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Quotient Structures of Full Fuzzy Parameterized Soft Groups and Homomorphism Theorems

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ABSTRACT

The current work develops the homomorphic and quotient structures of Full Fuzzy Parameterized Soft Groups (FFPS-groups), with the objective of establishing well-defined kernel, image, quotient, and isomorphism results within the FFPS framework. We investigate FFPS-homomorphisms and their kernels and images, establish the preservation of normal FFPS-subgroups under FFPS-isomorphisms, and introduce a quotient FFPS-group constructed through a fixed normal subgroup to eliminate ambiguity arising from parameter-dependent quotient constructions. The results show that the kernel of an FFPS-homomorphism is a normal FFPS-subgroup, its image forms an FFPS-subgroup, and the canonical quotient map is a surjective FFPS-homomorphism whose kernel is the induced normal FFPS-subgroup. These results yield a well-defined First Isomorphism Theorem for FFPS-groups. Furthermore, under reverse parameter–subgroup monotonicity, the paper establishes that variations in fuzzy parameter strength control subgroup inclusion and may determine algebraic properties such as commutativity. Thus, FFPS-groups provide a parameter-sensitive extension of classical group structures in which subgroup behaviour and algebraic properties vary with fuzzy parameter strength.

1 Introduction

The study of uncertainty in real life problems has developed through several related frameworks, including fuzzy set theory, soft set theory, fuzzy soft set theory, fuzzy soft expert set theory, deterministic soft set theory and fuzzy parameterized soft set theory. The introduction of fuzzy set by (Zadeh, 1965) provided a mathematical basis for representing graded membership rather than binary membership, thereby allowing objects to belong to a set to varying degrees. In (Molodtsov, 1999) soft set theory was introduced complementing Zadeh fuzzy set. In soft set theory Molodtsov introduce parameterized approach to uncertainty providing a general mathematical tool for dealing with objects that are vague, uncertain or not sharply defined. The hybridization of fuzzy and soft set theories created a foundation for modelling uncertainty both through degree-based membership and through parameter-dependent approximation.

Following Molodtsov's work, some operations and applications of soft sets have been studied by other authors like (Chen et al., 2005), (Maji et al., 2002, 2003). Maji et al., (2003) treated a soft set as a parameterized family of subsets of a universe, thereby allowing objects to be approximated through selected parameters rather than through a single universal description. This development was important because it gave researchers a flexible way of handling uncertainty without necessarily requiring probability distributions or membership functions at every level. In a related work the gluing concept of soft set with expert system was initiated in (Alkhazaleh and Salleh, 2011) to emphasize the

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due status of the opinions of all experts regarding taking any decision in decision-making system. The authors in (Uduak, et al., 2025) proposed a new concept, deterministic soft set by employing an external factors which aid in giving a systematic direction in the area of decision making problem.

The combination of fuzzy sets and soft sets produced fuzzy soft set theory. In this framework, the parameter set may remain crisp, but the approximate values are fuzzy subsets of the universe. Maji et al., (2001) have introduced the concept of fuzzy soft set, as a more general concept, and studied its properties. Also in 2011, (Çağman et al., 2011) further developed fuzzy soft set theory and its applications, showing how fuzzy soft structures can support decision-making in problems involving uncertain information. Their work helped to strengthen the theoretical and applied significance of fuzzy soft sets, especially in contexts where alternatives are evaluated under multiple uncertain criteria. In (Alkhazaleh and Salleh, 2014) the authors characterized fuzzy soft expert set and its application.

The algebraic study of soft sets later gave rise to soft groups. Aktaş and Çağman (2007) introduced soft groups and examined the relationship between soft set theory and algebraic structures. Their work showed that soft set theory could be meaningfully applied to group theory, thereby opening the way for further developments in soft algebra. In 2011, (Sezgin and Atagün, 2011) corrected several properties of soft groups that had previously been introduced in (Aktaş and Çağman, 2007). They also introduced the notion of normalistic soft groups. Subsequently, Aslam and Qurashi, (2012) developed additional concepts in the theory of soft groups, including cyclic soft groups, abelian soft groups, factor soft groups, and several related properties. In 2014, Aktaş and Özlü, (2014) introduced the concept of the order of a soft group and investigated its fundamental properties. Later, in 2019, (Nazmal, 2019) examined the homomorphic images and preimages of soft groups under the soft mapping defined in (Zorlutuna et. al., 2012). In (Alajlan and Alghamdi, 2023) the authors investigated the concepts in the soft group theory such as the center of the soft group, the kernel of soft homomorphism and soft automorphisms with their basic properties, and the characteristic soft subgroups of a given soft point group.

A further development came with fuzzy soft groups. Aygünoğlu and Aygün (2009) introduced fuzzy soft groups and studied their structural properties, including fuzzy soft group behaviour, homomorphism-related issues and normal fuzzy soft groups. This contribution is particularly important to the present work because it moved fuzzy soft set theory from general uncertainty modelling into algebraic structure, especially group theory. Furthermore, (Akin, 2021) proposed the concept of multi-fuzzy soft group. A definition of homomorphism of multi-fuzzy soft groups was presented and it was shown that the image (or pre-image) of a multi-fuzzy soft group is a multi-fuzzy soft group.

Despite the usefulness of fuzzy soft groups, one limitation remained: in many fuzzy soft structures, the parameters themselves are not fully fuzzified. This motivated the development of fuzzy parameterized and fully fuzzy parameterized soft structures. Çağman, et. al., (2010) introduced fuzzy parameterized fuzzy soft set theory and fuzzy parameterized soft set theory (Çağman, et. al.,2011a), in which both parameter membership and approximate descriptions play significant roles. Their work contributed to a more expressive soft-set model because it treated the parameter level itself as a site of uncertainty.

The concept of Full Fuzzy Parameterized Soft Set was later introduced by (Edeghagba and Muhammad, 2022). Their work improved earlier fuzzy parameterized soft set frameworks by fully fuzzifying the parameter set of a soft set and studying basic operations on such structures. This was a major step because it made the parameter set more than a crisp indexing device; instead, parameters became fuzzy descriptors capable of carrying graded semantic information.

Muhammad, et. al., (2023) further studied Full Fuzzy Parameterized Soft Set Theory with decision-making applications. Their study examined algebraic operations on full fuzzy parameterized soft sets and demonstrated the usefulness of the concept in decision-making through an algorithm and numerical example. This shows that fully fuzzy parameterized soft structures are not only abstract mathematical objects but may also be useful in applied contexts where choices depend on graded parameter influence.In a related work, (Edeghagba, et. al., 2024) applied the framework of fully fuzzy parametrization to the notion of soft expert set with decision-making applications.

The most direct foundation for the present paper is the work on Fully Fuzzy Parameterized Soft Groups. Edeghagba, et. al., (2024) introduced the concept of Fully Fuzzy Parameterized Soft Group, extending full fuzzy parameterized soft set theory to group structures. The work considered definitions, joint-intersection, meet-union, FFPS-subgroups, external products, homomorphisms, isomorphisms and factoring of FFPS-groups.

The present paper builds on this emerging FFPS-group literature by focusing more deeply on homomorphic and quotient structures. In the manuscript under consideration, an FFPS-homomorphism is defined as a pair (ϕ, ψ), where $(\phi \to G_2)$ is a group homomorphism and $\psi: \tilde{A} \to \tilde{B}$ is an order-preserving parameter map satisfying an approximation compatibility condition. The same manuscript defines the kernel and image of an FFPS-homomorphism and states that, when parameter sets are singletons, the notions reduce to the classical kernel and image of group theory.

However, quotient structures in FFPS-groups require special care. In classical group theory, quotient groups are formed using a fixed normal subgroup. In FFPS-groups, the situation is more delicate because approximations are parameter-dependent. If a quotient approximation is defined using a parameter-dependent subgroup say, $h_{\tilde{C}}(\hat{c})$, then different choices of $\hat{c}$ may produce different quotient subgroups. This means that the quotient may fail to be well-defined unless the normal subgroup used in the quotient is fixed or chosen by a canonical rule. The present paper resolves this by defining the quotient through a fixed normal subgroup $N \trianglelefteq G$, so that each quotient approximation is given by $q_{\tilde{A}}(\hat{a}) = f_{\tilde{A}}(\hat{a})N/N$. Equivalently, $q_{\tilde{A}}(\hat{a}) = \pi\big(f_{\tilde{A}}(\hat{a})\big)$, where $\pi \to G/N$ is the natural projection.

This construction has several advantages. First, the universe of the quotient FFPS-group is clearly fixed as (G/N). Second, every approximation lies naturally inside the same quotient group. Third, the canonical quotient map $\big(\pi, \mathrm{id}\tilde{A}\big): F\tilde{A} \to F_{\tilde{A}}/N_{\tilde{A}}$ becomes well-defined. Fourth, its kernel is precisely the (N)-induced FFPS-subgroup, whose approximation is given by $n_{\tilde{A}}(\hat{a}) = f_{\tilde{A}}(\hat{a}) \cap N$. This refinement strengthens the theoretical basis for quotient FFPS-groups and enhances the formulation of the First Isomorphism Theorem in the FFPS setting.

Another important issue addressed in this paper concerns the relationship between parameters and algebraic properties. In classical group theory, properties such as commutativity, normality and subgroup containment are treated as absolute. A subgroup is either normal or not normal; it is either commutative or non-commutative. In FFPS-groups, however, such properties may vary according to fuzzy parameter strength. The manuscript already recognises that parameters in FFPS-groups are not just labels but semantic constraints that govern subgroup approximations. Thus, FFPS-groups permit algebraic properties to arise, weaken or collapse as parameter values change.

This paper therefore contributes to literature showing the theoretical link between FFPS-groups and classical homomorphism theorems by establishing the conditions under which quotient and image structures correspond. In doing so, the study extends the foundational FFPS-group framework and gives a more rigorous basis for future work on FFPS-isomorphism theorems, parameter-sensitive algebraic properties and applications of FFPS-groups in uncertainty-based algebraic modelling.

2 Preliminary Definitions

In this section, we provide some basic relevant definitions.

Definition 1. *(Zadeh, 1965)[Fuzzy Set]: Let U be a non-empty universe set. A fuzzy set on U is a pair (μ, U), where $\mu: U \to [0,1]$ is a membership function. For each $x \in U$, the value $\mu(x)$ is called the degree of membership of x in the fuzzy set.*

Definition 2. *(Molodtsov, 1999)[Soft Set]: Let U be an initial universe set and let E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U, and let $A \subseteq E$. A pair (F, A) is called a soft set over U if $F: A \to \mathcal{P}(U)$ is a mapping. Thus, for each parameter $a \in A$, $F(a)$ is a subset of U.*

Definition 3. *(Maji, et al., 2001)[Fuzzy Soft Set]: Let U be a universe set and let E be a set of parameters. Let I^U denote the collection of all fuzzy subsets of U, where $I = [0,1]$. If $A \subseteq E$, then a pair (f, A) is called a fuzzy soft set over U if $f: A \to I^U$ is a mapping. Hence, for each $a \in A$, $f(a) = f_a: U \to [0,1]$ is a fuzzy subset of U.*

Definition 4. *(Cagman, et al., 2011a)[Fuzzy Parameterized Soft Set]: Let U be an initial universe set, E be a set of parameters, and let A be a fuzzy subset of E with membership function $\mu_A: E \to [0,1]$. A fuzzy parameterized soft set, briefly called an FP-soft set, over U is defined as $F_A = \left\{\left(\frac{\mu_A(x)}{x}, \gamma_A(x)\right): x \in E,\ \mu_A(x) \in [0,1],\ \gamma_A(x) \in \mathcal{P}(U)\right\}$, where $\gamma_A: E \to \mathcal{P}(U)$ is called the approximate function. The approximate function satisfies $\gamma_A(x) = \varnothing$* whenever $\mu_A(x) = 0$. *The value $\mu_A(x)$ represents the degree of importance or relevance of the parameter x. Thus, in an FP-soft set, the parameter set is fuzzified, while the approximate values are crisp subsets of the universe U.*

Definition 5. *(Edeghagba and Muhammad, 2021)[Full Fuzzy Parameterized Soft Set]: Let U be an initial universe set and let E be a set of parameters. Let $\tilde{E}$ denote the collection of all fuzzy subsets of E, and let $\tilde{A} \subseteq \tilde{E}$. A Full Fuzzy Parameterized Soft Set, briefly called an FFP-soft set, over U is a structure $F_{\tilde{A}} = \left\{\left(\hat{y}, F_{\tilde{A}}(\hat{y})\right): \hat{y} \in \tilde{A},\ F_{\tilde{A}}(\hat{y}) \in \mathcal{P}(U)\right\}$, where $F_{\tilde{A}}: \tilde{A} \to \mathcal{P}(U)$ is the approximation function. Each fuzzy parameter $\hat{y} \in \tilde{A}$ is a fuzzy subset of E and is represented by $\hat{y} = \left(\frac{\mu_{\hat{y}}(x_1)}{x_1}, \frac{\mu_{\hat{y}}(x_2)}{x_2}, \dots, \frac{\mu_{\hat{y}}(x_n)}{x_n}\right)$, where $E = \{x_1, x_2, \dots, x_n\}$ and $\mu_{\hat{y}}: E \to [0,1]$ is the membership function of $\hat{y}$. The approximation function satisfies $F_{\tilde{A}}(\hat{y}) = \varnothing$* whenever $\mu_{\hat{y}}(x) = 0$ for all $x \in E$. *Hence, in a Full Fuzzy Parameterized Soft Set, the parameters themselves are fuzzy sets over E, and each fuzzy parameter determines an approximate subset of the universe U.*

Definition 6. *(Rosenfeld, 1971)[Fuzzy Subgroup]: Let G be a group. A fuzzy subset μ of G is called a fuzzy subgroup of G if the following conditions hold for all $x, y \in G$:*

1. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$;
2. $\mu(x^{-1}) \geq \mu(x)$.

Equivalently, in many formulations, the second condition may be written as $\mu(x^{-1}) = \mu(x)$ for all $x \in G$.

Definition 7. *(Aktas and Cagman, 2007)[Soft Group]: Let G be a group and let (F, A) be a soft set over G. Then (F, A) (F, A) is called a soft group over G if, for every $a \in A$, $F(a) \leq G$. That is, each approximation $F(a)$ is a subgroup of G.*

Definition 8. *(Jun and Song, 2009)[Fuzzy Soft Group]: Let G be a group and let (f, A) be a fuzzy soft set over G. Then (f, A) is called a fuzzy soft group over G if, for every $a \in A$, the fuzzy set $f_a: G \to [0,1]$ is a fuzzy subgroup of G. Equivalently, for all $a \in A$ and all $x, y \in G$, the following conditions hold: $f_a(xy) \geq \min\{f_a(x), f_a(y)\}$, and $f_a(x^{-1}) \geq f_a(x)$. Thus, each approximate value f_a is a fuzzy subgroup in the sense of Rosenfeld .*

Definition 9. *(Rosenfeld, 1971)[Fuzzy Normal Subgroup]: Let G be a group. A fuzzy subgroup μ of G is called a fuzzy normal subgroup of G if $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$. Equivalently, $\mu(xy) = \mu(yx)$ for all $x, y \in G$.*

Definition 10. *(Edeghagba, et al., 2025)[Full Fuzzy Parameterized Soft Group]: Let G be a group, E be a set of parameters, and let $\tilde{E}$ be the collection of all fuzzy subsets of E. Let $\tilde{A} \subseteq \tilde{E}$. A Full Fuzzy Parameterized Soft Set $F_{\tilde{A}}$ over G is called a Full Fuzzy Parameterized Soft Group, briefly an FFPS-group, if the following conditions hold:*

1. *$F_{\tilde{A}}$ is a Full Fuzzy Parameterized Soft Set over G.*
2. *For every fuzzy parameter $\hat{x} \in \tilde{A}$, $F_{\tilde{A}}(\hat{x}) \leq G$. That is, each approximation $F_{\tilde{A}}(\hat{x})$ is a subgroup of G.*
3. *If the parameter set E is itself equipped with a group structure, then each fuzzy parameter $\hat{y} \in \tilde{A}$ may also be required to be a fuzzy subgroup of E.*

Definition 11. *(Edeghagba, et al., 2025)[FFPS-Homomorphism] Let $F_{\tilde{A}} = \left(F, \tilde{A}\right)$ and $G_{\tilde{B}} = \left(G, \tilde{B}\right)$ be two FFPS-groups over the groups G_1 and G_2, with approximate functions $f_{\tilde{A}}$ and $g_{\tilde{B}}$ respectively. An FFPS-homomorphism from*

$F_{\tilde{A}}$ to $G_{\tilde{B}}$ is a pair $(\varphi,\psi): F_{\tilde{A}} \to G_{\tilde{B}}$, where $\varphi: G_1 \to G_2$ is a group homomorphism, $\psi: \tilde{A} \to \tilde{B}$ is a parameter order-preserving mapping such that for all $\hat{a}_1, \hat{a}_2 \in \tilde{A}$, $\hat{a}_1 \leq \hat{a}_2 \Rightarrow \psi(\hat{a}_1) \leq \psi(\hat{a}_2)$. and for all $\hat{a} \in \tilde{A}$,

$$\varphi\big(f_{\tilde{A}}(\hat{a})\big) \leq g_{\tilde{B}}\big(\psi(\hat{a})\big) \tag{1}$$

Therefore (φ,ψ) an FFPS-homomorphism.

3 Result

Definition 12. *Let $(\varphi,\psi): F_{\tilde{A}} \to G_{\tilde{B}}$ be an FFPS-homomorphism from $F_{\tilde{A}}$ to $G_{\tilde{B}}$. Then the image of (φ,ψ) is defined as the FFPS-set $\operatorname{Im}(\varphi,\psi) = I_{\psi(\tilde{A})}$ over the group G_2, with approximation function $i_{\psi(\tilde{A})}: \psi(\tilde{A}) \to \mathcal{P}(G_2)$ defined by $i_{\psi(\tilde{A})}\big(\psi(\hat{a})\big) = \varphi\big(f_{\tilde{A}}(\hat{a})\big)$, for all $\hat{a} \in \tilde{A}$. The image is denoted by $\operatorname{Im}(\varphi,\psi)$.*

If the parameter sets are singletons, then these notions coincide with the classical kernel and image.

Example 1. *Let $G_1 = \mathbb{Z}_6$ and $G_2 = \mathbb{Z}_3$ under addition and $F_{\tilde{A}}$ over G_1 and $G_{\tilde{B}}$ over G_2, where $\tilde{A} = \{\hat{a}_1, \hat{a}_2\}$ and $\tilde{B} = \{\hat{b}_1, \hat{b}_2\}$. Assume their approximation to be $f_{\tilde{A}}(\hat{a}_1) = \mathbb{Z}_6$ and $f_{\tilde{A}}(\hat{a}_2) = \{0,3\}$, and $g_{\tilde{B}}(\hat{b}_1) = \mathbb{Z}_3$ and $g_{\tilde{B}}(\hat{b}_2) = \{0\}$. Define $\varphi: G_1 \longrightarrow G_2$ by $\varphi(x) = x \bmod 3$ and define the parameter mapping $\psi: \tilde{A} \longrightarrow \tilde{B}$ with $\psi(\hat{a}_1) = \hat{b}_1$ and $\psi(\hat{a}_2) = \hat{b}_2$. Clearly, φ is a group homomorphism. Moreover, $\varphi\big(f_{\tilde{A}}(\hat{a}_1)\big) = \varphi(\mathbb{Z}_6) = \mathbb{Z}_3 = g_{\tilde{B}}(\hat{b}_1)$ while $\varphi\big(f_{\tilde{A}}(\hat{a}_2)\big) = \varphi(\{0,3\}) = \{0\} = g_{\tilde{B}}(\hat{b}_2)$. Hence (φ,ψ) is an FFPS-homomorphism. Now by Definition 12, the image $\operatorname{Im}(\varphi,\psi)$ has approximation function $i_{\tilde{B}}(\hat{b}) = \bigcup_{\substack{\hat{a}\in\tilde{A} \\ \psi(\hat{a})=\hat{b}}} \varphi\big(f_{\tilde{A}}(\hat{a})\big)$. It follows that $i_{\tilde{B}}(\hat{b}_1) = \varphi(\mathbb{Z}_6) = \mathbb{Z}_3$ and $i_{\tilde{B}}(\hat{b}_2) = \varphi(\{0,3\}) = \{0\}$. Hence $\operatorname{Im}(\varphi,\psi) = \{(\hat{b}_1, \mathbb{Z}_3), (\hat{b}_2, \{0\})\}$*

Proposition 1. *Let $(\varphi,\psi): F_{\tilde{A}} \to G_{\tilde{B}}$ be an FFPS-homomorphism. Then the kernel $\ker(\varphi,\psi)$ is a normal FFPS-subgroup of $F_{\tilde{A}}$.*

Proof. Given that the kernel of (φ,ψ) is the FFPS-set $\ker(\varphi,\psi) = K_{\tilde{A}}$, with approximation function $k_{\tilde{A}}: \tilde{A} \to \mathcal{P}(G_1)$, where G_1 is the universe set for $F_{\tilde{A}}$. Now consider that for all $\hat{a} \in \tilde{A}$, $k_{\tilde{A}}(\hat{a}) = \{x \in f_{\tilde{A}}(\hat{a}) \mid \varphi(x) = e_{G_2}\}$, then it is the case that $k_{\tilde{A}}(\hat{a}) = f_{\tilde{A}}(\hat{a}) \cap \ker\varphi$. Therefore, $k_{\tilde{A}}(\hat{a}) \leq f_{\tilde{A}}(\hat{a})$ for all $\hat{a} \in \tilde{A}$. Hence $\ker(\varphi,\psi)$ is an FFPS-subgroup of $F_{\tilde{A}}$. Next we show that $\ker(\varphi,\psi)$ is normal FFPS-subgroup of $F_{\tilde{A}}$. Suppose $x \in k_{\tilde{A}}(\hat{a})$, then $\varphi(x) = e_{G_2}$ for $\hat{a} \in \tilde{A}$. Let $g \in f_{\tilde{A}}(\hat{a})$ and since φ is a homomorphism

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} = \varphi(g)e_{G_2}\varphi(g)^{-1} = e_{G_2},$$

for $\varphi(x) = e_{G_2}$. Clearly, $gxg^{-1} \in f_{\tilde{A}}(\hat{a})$ since $gxg^{-1} \in \ker\varphi$ and that $f_{\tilde{A}}(\hat{a})$ is a subgroup of G_1. Then it is the case that $gxg^{-1} \in f_{\tilde{A}}(\hat{a}) \cap \ker\varphi = k_{\tilde{A}}(\hat{a})$. Then we have that for all $g \in f_{\tilde{A}}(\hat{a})$, $g\, k_{\tilde{A}}(\hat{a})\, g^{-1} = k_{\tilde{A}}(\hat{a})$, showing that $k_{\tilde{A}}(\hat{a})$ is a normal subgroup of $f_{\tilde{A}}(\hat{a})$ for every $\hat{a} \in \tilde{A}$. This concludes the proof that $\ker(\varphi,\psi)$ is a normal FFPS-subgroup of $F_{\tilde{A}}$. □

Proposition 2. *Let $(\varphi,\psi): F_{\tilde{A}} \to G_{\tilde{B}}$ be an FFPS-homomorphism. Then the image $\operatorname{Im}(\varphi,\psi)$ is an FFPS-subgroup of $G_{\tilde{B}}$.*

Proof. Suppose the image of the FFPS-homomorphism (φ,ψ) is the FFPS-set $\operatorname{Im}(\varphi,\psi) = I_{\tilde{B}}$ with approximation function $i_{\tilde{B}}: \tilde{B} \to \mathcal{P}(G_2)$, where G_2 is the universe set for $G_{\tilde{B}}$. Let $i_{\tilde{B}}(\hat{b})$ be define as

$$i_{\tilde{B}}(\hat{b}) = \begin{cases} \varphi\big(f_{\tilde{A}}(\hat{a})\big), & \text{if } \hat{b} = \psi(\hat{a}) \text{ for some } \hat{a} \in \tilde{A}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

for all $\hat{b} \in \tilde{B}$. In the first case we show that $i_{\tilde{B}}(\hat{b})$ is a subgroup of G_2. Consider $\hat{b} \in \tilde{B}$ such that no $\hat{a} \in \tilde{A}$ for which $\psi(\hat{a}) = \hat{b}$, thus $i_{\tilde{B}}(\hat{b}) = \varnothing$. This holds in an FFPS-set nonetheless. Suppose $i_{\tilde{B}}(\hat{b}) \neq \emptyset$ then the set $\varphi\big(f_{\tilde{A}}(\hat{a})\big)$ is a subgroup of G_2, since $\varphi: G_1 \to G_2$ is a homomorphism and $f_{\tilde{A}}(\hat{a})$ is a subgroup of G_1. Hence $i_{\tilde{B}}(\hat{b})$ is a subgroup of G_2. Next we show that $i_{\tilde{B}}(\hat{b}) \le g_{\tilde{B}}(\hat{b})$ for all $\hat{b} \in \tilde{B}$. Choose $\hat{a} \in \tilde{A}$ such that $\psi(\hat{a}) = \hat{b}$. Therefore, $\varphi(f_{\tilde{A}}(\hat{a})) \le g_{\tilde{B}}(\psi(\hat{a})) = g_{\tilde{B}}(\hat{b})$, by the conditions of FFPS-homomorphism. Then it follows that $i_{\tilde{B}}(\hat{b}) \le g_{\tilde{B}}(\hat{b})$. Thus, for all $\hat{b} \in \tilde{B}$, there exists $\hat{a} \in \tilde{A}$ with $\psi(\hat{a}) = \hat{b}$ where $i_{\tilde{B}}(\hat{b}) \le g_{\tilde{B}}(\hat{b})$, then $\mathrm{Im}(\varphi, \psi) \le G_{\tilde{B}}$. It then follows that $\mathrm{Im}(\varphi, \psi)$ is an FFPS-subgroup of $G_{\tilde{B}}$. □

Definition 13. *Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be FFPS-groups over the groups G_1 and G_2, respectively, and let $(\varphi, \psi): F_{\tilde{A}} \to G_{\tilde{B}}$ be an FFPS-homomorphism.*

1. *If the mapping $\varphi: G_1 \to G_2$ is a monomorphism and the parameter mapping $\psi: \tilde{A} \to \tilde{B}$ is injective then an FFPS-homomorphism (φ, ψ) is called an FFPS-monomorphism. This implies, $F_{\tilde{A}}$ is FFPS-embedded in $G_{\tilde{B}}$.*

2. *If the mapping $\varphi: G_1 \to G_2$ is a epimorphism and the parameter mapping $\psi: \tilde{A} \to \tilde{B}$ is surjective then an FFPS-homomorphism (φ, ψ) is called an FFPS-epimorphism. This implies, that every approximation in $G_{\tilde{B}}$ is induced by some approximation in $F_{\tilde{A}}$,*

3. *If an FFPS-homomorphism (φ, ψ) is both an FFPS-monomorphism and an FFPS-epimorphism, then it is called an FFPS-isomorphism. This means the mapping $\varphi: G_1 \to G_2$ is a bijective, the parameter mapping $\psi: \tilde{A} \to \tilde{B}$ is bijective and for all $\hat{a} \in \tilde{A}$,*
$$\varphi(f_{\tilde{A}}(\hat{a})) = g_{\tilde{B}}(\psi(\hat{a})) \qquad (2)$$
holds. This is denoted by $F_{\tilde{A}} \cong G_{\tilde{B}}$ and we say that $F_{\tilde{A}}$ and $G_{\tilde{B}}$ are FFPS-isomorphic.

Equation (2) is the approximation compatibility equation.

Generally, the order of parameter sets needs not be equal as presented but in the definition of FFPS-isomorphic this is a requirement.

Example 2. *Let $G_1 = \mathbb{Z}_4$ under addition and $G_2 = \{1, -1, i, -i\}$ under multiplication and $F_{\tilde{A}}$ over G_1 and $G_{\tilde{B}}$ over G_2, where $\tilde{A} = \{\hat{a}_1, \hat{a}_2\}$ and $\tilde{B} = \{\hat{b}_1, \hat{b}_2\}$. Assume their approximation to be $f_{\tilde{A}}(\hat{a}_1) = \mathbb{Z}_4$ and $f_{\tilde{A}}(\hat{a}_2) = \{0,2\}$, and $g_{\tilde{B}}(\hat{b}_1) = \{1, i, -1, -i\}$ and $g_{\tilde{B}}(\hat{b}_2) = \{1, -1\}$. Define $\varphi: G_1 \longrightarrow G_2$ by $\varphi(x) = i^n$ where $n \in \mathbb{Z}_4$ and the parameter mapping $\psi: \tilde{A} \longrightarrow \tilde{B}$ by $\psi(\hat{a}_1) = \hat{b}_1$ and $\varphi(\hat{a}_2) = \hat{b}_2$. Clearly, φ is a bijective group homomorphism, and ψ is bijective. Then, $\varphi(f_{\tilde{A}}(\hat{a}_1)) = \varphi(\mathbb{Z}_4) = \{1, i, -1, -i\} = g_{\tilde{B}}(\psi(\hat{a}_1))$ while $\varphi(f_{\tilde{A}}(\hat{a}_2)) = \varphi(\{0,2\}) = \{0\} = g_{\tilde{B}}(\psi(\hat{a}_2))$. Hence, $\varphi(f_{\tilde{A}}(\hat{a})) = g_{\tilde{B}}(\psi(\hat{a}))$ for all $\hat{a} \in \tilde{A}$. Since both φ and ψ are injective, (φ, ψ) is an FFPS-monomorphism. Since both mappings are also surjective, (φ, ψ) is an FFPS-epimorphism.*

Proposition 3. *Suppose $(\varphi, \psi): F_{\tilde{A}} \to G_{\tilde{B}}$ is an FFPS-isomorphism, and $H_{\tilde{C}}$ is a normal FFPS-subgroup of $F_{\tilde{A}}$, then $(\varphi, \psi)(H_{\tilde{C}})$ is a normal FFPS-subgroup of $G_{\tilde{B}}$.*

Proof. Suppose (φ, ψ) is an FFPS-isomorphism, then for all $\hat{a} \in \tilde{A}$ $\varphi(f_{\tilde{A}}(\hat{a})) = g_{\tilde{B}}(\psi(\hat{a}))$. Let $H_{\tilde{C}}$ be a normal FFPS-subgroup of $F_{\tilde{A}}$ and define $(\varphi, \psi)(H_{\tilde{C}}) = K_{\psi(\tilde{C})}$ where $k_{\psi(\tilde{C})}: \psi(\tilde{C}) \to \mathcal{P}(G_2)$ is given as $k_{\psi(\tilde{C})}(\psi(\hat{c})) = \varphi(h_{\tilde{C}}(\hat{c}))$, for $\hat{c} \in \tilde{C}$. In the first case, since φ is an isomorphism and $h_{\tilde{C}}(\hat{c})$ is a subgroup of G_1 for all $\hat{c} \in \tilde{C}$, then it follows that $k_{\psi(\tilde{C})}(\psi(\hat{c})) \le G_2$. Also, by definition since $H_{\tilde{C}}$ is an FFPS-subgroup of $F_{\tilde{A}}$ it follows that for all $\hat{c} \in \tilde{C}$ there exists $\hat{a} \in \tilde{A}$ with $\hat{a} \ge \hat{c}$ such that $h_{\tilde{C}}(\hat{c}) \le f_{\tilde{A}}(\hat{a})$. Then

$$K_{\psi(\tilde{C})}\left(\psi(\tilde{C})\right) = \varphi(h_{\tilde{C}}(\hat{c})) \le \varphi(f_{\tilde{A}}(\hat{a}))$$

$$\le g_{\tilde{B}}(\psi(\hat{a}))(by\ equation\ 2)$$

where $\psi(\hat{a}) \geq \psi(\hat{c})$. Therefore, $(\varphi,\psi)(H_{\tilde{C}}) \leq G_{\tilde{B}}$. Next by hypothesis $H_{\tilde{C}}$ is a normal FFPS-subgroup of $F_{\tilde{A}}$, for each $\hat{c} \in \tilde{C}$ there exists $\hat{a} \in \tilde{A}$ with $\hat{a} \geq \hat{c}$ such that $g\, h_{\tilde{C}}(\hat{c})\, g^{-1} = h_{\tilde{C}}(\hat{c})$ for all $g \in f_{\tilde{A}}(\hat{a})$. Hence for $x_2 \in g_{\tilde{B}}(\psi(\hat{a}))$ and φ is surjective, there exists $g \in f_{\tilde{A}}(\hat{a})$ such that $\varphi(g) = y$. Let $x_1 \in h_{\tilde{C}}(\hat{c})$, then $x_2\varphi(x_1)x_2^{-1} = \varphi(g)\varphi(x_1)\varphi(g)^{-1} = \varphi(gx_1g^{-1}) \in \varphi(h_{\tilde{C}}(\hat{c}))$. Therefore, $x_2k_{\psi(\tilde{C})}(\psi(\hat{c}))x_2^{-1} = k_{\psi(\tilde{C})}(\psi(\hat{c}))$, for all $x_2 \in g_{\tilde{B}}(\psi(\hat{a}))$. Thus, $k_{\psi(\tilde{C})}(\psi(\hat{c}))$ is a normal subgroup of $g_{\tilde{B}}(\psi(\hat{a}))$ for every $\hat{c} \in \tilde{C}$. Finally, $(\varphi,\psi)(H_{\tilde{C}})$ is a normal FFPS-subgroup of $G_{\tilde{B}}$. □

Definition 14. *Let $F_{\tilde{A}}$ be an FFPS-group over a group G, with approximation function $f_{\tilde{A}}: \tilde{A} \to \mathcal{P}(G)$. Let $N \trianglelefteq G$ be a fixed normal subgroup of G. For each $\hat{a} \in \tilde{A}$, define $n_{\tilde{A}}(\hat{a}) = f_{\tilde{A}}(\hat{a}) \cap N$. Then $N_{\tilde{A}}$, with approximation function $n_{\tilde{A}}$, is called the N-induced normal FFPS-subgroup of $F_{\tilde{A}}$. The quotient FFPS-group of $F_{\tilde{A}}$ by $N_{\tilde{A}}$, denoted by $F_{\tilde{A}}/N_{\tilde{A}}$, is the FFPS-set over the quotient group G/N, with parameter set $\tilde{A}$, and approximation function $q_{\tilde{A}}: \tilde{A} \to \mathcal{P}(G/N)$ defined by $q_{\tilde{A}}(\hat{a}) = \pi(f_{\tilde{A}}(\hat{a})) = f_{\tilde{A}}(\hat{a})N/N$, where $\pi: G \to G/N$ is the natural projection.*

Remark 1. *For each $\hat{a} \in \tilde{A}$, $q_{\tilde{A}}(\hat{a}) = f_{\tilde{A}}(\hat{a})N/N \cong f_{\tilde{A}}(\hat{a})/(f_{\tilde{A}}(\hat{a}) \cap N)$. Thus, the quotient is defined by a fixed normal subgroup $N \trianglelefteq G$.*

Proposition 4. *Let $F_{\tilde{A}}$ be an FFPS-group over a group G, $N_{\tilde{A}}$ a normal FFPS-subgroup of $F_{\tilde{A}}$ and $N \trianglelefteq G$. Then the quotient $F_{\tilde{A}}/N_{\tilde{A}}$ is an FFPS-group over G/N.*

Proof. Let the map $\pi: G \to G/N$ be the natural projection. By definition each approximation $f_{\tilde{A}}(\hat{a})$ in $F_{\tilde{A}}$ is a subgroup of G. Therefore, $q_{\tilde{A}}(\hat{a}) = \pi(f_{\tilde{A}}(\hat{a}))$ is a subgroup of G/N, because the image of a subgroup under a group homomorphism is a subgroup. Hence, $q_{\tilde{A}}(\hat{a}) \leq G/N$ for every $\hat{a} \in \tilde{A}$. Next we show that $F_{\tilde{A}}/N_{\tilde{A}}$ is an FFPS-group. Assume that $\hat{a}_1 \leq \hat{a}_2$ and since $F_{\tilde{A}}$ satisfies reverse monotonicity, it follows that $f_{\tilde{A}}(\hat{a}_2) \leq f_{\tilde{A}}(\hat{a}_1)$. Therefore, by the definition of π, then we have $\pi(f_{\tilde{A}}(\hat{a}_2)) \leq \pi(f_{\tilde{A}}(\hat{a}_1))$. Therefore, $q_{\tilde{A}}(\hat{a}_2) \leq q_{\tilde{A}}(\hat{a}_1)$. Hence, the quotient approximation function $q_{\tilde{A}}$ also satisfies reverse monotonicity. Consequently, $F_{\tilde{A}}/N_{\tilde{A}}$ is an FFPS-group over G/N. □

Corollary 1. *Let $F_{\tilde{A}}$ be an FFPS-group over G and $N_{\tilde{A}}$ a normal FFPS-subgroup of $F_{\tilde{A}}$ for $N \trianglelefteq G$. Then the canonical quotient map $(\pi, \mathrm{id}_{\tilde{A}}): F_{\tilde{A}} \to F_{\tilde{A}}/N_{\tilde{A}}$ is a surjective FFPS-homomorphism, and* $\ker(\pi, \mathrm{id}_{\tilde{A}}) = N_{\tilde{A}}$.

Corollary 1 is the canonical FFPS-Homomorphism.

Proof. For each $\hat{a} \in \tilde{A}$, $\pi(f_{\tilde{A}}(\hat{a})) = q_{\tilde{A}}(\hat{a})$. Thus, $\pi(f_{\tilde{A}}(\hat{a})) \leq q_{\tilde{A}}(\hat{a})$, so $(\pi, \mathrm{id}_{\tilde{A}})$ is an FFPS-homomorphism. Clearly, this is surjective onto the quotient approximations by construction, since $q_{\tilde{A}}(\hat{a}) = \pi(f_{\tilde{A}}(\hat{a}))$. Finally, the kernel approximation at $\hat{a}$ is

$$\ker(\pi, \mathrm{id}_{\tilde{A}})(\hat{a}) = \{x \in f_{\tilde{A}}(\hat{a}): \pi(x) = N\}.$$

But $\pi(x) = N \quad \Leftrightarrow \quad x \in N$. Therefore,$\ker(\pi, \mathrm{id}_{\tilde{A}})(\hat{a}) = f_{\tilde{A}}(\hat{a}) \cap N = n_{\tilde{A}}(\hat{a})$. Hence, $\ker(\pi, \mathrm{id}_{\tilde{A}}) = N_{\tilde{A}}$. □

Example 3. *We consider the symmetric group S_3 on the set $\{1,2,3\}$ and let $G = S_3$ Then $S_3 = \{e, (12), (13), (23), (123), (132)\}$. Consider the alternating group $A_3 = \{e, (123), (132)\}$ which is a normal subgroup of S_3 and let $N = A_3$. Therefore, $N = A_3$ is a fixed normal subgroup of $G = S_3$. Let $\tilde{A} = \{\hat{a}_0, \hat{a}_1, \hat{a}_2\}$ be a fuzzy parameter set with the order $\hat{a}_0 \leq \hat{a}_1 \leq \hat{a}_2$. Consider the approximation function $f_{\tilde{A}}: \tilde{A} \to \mathcal{P}(S_3)$ defined by $f_{\tilde{A}}(\hat{a}_0) = S_3$, $f_{\tilde{A}}(\hat{a}_1) = A_3$ and $f_{\tilde{A}}(\hat{a}_2) = \{e\}$. Thus $F_{\tilde{A}} = \{S_3, A_3, \{e\}\}$. Clearly, since $\hat{a}_0 \leq \hat{a}_1 \leq \hat{a}_2$, and $f_{\tilde{A}}(\hat{a}_2) \leq f_{\tilde{A}}(\hat{a}_1) \leq f_{\tilde{A}}(\hat{a}_0)$, then $F_{\tilde{A}}$ is an FFPS-group over S_3. Next, since $N = A_3 \trianglelefteq S_3$, we define the N-induced normal FFPS-subgroup $N_{\tilde{A}}$ by $n_{\tilde{A}}(\hat{a}) = f_{\tilde{A}}(\hat{a}) \cap N$. Then $n_{\tilde{A}}(\hat{a}_0) = f_{\tilde{A}}(\hat{a}_0) \cap A_3 = S_3 \cap A_3 = A_3$, $n_{\tilde{A}}(\hat{a}_1) = f_{\tilde{A}}(\hat{a}_1) \cap A_3 = A_3 \cap A_3 = A_3$, and $n_{\tilde{A}}(\hat{a}_2) = f_{\tilde{A}}(\hat{a}_2) \cap A_3 = \{e\} \cap A_3 = \{e\}$. Thus, $N_{\tilde{A}} = \{A_3, \{e\}\}$.*

Therefore, the quotient group is $G/N = S_3/A_3 = \{A_3, (12)A_3\}$ and $\pi: S_3 \to S_3/A_3$ is the natural projection. The approximate function $q_{\tilde{A}}: \tilde{A} \to \mathcal{P}(S_3/A_3)$ is defined by $q_{\tilde{A}}(\hat{a}) = \pi(f_{\tilde{A}}(\hat{a})) = f_{\tilde{A}}(\hat{a})A_3/A_3$.

For $\hat{a}_0$, we have $q_{\tilde{A}}(\hat{a}_0) = f_{\tilde{A}}(\hat{a}_0)A_3/A_3 = S_3A_3/A_3 = S_3/A_3$. Therefore, $q_{\tilde{A}}(\hat{a}_0) = \{A_3, (12)A_3\}$.

Next, for $\hat{a}_1$, we have $q_{\tilde{A}}(\hat{a}_1) = f_{\tilde{A}}(\hat{a}_1)A_3/A_3 = A_3A_3/A_3 = A_3/A_3$. Therefore, $q_{\tilde{A}}(\hat{a}_1) = \{A_3\}$.

Lastly, we consider for $\hat{a}_2$, we have $q_{\tilde{A}}(\hat{a}_2) = f_{\tilde{A}}(\hat{a}_2)A_3/A_3 = \{e\}A_3/A_3 = A_3/A_3$. Therefore, $q_{\tilde{A}}(\hat{a}_2) = \{A_3\}$.

Since $\hat{a}_0 \leq \hat{a}_1 \leq \hat{a}_2$, and $q_{\tilde{A}}(\hat{a}_2) = \{A_3\} \leq \{A_3\} = q_{\tilde{A}}(\hat{a}_1) \leq S_3/A_3 = q_{\tilde{A}}(\hat{a}_0)$, the quotient approximation function satisfies reverse monotonicity. Hence, $F_{\tilde{A}}/N_{\tilde{A}} = \{\{A_3, (12)A_3\}, \{A_3\}\}$ is an FFPS-group over S_3/A_3. Next the canonical quotient map $(\pi, \mathrm{id}_{\tilde{A}}): F_{\tilde{A}} \to F_{\tilde{A}}/N_{\tilde{A}}$. defined by $\pi\left(f_{\tilde{A}}(\hat{a})\right) = q_{\tilde{A}}(\hat{a})$ for every $\hat{a} \in \tilde{A}$. Clearly, $(\pi, \mathrm{id}_{\tilde{A}})$ is a surjective FFPS-homomorphism.
Furthermore, for each $\hat{a} \in \tilde{A}$, $\ker(\pi, \mathrm{id}_{\tilde{A}})(\hat{a}) = \{x \in f_{\tilde{A}}(\hat{a}): \pi(x) = A_3\}$. But $\pi(x) = A_3 \iff x \in A_3$. Therefore,$\ker(\pi, \mathrm{id}_{\tilde{A}})(\hat{a}) = f_{\tilde{A}}(\hat{a}) \cap A_3 = n_{\tilde{A}}(\hat{a})$. Hence, $\ker(\pi, \mathrm{id}_{\tilde{A}}) = N_{\tilde{A}}$.

Remark 2. *Suppose the quotient approximation used depends on an arbitrary choice of $\hat{c} \in \tilde{C}$, then using the same parameter $\hat{a}_0$, with the choice $h_{\tilde{C}}(\hat{c}_1) = A_3$, then $f_{\tilde{A}}(\hat{a}_0)/h_{\tilde{C}}(\hat{c}_1) = S_3/A_3$, which is of order 2, and the choice $h_{\tilde{C}}(\hat{c}_2) = \{e\}$, then $f_{\tilde{A}}(\hat{a}_0)/h_{\tilde{C}}(\hat{c}_2) = S_3/\{e\} \cong S_3$, which is of order 6.*

Thus, the same approximation $f_{\tilde{A}}(\hat{a}_0) = S_3$ may produce two different quotient groups: S_3/A_3 and *$S_3/\{e\}$. Therefore, the quotient is not well-defined unless the normal subgroup used for the quotient is fixed.*

Theorem 1 (First Isomorphism Theorem). *Let $(\varphi, \psi): F_{\tilde{A}} \to G_{\tilde{B}}$ be an FFPS-homomorphism from FFPS-group $F_{\tilde{A}}$ into the FFPS-group $G_{\tilde{B}}$ over the groups G_1 and G_2, respectively and $(\pi, \mathrm{id}_{\tilde{A}}): F_{\tilde{A}} \to F_{\tilde{A}}/\ker(\varphi, \psi)$, be the FFPS-quotient map. Then there is an FFPS-isomorphism $\overline{(\varphi, \psi)}: F_{\tilde{A}}/\ker(\varphi, \psi) \to \mathrm{Im}(\varphi, \psi)$ such that the diagram commutes*

$$\begin{tikzcd}
F_{\tilde{A}} \arrow[rr, "\varphi_\psi"] \arrow[d, "(\pi_{\mathrm{id}_{\tilde{A}}})"'] & & \mathrm{Im}(\varphi, \psi) \\
F_{\tilde{A}}/\ker(\varphi, \psi) \arrow[urr, "\overline{(\varphi_\psi)}"'] & &
\end{tikzcd}$$

i.e. so that $(\varphi, \psi) = \overline{(\varphi, \psi)} \circ (\pi, \mathrm{id}_{\tilde{A}})$, where $\varphi_\psi = (\varphi, \psi)$, $\pi_{\mathrm{id}_{\tilde{A}}} = (\pi, \mathrm{id}_{\tilde{A}})$ and $\overline{(\varphi_\psi)} = \overline{(\varphi, \psi)}$.Therefore, $F_{\tilde{A}}/\ker(\varphi, \psi) \cong \mathrm{Im}(\varphi, \psi)$.

Proof. By Proposition 1, the kernel $\ker(\varphi, \psi)$ is a normal FFPS-subgroup of $F_{\tilde{A}}$ and by Proposition 4 $F_{\tilde{A}}/\ker(\varphi, \psi)$ is a quotient FFPS-group and is well define. Next, let the mapping $\overline{\varphi}: G_1/\ker\varphi \to \varphi(G_1)$ be given by $\overline{\varphi}(g \ker\varphi) = \varphi(g)$. This map is well defined, since $g_1\ker\varphi = g_2\ker\varphi$ implies $\varphi(g_1) = \varphi(g_2)$. Define $\overline{\psi}: \tilde{A} \to \psi(\tilde{A})$ by $\overline{\psi}(\hat{a}) = \psi(\hat{a})$. Therefore, for each $\hat{a} \in \tilde{A}$, $\overline{\varphi}\left(q_{\tilde{A}}(\hat{a})\right) = \varphi\left(f_{\tilde{A}}(\hat{a})\right) \subseteq g_{\tilde{B}}\left(\psi(\hat{a})\right) = g_{\tilde{B}}\left(\overline{\psi}(\hat{a})\right)$. Thus $\overline{(\varphi, \psi)}$ is an FFPS-homomorphism. Therefore, by the classical First Isomorphism the map $\overline{\varphi}$ is bijective and $\overline{\psi}$ is bijective onto $\psi(\tilde{A})$, then $\overline{(\varphi, \psi)}$ is an FFPS-isomorphism. Hence, $(\varphi, \psi) = \overline{(\varphi, \psi)} \circ (\pi, \mathrm{id}_{\tilde{A}})$ and this completes the proof. □

Axiom 1. *Let $\tilde{F}_A$ be an FFPS-group over a group G, and $(\tilde{A}, \leq)$ be a partially ordered set of fuzzy parameters. Then the approximation function $f_{\tilde{A}}: \tilde{A} \to \mathcal{P}(G)$ satisfies the following properties:*

1. *For all $\hat{a}_1, \hat{a}_2 \in \tilde{A}$, $\hat{a}_1 \leq \hat{a}_2 \Rightarrow f_{\tilde{A}}(\hat{a}_2) \leq f_{\tilde{A}}(\hat{a}_1)$, (Reverse monotonicity).*
2. *If $\{e_G\} \in F_{\tilde{A}}$, then there exists a maximal element $\hat{a}_{\max} \in \tilde{A}$ such that $f_{\tilde{A}}(\hat{a}_{\max}) = \{e_G\}$.(Maximal trivial parameter)*
3. *For all $\hat{a} \in \tilde{A}$ with $\hat{a} < \hat{a}_{\max}$, $f_{\tilde{A}}(\hat{a}) \gneqq \{e_G\}$. (Strict enlargement below the maximum)*

Proposition 5. *Let (φ, ψ) be an FFPS-homomorphism from $F_{\tilde{A}}$ to $G_{\tilde{B}}$ FFPS-groups over groups G_1 and G_2 respectively. Suppose that the approximation function $f_{\tilde{A}}: \tilde{A} \to \mathcal{P}(G_1)$ satisfies the reverse monotonicity condition $\hat{a} \leq \hat{b} \Longrightarrow f_{\tilde{A}}(\hat{b}) \leq f_{\tilde{A}}(\hat{a})$, for all $\hat{a}, \hat{b} \in \tilde{A}$.*

Then the following statements hold:

1. *for $\hat{a} \in \tilde{A}$ such that $f_{\tilde{A}}(\hat{a}) \leq ker\varphi$, and $\hat{a} \leq \hat{b}$ then $f_{\tilde{A}}(\hat{b}) \leq ker\varphi$.*
2. *for $\hat{x}, \hat{y} \in \tilde{A}$, $\hat{x} \leq \hat{y} \Longrightarrow \varphi(f_{\tilde{A}}(\hat{y})) \leq \varphi(f_{\tilde{A}}(\hat{x}))$.*

Proof. 1. Define

$$\mathcal{K}_{\varphi} = \{\hat{x} \in \tilde{A}: f_{\tilde{A}}(\hat{x}) \leq \ker\phi\}.$$

Let $\hat{a} \in \mathcal{K}_{\phi}$. By definition of $\mathcal{K}_{\varphi}$, it follows that $f_{\tilde{A}}(\hat{a}) \leq K_{\varphi}$. Next, given that $\hat{a} \leq \hat{b}$ and $f_{\tilde{A}}(\hat{a}) \leq \ker\phi$ and since $f_{\tilde{A}}$ satisfies reverse monotonicity, $f_{\tilde{A}}(\hat{b}) \leq f_{\tilde{A}}(\hat{a})$, then it follows that $f_{\tilde{A}}(\hat{b}) \leq \ker\phi$. Hence,$\hat{b} \in \mathcal{K}_{\varphi}$.

2. Next, let $\hat{x}, \hat{y} \in \tilde{A}$ and given that $\hat{x} \leq \hat{y}$. By reverse monotonicity, $f_{\tilde{A}}(\hat{y}) \leq f_{\tilde{A}}(\hat{x})$. Since $\varphi: G_1 \to G_2$ is a group homomorphism, the image of a subgroup under φ is a subgroup, and inclusion of subgroups is preserved under φ. Therefore, $\varphi(f_{\tilde{A}}(\hat{y})) \leq \varphi(f_{\tilde{A}}(\hat{x}))$. Hence, $\hat{x} \leq \hat{y} \Longrightarrow \varphi(f_{\tilde{A}}(\hat{y})) \leq \varphi(f_{\tilde{A}}(\hat{x}))$, and this completes the proof. □

Proposition 6. *Suppose $F_{\tilde{A}}$ is an FFPS-group over a non-commutative group G with parameter set E and $\tilde{A}$ be a fuzzy parameter set on E. If $\hat{a} \leq \hat{b}$ in $\tilde{A}$, such that $\hat{a}(e) \leq 1$ and $\hat{b}(e) = 1$, then $f_{\tilde{A}}(\hat{b})$ is a commutative subgroup of G while $f_{\tilde{A}}(\hat{a})$ is a non-commutative subgroup of G, where e represent commutativity.*

Proof. Assume $\hat{b}(e) = 1$, then by the fuzzy value it follows that $f_{\tilde{A}}(\hat{b})$, the approximated subgroup of G by $\hat{b}$ is a commutative subgroup such that $uv = vu$ for all $u, v \in f_{\tilde{A}}(\hat{b})$. By definition since $\hat{a} \leq \hat{b}$, it implies that $f_{\tilde{A}}(\hat{b}) \leq f_{\tilde{A}}(\hat{a})$, therefore, since $\hat{a}(e) \leq 1$ it is the case that there are non-commuting elements in $f_{\tilde{A}}(\hat{a})$ such that $uv \neq vu$ for some $u, v \in f_{\tilde{A}}(\hat{a})$, implying that $f_{\tilde{A}}(\hat{a})$ is a non-commutative subgroup of G. This completes the proof. □

Clearly, Proposition 6 shows that commutativity depends on parameters, not just elements. Therefore, comparing FFPS-group to classical, soft, or fuzzy soft groups, FFPS-groups allow algebraic properties to arise or collapse uniquely through variation in parameter strength.

Corollary 2. *Let $F_{\tilde{A}}$ is an FFPS-group over a non-commutative group G. Then there exist a threshold $\alpha \in (0,1)$ such that for a fuzzy parameter $\hat{a} \in \tilde{A}$ over the parameter $e \in E$, $\hat{a}(e) \geq \alpha \Rightarrow f_{\tilde{A}}(\hat{a})$ is a commutative subgroup of G and* $\hat{a}(e) < \alpha \Rightarrow f_{\tilde{A}}(\hat{a})$ is a non-commutative subgroup of G.

Proof. By Proposition 6, the commutativity of an approximated subgroup depends on the strength of the fuzzy parameter associated with commutativity. Hence, there exists a threshold value, say α, such that when the membership degree of a fuzzy parameter satisfies $\hat{a}(e) \geq \alpha$, the corresponding approximation $f_{\tilde{A}}(\hat{a})$ is commutative. On the other hand, when $\hat{a}(e) < \alpha$, the weaker commutativity condition allows a larger approximation which may contain non-commuting elements, and hence $f_{\tilde{A}}(\hat{a})$ is non-commutative. Therefore, α serves as a threshold separating the commutative and non-commutative approximations. □

Example 4. *Let $G = S_3$ be the symmetric group of degree three, $E = \{e_1, e_2, e_3\}$ be the set of parameters with e_1 representing normality, e_2 representing commutativity, and e_3 representing centerness, and $\tilde{A}$ is a finite subset of $\tilde{E}$ ($\tilde{A} \tilde{\subseteq} \tilde{E}$), and where $\tilde{E}$ is a collection of all fuzzy subsets of E. For $\hat{a}, \hat{b} \in \tilde{A}$ define fuzzy set $\hat{a} = \{\frac{1}{e_1}, \frac{1}{e_2}, \frac{1}{e_3}\}$ and $\hat{b} = \{\frac{0.3}{e_1}, \frac{1}{e_2}, \frac{0.35}{e_3}\}$. Obviously, $\hat{b} \leq \hat{a}$. Given the approximation function $f_{\tilde{A}}(\hat{a}) = \{e\}$ and $f_{\tilde{A}}(\hat{b}) = \{e, (1,2)\}$. Clearly, the choice of subgroups by the approximation depends on the fuzzy values of the fuzzy parameters. In this case fuzzy parameter $\hat{a}$ assigns full membership degree to all the given* *algebraic properties represented by the parameters and so the identity element in S_3 is the only element that satisfies these properties to this degrees. On the other hand fuzzy parameter $\hat{b}$ assigns lesser membership to all algebraic properties represented by the parameters, therefor by the reverse-order property of the approximation function, it follows that the subgroup approximation of $\hat{b}$, $f_{\tilde{A}}(\hat{b})$ must be a supergroup of $f_{\tilde{A}}(\hat{a})$. Thus weakening the parameter reduces the algebraic properties while increasing the subgroup size.*

Hence FFPS-groups are parameter controlled.

4 Conclusion

This paper examined homomorphism, kernel, image and quotient structures in Full Fuzzy Parameterized Soft Groups. Building on existing developments in fuzzy sets, soft sets, fuzzy soft groups and fully fuzzy parameterized soft structures, the study showed that FFPS-groups provide a flexible algebraic framework in which subgroup behaviour is controlled by fuzzy parameters.
The study observed that defining quotient approximations through arbitrary choices of parameter-dependent normal subgroups can lead to ambiguity, because different parameters may generate different quotient structures. To avoid this problem, the paper proposed quotient construction based on a fixed normal subgroup $N \trianglelefteq G$. Under this approach, each quotient approximation is defined by $q_{\tilde{A}}(\hat{a}) = f_{\tilde{A}}(\hat{a})N/N$, and the quotient FFPS-group is formed over the fixed quotient group G/N.

The paper also introduced the canonical FFPS-quotient map and showed that its kernel is the N-induced normal FFPS-subgroup. This result strengthens the foundation for developing isomorphism theorems in FFPS-group theory. In particular, it supports a cleaner formulation of the First Isomorphism Theorem by ensuring that the quotient object is well-defined before being compared with the image of an FFPS-homomorphism.

Overall, the results demonstrate that FFPS-groups are not merely extensions of classical groups, soft groups or fuzzy soft groups. Rather, they provide a richer algebraic environment in which the structure of approximated subgroups depends on fuzzy parameter values. This makes FFPS-groups suitable for further theoretical investigation and possible applications in areas where algebraic structure, uncertainty, graded properties and parameter-based approximation occur simultaneously. Future studies may extend these results to Second and Third Isomorphism Theorems.

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