improvement is observed. General, the results confirm that while all the SG methods successfully converge to the targeted solution, the higher-order methods are more effective, requiring fewer iterations and exhibiting faster and more stable convergence than the classical Newton-based approaches.

6 Conclusion

To solve fractional-order boundary value problems without discretization error, a continuous shooting technique based on the Adomian decomposition method concept has been developed. Second-order NM and higher order (HM, MFOM) root-finding algorithms were added to the final scheme to improve the shooting angle. To evaluate the correctness of the method using metrics like absolute error and tolerance value, it was applied to a few numerical samples. The strategy performed well according to all criteria, but as the shooting angle was changed, the method's performance improved. Out of all the root-finding algorithms, the HM's results demonstrated the best performance. Because it converges faster than others. The continuous shooting technique based on the Adomian decomposition method helps to avoid discretization errors and provides results as polynomials to get solutions at all places; therefore, it is advised to adopt this strategy while using the shooting technique.

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