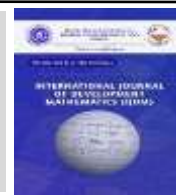




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An Optimized Computational Method for Solving Quadratic Riccati Differential Equations

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ABSTRACT

This study presents the derivation, analysis and implementation of an Optimized Computational Method (OCM) for the numerical solution of Quadratic Riccati Differential Equations (QRDEs). The OCM was developed using a power series polynomial as the basis function through interpolation and collocation techniques with an optimized mesh point introduced within a one-step interval. The optimization procedure was employed to determine the new mesh point, thereby enhancing the approximation capability of the resulting numerical scheme. Fundamental properties of the method, including order, consistency, zero-stability, convergence and absolute stability were investigated. The analysis revealed that an OCM possesses a uniform order of seven, confirming its high accuracy and small truncation error. Furthermore, the method was shown to be consistent, zero-stable and convergent according to the Dahlquist convergence criterion. Stability analysis using the boundary locus technique established that the method is A_α -stable. To validate its performance, an OCM was applied to three test problems involving QRDEs and the results were compared with existing methods reported in the literature. The numerical solutions obtained by an OCM showed excellent agreement with the exact solutions throughout the integration interval. Comparative analysis demonstrated that an OCM consistently outperformed the existing methods in terms of accuracy and computational reliability. The results confirm that an OCM is an efficient, stable and highly accurate numerical technique for solving QRDEs arising in scientific and engineering applications.

1 Introduction

The Riccati differential equation (RDE), named after Count Jacopo Francesco Riccati (1676–1754), is a nonlinear ordinary differential equation with extensive applications across scientific and engineering disciplines. It has a profound link to the one-dimensional static Schrödinger equation and serves as a crucial tool in modeling random processes, optimal control, diffusion problems and stochastic realization theory (Nasr Al-Din, 2020a). The equation's versatility extends to modern fields such as financial mathematics, network synthesis and nonlinear partial differential equations, particularly in solving solitary wave problems where solutions are expressed as polynomials in elementary functions that satisfy a projective Riccati equation (Nasr Al-Din, 2020b; Vinod and Dimple, 2016). Over time, its scope has broadened, encompassing various physical phenomena like spring-mass oscillations, electrical circuits, beam bending, chemical reactions, pendulum dynamics and rotational motion (Baghchehjoughi et al., 2014). Researchers

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such as Baghchehjoughi et al. (2014) have emphasized its role as a unifying model for both classical and modern systems, demonstrating its enduring relevance in applied mathematics and physics.

Due to its nonlinear nature, solving the Riccati equation remains a central challenge, drawing significant attention from mathematicians and engineers. The equation's solution process often parallels that of second-order inhomogeneous linear differential equations, where obtaining one particular solution leads to the general form (Mısıır, 2023). Consequently, numerous analytical and numerical methods have been proposed to address it effectively. Tan and Abbasbandy (2018) utilized the Homotopy Analysis Method (HAM), Batiha (2015) developed the multistage variational iteration method, while File and Aga (2016) applied the fourth-order Runge-Kutta method. Additional techniques include the modified Laplace Adomian decomposition method (Vinod and Dimple, 2016), fifth-order predictor-corrector and Bezier curve methods (Ghomanjani and Khorram, 2015) and Genocchi polynomial-based algorithms (Ghomanjani and Shateyi, 2020). Recent advances also involve the application of Legendre scaling functions for converting Riccati problems into algebraic systems (File and Aga, 2016). Despite some methodological limitations, these approaches have significantly improved the accuracy and efficiency of solving the Quadratic Riccati Differential Equation (QRDE), reaffirming its importance in areas such as robust stabilization, stochastic realization theory and modern financial modeling (Riaz et al., 2015; Vahidi and Didgar, 2018).

The quadratic Riccati differential equation, categorized as an ordinary differential equation (ODE) of the form;

$$\frac{d\gamma}{d\xi} = a(\xi)\gamma(t) + b(\xi)\gamma^2(\xi) + c(\xi), \gamma(\xi_0) = \gamma_0 \quad (1.1)$$

The quadratic Riccati differential equation (1.1) is consider in this study.

where $\gamma(\xi)$ is the unknown function and $a(\xi)$, $b(\xi)$, $c(\xi)$ are continuous functions with $c(\xi) \neq 0$ and ξ_0, γ_0 are arbitrary constants of ξ . The term $c(\xi)\gamma^2(\xi)$ named for its quadratic term, this equation poses analytical challenges, often lacking closed-form solution.

The QRDE as expressed in equation (1.1) can alternatively be represented by the following equation:

$$\frac{d\gamma}{d\xi} = f(\xi, \gamma) \quad (1.2)$$

It's crucial to emphasize that equation (1.1) must adhere to the existence and uniqueness theorem outlined as follows:

The Variational Iteration Method (VIM) is a robust analytical technique employed to solve both linear and nonlinear differential equations, including ordinary and partial types, by iteratively refining an initial approximation until convergence is achieved, making it highly effective for approximating solutions to Quadratic Riccati Differential Equations (QRDEs) (Batiha et al., 2017). Based on linearization and progressive refinement, VIM provides rapid convergence and a systematic framework for tackling complex nonlinear problems in mathematics, physics and engineering (Ghorbani and Momani, 2020). Building on this foundation, the Multistage Variational Iteration Method (MVIM) extends VIM by introducing multiple stages of refinement to enhance accuracy and stability in solving nonlinear equations (Hashim et al., 2016). Specifically tailored for QRDEs, MVIM uses successive iterations that build on previous results to achieve faster convergence and higher precision, although this improvement demands more computational effort and careful selection of step sizes and initial approximations to maintain efficiency (Ghorbani and Momani, 2020; Riaz et al., 2021).

Beyond VIM and MVIM, other analytical and numerical methods have also been developed to tackle QRDEs effectively. These include the Differential Transform Method (DTM), which converts differential equations into algebraic forms for easier computation (Sunday, 2017; Kamoh et al., 2020) and the Fourth-Order Runge-Kutta Method (RK4), known for its accuracy and computational balance (File and Aga, 2016). The Chebyshev Wavelets Method (CWM) employs orthogonal wavelets for localized function representation (Wase et al., 2021; Jain, 2014), while the Hybrid Functions and Tau Method (HFTM) combines hybrid orthogonal functions with the Tau technique for efficient algebraic transformation (Vahidi and Didgar, 2018). Additionally, methods such as the Non-Standard Finite Difference Method (NSFDM) (Sunday et al., 2017; Riaz et al., 2015), Homotopy Type Methods (HTMs) (Odibat and Momani, 2014; Saed, 2020) and the Adomian Decomposition Method (ADM) (Agbata et al., 2022; Wenjin and Yanni, 2020) have further enriched the solution landscape for QRDEs. Collectively, these methodologies underscore ongoing advancements aimed at improving accuracy, convergence and stability in solving nonlinear differential equations (Man et al., 2019; Agbata et al., 2021, 2022).

2 Methodology

2.1 Derivation of an Optimized Computational Method (OCM)

An Optimized Computational Method (OCM) was derived using a power series polynomial as the basis function, through interpolation and collocation procedures by optimizing one point on one step interval. The optimized mesh point are $\mu, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$.

The polynomial

$$\gamma(\xi) = h \sum_{i=0}^{\rho+\varphi-1} \alpha_i \xi^i \quad (2.1)$$

is considered as an approximate solution. The first differentiation of (2.1) give

$$\frac{d\gamma}{d\xi} = h \sum_{i=0}^{\rho+\varphi-1} i \alpha_i \xi^{i-1} \quad (2.2)$$

Where $\alpha \in \mathfrak{R}$, ρ, φ are distinct number of interpolation and collocation and $\gamma(\xi)$ is continuously differential. Let the solution of (1.2) be sought on the integration interval $[u, v]$ with a constant step-size h defined by $h = \xi_{n+1} - \xi_n, n = 0, 1, \dots, N$.

Substituting equation (2.2) in (1.2) gives,

$$f(\xi, \gamma) = h \sum_{i=0}^{\rho+\varphi-1} i \alpha_i \xi^{i-1} \quad (2.3)$$

In order to obtain the exact value of an optimized point, evaluate

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 1 \\ 0 & \mu & \frac{(\frac{1}{3})^2}{2!} & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{2}{3})^2}{2!} & \frac{(\frac{5}{6})^2}{2!} & \frac{(1)^2}{2!} \\ 0 & \frac{(\mu)^2}{2!} & \frac{(\frac{1}{3})^3}{3!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{2}{3})^3}{3!} & \frac{(\frac{5}{6})^3}{3!} & \frac{(1)^3}{3!} \\ 0 & \frac{(\mu)^3}{3!} & \frac{(\frac{1}{3})^4}{4!} & \frac{(\frac{1}{2})^4}{4!} & \frac{(\frac{2}{3})^4}{4!} & \frac{(\frac{5}{6})^4}{4!} & \frac{(1)^4}{4!} \\ 0 & \frac{(\mu)^4}{4!} & \frac{(\frac{1}{3})^5}{5!} & \frac{(\frac{1}{2})^5}{5!} & \frac{(\frac{2}{3})^5}{5!} & \frac{(\frac{5}{6})^5}{5!} & \frac{(1)^5}{5!} \\ 0 & \frac{(\mu)^5}{5!} & \frac{(\frac{1}{3})^6}{6!} & \frac{(\frac{1}{2})^6}{6!} & \frac{(\frac{2}{3})^6}{6!} & \frac{(\frac{5}{6})^6}{6!} & \frac{(1)^6}{6!} \end{bmatrix}^{-1} \begin{bmatrix} \frac{(1)^1}{1!} \\ \frac{(1)^2}{2!} \\ \frac{(1)^3}{3!} \\ \frac{(1)^4}{4!} \\ \frac{(1)^5}{5!} \\ \frac{(1)^6}{6!} \\ \frac{(1)^7}{7!} \end{bmatrix} \quad (2.4)$$

To obtain

$$\begin{bmatrix} 1 & -\frac{71}{84(6\mu-5)(\mu-1)(3\mu-1)(2\mu-1)(3\mu-2)} & \frac{3}{280} \frac{511\mu-52}{3\mu-1} & -\frac{2}{105} \frac{413\mu-29}{2\mu-1} & \frac{9}{140} \frac{203\mu-17}{3\mu-2} & -\frac{6}{35} \frac{77\mu-5}{6\mu-5} & \frac{1}{168} \frac{77\mu-6}{\mu-1} \end{bmatrix}^T \quad (2.5)$$

Simplify equation (2.5) on (2.4) one by one to obtained

$$\left. \begin{aligned} & \frac{6}{35(6\mu-5)}(77\mu-5) - \frac{9}{140(3\mu-2)}(203\mu-17) + \frac{2}{105(2\mu-1)}(413\mu-29) - \frac{3}{280(3\mu-1)}(511\mu-52) - \frac{1}{168} \frac{77\mu-6}{\mu-1} + \frac{71}{84(2\mu-1)(3\mu-1)(3\mu-2)(6\mu-5)(\mu-1)} + \frac{1}{2} \\ & \frac{1}{7(6\mu-5)}(77\mu-5) - \frac{3}{70(3\mu-2)}(203\mu-17) + \frac{1}{105(2\mu-1)}(413\mu-29) - \frac{1}{280(3\mu-1)}(511\mu-52) - \frac{1}{168} \frac{77\mu-6}{\mu-1} + \frac{71}{84(2\mu-1)(3\mu-1)(3\mu-2)(6\mu-5)(\mu-1)} + \frac{1}{6} \\ & \frac{5}{84(6\mu-5)}(77\mu-5) - \frac{1}{70(3\mu-2)}(203\mu-17) + \frac{1}{420(2\mu-1)}(413\mu-29) - \frac{1}{1680(3\mu-1)}(511\mu-52) - \frac{1}{336} \frac{77\mu-6}{\mu-1} + \frac{71}{168(2\mu-1)(3\mu-1)(3\mu-2)(6\mu-5)(\mu-1)} + \frac{1}{24} \\ & \frac{25}{1512(6\mu-5)}(77\mu-5) - \frac{1}{315(3\mu-2)}(203\mu-17) + \frac{1}{2520(2\mu-1)}(413\mu-29) - \frac{1}{15120(3\mu-1)}(511\mu-52) - \frac{1}{1008} \frac{77\mu-6}{\mu-1} + \frac{71}{504(2\mu-1)(3\mu-1)(3\mu-2)(6\mu-5)(\mu-1)} + \frac{1}{120} \\ & \frac{125}{36288(6\mu-5)}(77\mu-5) - \frac{1}{1890(3\mu-2)}(203\mu-17) + \frac{1}{20160(2\mu-1)}(413\mu-29) - \frac{1}{181440(3\mu-1)}(511\mu-52) - \frac{1}{4032} \frac{77\mu-6}{\mu-1} + \frac{71}{2016(2\mu-1)(3\mu-1)(3\mu-2)(6\mu-5)(\mu-1)} + \frac{1}{720} \\ & \frac{125}{217728(6\mu-5)}(77\mu-5) - \frac{1}{141775(3\mu-2)}(203\mu-17) + \frac{1}{201600(2\mu-1)}(413\mu-29) - \frac{1}{2721600(3\mu-1)}(511\mu-52) - \frac{1}{20160} \frac{77\mu-6}{\mu-1} + \frac{71}{10080(2\mu-1)(3\mu-1)(3\mu-2)(6\mu-5)(\mu-1)} + \frac{1}{5040} \\ & \frac{625}{7838208(6\mu-5)}(77\mu-5) - \frac{1}{127575(3\mu-2)}(203\mu-17) + \frac{1}{2419200(2\mu-1)}(413\mu-29) - \frac{1}{48988800(3\mu-1)}(511\mu-52) - \frac{1}{120960} \frac{77\mu-6}{\mu-1} + \frac{71}{60480(2\mu-1)(3\mu-1)(3\mu-2)(6\mu-5)(\mu-1)} + \frac{1}{40320} \end{aligned} \right\} \quad (2.6)$$

Evaluate equation (2.6) to get

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6531840}(71\mu-5) \end{bmatrix}^T \quad (2.7)$$

Equate equation (2.7) to zero and solve for exact value of μ as

$$\frac{1}{6531840}(71\mu-5) = 0 \quad (2.8)$$

Therefore the optimized point is $\mu = \frac{5}{71}$.

Now, interpolate equation (2.1) at optimized point, μ and collocate equation (2.3) at points

$0, \frac{5}{71}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$ to give,

$AX = B$

$$(2.9)$$

Where

$$A = \begin{bmatrix} 1 & \xi_{n+\frac{5}{71}} & \xi_{n+\frac{5}{71}}^2 & \xi_{n+\frac{5}{71}}^3 & \xi_{n+\frac{5}{71}}^4 & \xi_{n+\frac{5}{71}}^5 & \xi_{n+\frac{5}{71}}^6 & \xi_{n+\frac{5}{71}}^7 \\ 0 & 1 & 2\xi_n & 3\xi_n^2 & 4\xi_n^3 & 5\xi_n^4 & 6\xi_n^5 & 7\xi_n^6 \\ 0 & 1 & 2\xi_{n+\frac{5}{71}} & 3\xi_{n+\frac{5}{71}}^2 & 4\xi_{n+\frac{5}{71}}^3 & 5\xi_{n+\frac{5}{71}}^4 & 6\xi_{n+\frac{5}{71}}^5 & 7\xi_{n+\frac{5}{71}}^6 \\ 0 & 1 & 2\xi_{n+\frac{1}{3}} & 3\xi_{n+\frac{1}{3}}^2 & 4\xi_{n+\frac{1}{3}}^3 & 5\xi_{n+\frac{1}{3}}^4 & 6\xi_{n+\frac{1}{3}}^5 & 7\xi_{n+\frac{1}{3}}^6 \\ 0 & 1 & 2\xi_{n+\frac{1}{2}} & 3\xi_{n+\frac{1}{2}}^2 & 4\xi_{n+\frac{1}{2}}^3 & 5\xi_{n+\frac{1}{2}}^4 & 6\xi_{n+\frac{1}{2}}^5 & 7\xi_{n+\frac{1}{2}}^6 \\ 0 & 1 & 2\xi_{n+\frac{2}{3}} & 3\xi_{n+\frac{2}{3}}^2 & 4\xi_{n+\frac{2}{3}}^3 & 5\xi_{n+\frac{2}{3}}^4 & 6\xi_{n+\frac{2}{3}}^5 & 7\xi_{n+\frac{2}{3}}^6 \\ 0 & 1 & 2\xi_{n+\frac{5}{6}} & 3\xi_{n+\frac{5}{6}}^2 & 4\xi_{n+\frac{5}{6}}^3 & 5\xi_{n+\frac{5}{6}}^4 & 6\xi_{n+\frac{5}{6}}^5 & 7\xi_{n+\frac{5}{6}}^6 \\ 0 & 1 & 2\xi_{n+1} & 3\xi_{n+1}^2 & 4\xi_{n+1}^3 & 5\xi_{n+1}^4 & 6\xi_{n+1}^5 & 7\xi_{n+1}^6 \end{bmatrix}$$

$$X = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7]^T, B = \left[y_{n+\frac{5}{71}}, f_n, f_{n+\frac{5}{71}}, f_{n+\frac{1}{3}}, f_{n+\frac{1}{2}}, f_{n+\frac{2}{3}}, f_{n+\frac{5}{6}}, f_{n+1} \right]^T$$

Solving (2.9) at interpolation and collocation points to get the continuous form of OCM as,

$$y(\xi) = \alpha_{\frac{5}{71}}(\xi)y_{n+\frac{5}{71}} + h \left[\beta_0(\xi)f_n + \beta_{\frac{5}{71}}(\xi)f_{n+\frac{5}{71}} + \beta_{\frac{1}{3}}(\xi)f_{n+\frac{1}{3}} + \beta_{\frac{1}{2}}(\xi)f_{n+\frac{1}{2}} + \beta_{\frac{2}{3}}(\xi)f_{n+\frac{2}{3}} + \beta_{\frac{5}{6}}(\xi)f_{n+\frac{5}{6}} + \beta_1(\xi)f_{n+1} \right] \quad (2.10)$$

Where

$$\left. \begin{aligned} \alpha_{\frac{5}{71}}(\xi) &= 1 \\ \beta_0(\xi) &= \frac{1}{4200}(41 + 4282\tau - 39567\tau^2 + 130140\tau^3 - 181368\tau^4 + 92016\tau^5)(\tau-1)^2 \\ \beta_{\frac{5}{71}}(\xi) &= -\frac{128100283921}{651398748000}(1 + 2\tau - 67\tau^2 + 270\tau^3 - 408\tau^4 + 216\tau^5)(\tau-1)^2 \\ \beta_{\frac{1}{3}}(\xi) &= \frac{9}{7840}(-233 - 466\tau - 5949\tau^2 + 58218\tau^3 - 121320\tau^4 + 76680\tau^5)(\tau-1)^2 \\ \beta_{\frac{1}{2}}(\xi) &= -\frac{4}{6405}(211 + 422\tau - 9867\tau^2 + 126144\tau^3 - 319230\tau^4 + 230040\tau^5)(\tau-1)^2 \\ \beta_{\frac{2}{3}}(\xi) &= \frac{27}{35560}(-199 - 358\tau - 5787\tau^2 + 63684\tau^3 - 183000\tau^4 + 153360\tau^5)(\tau-1)^2 \\ \beta_{\frac{5}{6}}(\xi) &= -\frac{36}{11375}(71 + 142\tau - 207\tau^2 + 5520\tau^3 - 15318\tau^4 + 15336\tau^5)(\tau-1)^2 \\ \beta_1(\xi) &= \frac{1}{4620}(246 + 246\tau + 1121\tau^2 - 1165\tau^3 + 45501\tau^4 - 69180\tau^5 + 38340\tau^6)(\tau-1)^2 \end{aligned} \right\} \quad (2.11)$$

where $\tau = \frac{\xi - \xi_n}{h}$.

Evaluating (2.10) at the points $\xi = \left[\frac{5}{71}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \right]$ to gives new OCM as

$$\begin{aligned}
& \left[\begin{array}{c} \gamma_{n+\frac{5}{71}} \\ \gamma_{n+\frac{1}{3}} \\ \gamma_{n+\frac{1}{2}} \\ \gamma_{n+\frac{2}{3}} \\ \gamma_{n+\frac{5}{6}} \\ \gamma_{n+1} \end{array} \right] = \gamma_n + \left[\begin{array}{c} 619996594255 \\ 21520847698728 \\ 437 \\ 68040 \\ 53 \\ 13440 \\ 233 \\ 42525 \\ 745 \\ 217728 \\ 41 \\ 4200 \end{array} \right] f_n + \left[\begin{array}{c} 16601702203 \\ 369994488864 \\ 6020713344287 \\ 31657979152800 \\ 128100283921 \\ 694825331200 \\ 3714908233709 \\ 19786236970500 \\ 3714908233709 \\ 20261106657792 \\ 128100283921 \\ 651398748000 \end{array} \right. \\
& \left. \begin{array}{c} 1633472656875 \\ 200861245188128 \\ 5387 \\ 23520 \\ 38511 \\ 125440 \\ 2579 \\ 8820 \\ 23125 \\ 75264 \\ 2097 \\ 7840 \end{array} \right. \begin{array}{c} 1398585537500 \\ 164096463702801 \\ 65972 \\ 518805 \\ 929 \\ 51240 \\ 42496 \\ 518805 \\ 34375 \\ 830088 \\ 844 \\ 6405 \end{array} \begin{array}{c} 4636156381875 \\ 911049219246152 \\ 20729 \\ 320040 \\ 24489 \\ 568960 \\ 5191 \\ 40005 \\ 263125 \\ 1024128 \\ 4833 \\ 35560 \end{array} \begin{array}{c} 19573208748 \\ 11657125836811 \\ 404 \\ 20475 \\ 261 \\ 18200 \\ 256 \\ 11375 \\ 311 \\ 6552 \\ 2556 \\ 11375 \end{array} \begin{array}{c} 4686944375 \\ 19727443723834 \\ 1493 \\ 561330 \\ 37 \\ 18480 \\ 793 \\ 280665 \\ 625 \\ 1796256 \\ 41 \\ 770 \end{array} \right] \left[\begin{array}{c} f_{n+\frac{5}{71}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{5}{6}} \\ f_{n+1} \end{array} \right] \quad (2.12)
\end{aligned}$$

3 Basic Properties of an Optimized Computational Method (OCM)

In this section, the basic properties of an Optimized Computational Method (OCM) was numerically analyzed. These properties are order and error constant, consistency, convergence, zero-stability and region of absolute stability.

3.1 Order and Error Constant of OCM

Consider the linear operator $\ell\{\gamma(\xi); h\}$ associated with OCM be defined as,

$$\ell\{\gamma(\xi); h\} = \alpha_{\frac{5}{71}}(\xi) \gamma_{n+\frac{5}{71}} + h \sum_{j=0}^2 \beta_j(\xi) f_{n+j}, \quad j = 0, \frac{5}{71}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \quad (3.1)$$

Expanding (3.1) by Taylor series and comparing the coefficients of h yield,

$$\ell\{\gamma(\xi); h\} = c_0 \gamma(\xi) + c_1 h \gamma'(\xi) + c_2 h^2 \gamma''(\xi) + \dots + c_p h^p \gamma^{(p)}(\xi) + \dots c_{p+1} h^{p+1} \gamma^{(p+1)}(\xi) + \dots \quad (3.2)$$

Definition 3.1: let L be the linear operator with the related continuous method (3.1) are supposed to be of order p if

$c_0 = c_1 = c_2 = \dots = c_p = 0$ and $c_{p+1} \neq 0$. c_{p+1} is refers to the error constant and the truncation error is known by

$$t_{n+k} = c_{p+1} h^{(p+1)}(\xi_n) + O(h^{p+2}) \quad (3.3)$$

Now, according to Sunday, (2017),

$$\begin{aligned}
\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \bar{c}_5 = \bar{c}_6 = \bar{c}_7 = 0, \\
\bar{c}_8 = [7.8458 \times 10^{-10} \quad -7.1737 \times 10^{-09} \quad -5.8600 \times 10^{-09} \quad -7.0177 \times 10^{-09} \quad -5.0075 \times 10^{-09} \quad -1.2630 \times 10^{-09}] \quad (3.4)
\end{aligned}$$

Therefore, an OCM is of uniform order seven.

3.2. Consistency of OCM

Now that the order $p \geq 7$, therefore, OCM is consistent (Sunday, 2017).

3.3. Zero stability and Convergence of OCM

Definition 3.2: The of an OCM is expected to be zero-stable, if the roots $z_s, s = 1, 2, \dots, k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(zA - E)$ satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h \rightarrow 0$, $\rho(z) = z^{r-\mu}(z-1)^\mu$ where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ and E , (Sunday, 2017).

Then our method,

$$\rho(z) = z \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] - \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] = 0$$

$\rho(z) = z^5(z-1) = 0$. Therefore, OCM is zero-stable. By the spirit of Dahlquist, it is convergent.

3.4 Absolute Stability of an OCM

Definition 3.3: According to Sunday and Philip, (2018) the boundary locus method was used to obtain the stability polynomial as

$$\begin{aligned} \bar{h}(w) = & \left(-\frac{11}{3220560} w^5 + \frac{5}{3864672} w^6 \right) h^6 + \left(-\frac{229}{3864672} w^6 - \frac{481}{4830840} w^5 \right) h^5 + \left(-\frac{3499}{2147040} w^5 + \frac{7627}{6441120} w^6 \right) h^4 \\ & + \left(-\frac{4583}{322056} w^6 - \frac{701}{40257} w^5 \right) h^3 + \left(-\frac{13081}{107352} w^5 + \frac{11605}{54} w^6 \right) h^2 + \left(\frac{725}{1491} w^6 - \frac{766}{1491} w^5 \right) h - w^5 + w^6 \end{aligned} \quad (3.5)$$

The polynomial (3.5) was used to plot the region as

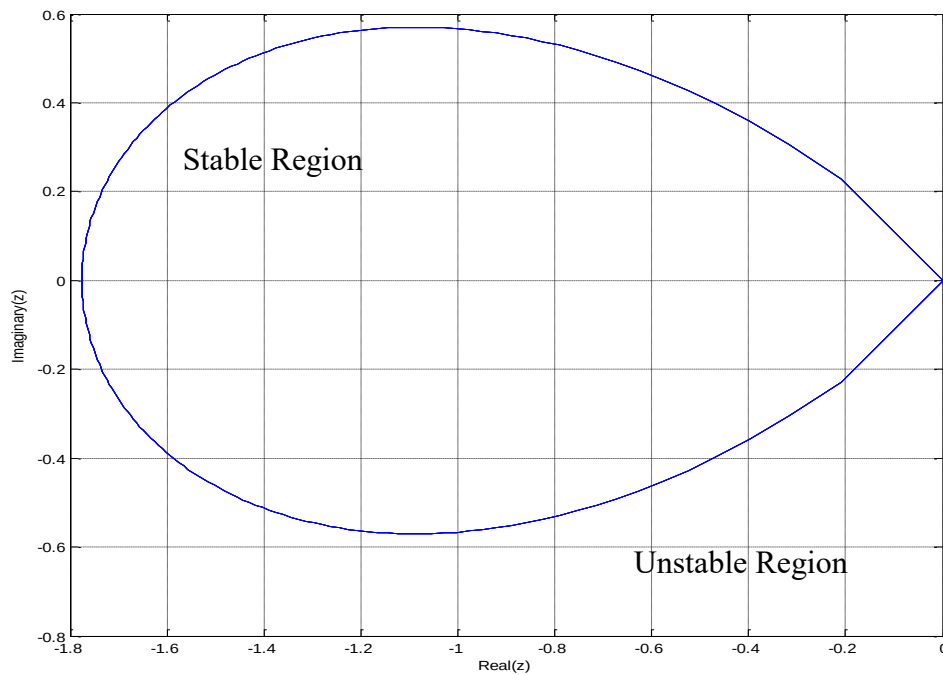


Figure 3.1: The Stability Region of an OCM.

4 Numerical Simulation of an Optimized Computational Method (OCM)

An Optimized Computational Method (OCM) was used to solve Quadratic Riccati Differential Equations (QRDEs) of the form (1.1). The corresponding numerical results presented are tabulated to show a comparison with existing methods in terms of errors. The following acronyms were used in tables.

Acronyms	Meaning
ξ	Point of Evaluation for time
EOCM	Error in an Optimized Computational Method
ES17	Error in Sunday, (2017)
ESP18	Error in Sunday and Philip, (2018)
EKS19	Error in Kashkari and Syam, (2019)
EKe21	Error in Kamoh, et al. (2021)

EI20

Error in Isah *et al.* (2020)**Example 4.1**

Consider the QRDE of the form,

$$\frac{d\gamma}{d\xi} = \gamma^2(\xi) - 1, \gamma(0) = 0 \quad (4.1)$$

whose exact solution is given by,

$$\gamma(\xi) = -\tanh(\xi) \quad (4.2)$$

Source: (Sunday, 2017; Sunday & Philip, 2018).

Example 4.2

Consider the QRDE of the form,

$$\frac{d\gamma}{d\xi} = -10[\gamma(\xi) - 1]^2, \gamma(0) = 2, h = 0.01 \quad (4.3)$$

with the exact solution given by

$$\gamma(\xi) = 1 + \frac{1}{1 + 10\xi} \quad (4.4)$$

Source: (Kamoh, *et al.*, 2021; Kashkari and Syam, 2019).**Example 4.3**

Consider the QRDE of the form,

$$\frac{d\gamma}{d\xi} = 4 - 4\gamma(\xi) + \gamma^2(\xi) = 0, \gamma(0) = 1 \quad (4.5)$$

Whose exact solution is

$$\gamma(\xi) = \frac{2\xi - 1}{\xi - 1} \quad (4.6)$$

Source: (Isah *et al.*, 2020).

Table 4.1: Numerical Results of Example 4.1

ξ	Exact Solution	Computed Solution	EOCM	ES17	ESP18
0.100	-0.09966799462495581712	-0.09966799462495581590	1.22000e-18	1.14769e-14	6.93889e-17
0.200	-0.19737532022490400074	-0.19737532022490399904	1.70000e-18	6.71407e-14	8.32667e-17
0.300	-0.29131261245159090582	-0.29131261245159090455	1.27000e-18	1.83409e-13	0.00000e00
0.400	-0.37994896225522488527	-0.37994896225522488492	3.50000e-19	3.38563e-13	2.22045e-16
0.500	-0.46211715726000975850	-0.46211715726000975896	4.60000e-19	4.86111e-13	1.11022e-16
0.600	-0.53704956699803528586	-0.53704956699803528666	8.00000e-19	5.79869e-13	2.22045e-16
0.700	-0.60436777711716349631	-0.60436777711716349702	7.10000e-19	5.94747e-13	2.22045e-16
0.800	-0.66403677026784896368	-0.66403677026784896409	4.10000e-19	5.32907e-13	5.55112e-16
0.900	-0.71629787019902442081	-0.71629787019902442090	9.00000e-20	4.16001e-13	4.44089e-16
1.000	-0.76159415595576488812	-0.76159415595576488810	2.00000e-20	2.74447e-13	2.22045e-16

Table 4.2: Numerical Results of Example 4.2

ξ	Exact Solution	Computed Solution	EOCM	EKS19	EKe21
0.100	1.9090909090909090909	1.909090909090907951	1.14000e-17	1.55825e-06	1.75330e-10
0.200	1.8333333333333333333	1.833333333333337629	4.30000e-17	2.39975e-06	2.32000e-10
0.300	1.7692307692307692308	1.76923076923076920758	2.32000e-17	2.83045e-06	2.41150e-10
0.400	1.7142857142857142857	1.71428571428571485212	5.66400e-16	3.02094e-06	2.31400e-10
0.500	1.6666666666666666667	1.6666666666666621863	4.48100e-16	3.06956e-06	2.14840e-10
0.600	1.6250000000000000000	1.6250000000000076000	7.60000e-16	3.03457e-06	1.96600e-10
0.700	1.5882352941176470588	1.58823529411764721780	1.59000e-16	2.95115e-06	1.78870e-10
0.800	1.5555555555555555556	1.555555555555583270	2.77100e-16	2.84088e-06	1.62500e-10
0.900	1.5263157894736842105	1.52631578947368454122	3.30700e-16	2.71713e-06	1.47730e-10
1.000	1.5000000000000000000	1.5000000000000041597	4.16000e-16	2.58816e-06	1.34570e-10

Table 4.3: Numerical Results of Example 4.3

ξ	Exact Solution	Computed Solution	EOCM	EI20
0.100	0.9898989898989898989	0.9898989898989898989	0.0000e00	1.11400e-04
0.200	0.97959183673469387755	0.97959183673469387755	0.0000e00	1.85700e-03
0.300	0.96907216494845360825	0.96907216494845360825	0.0000e00	2.53600e-04
0.400	0.9583333333333333333	0.9583333333333333333	0.0000e00	7.14800e-05
0.500	0.94736842105263157895	0.94736842105263157895	0.0000e00	9.72000e-06
0.600	0.93617021276595744681	0.93617021276595744681	0.0000e00	2.06400e-06
0.700	0.92473118279569892473	0.92473118279569892473	0.0000e00	2.80400e-07
0.800	0.91304347826086956522	0.91304347826086956522	0.0000e00	5.29200e-08
0.900	0.90109890109890109890	0.90109890109890109890	0.0000e00	7.16300e-09
1.000	0.8888888888888888889	0.8888888888888888889	0.0000e00	1.09300e-09

4.3 Discussion of Results

An Optimized Computational Method (OCM) was developed through the application of interpolation and collocation techniques using a power series polynomial as the trial function. A key feature of the derivation is the introduction of an optimized mesh point within a one-step interval, which was obtained through a systematic optimization process. By substituting the polynomial approximation and its derivative into the governing differential equation, an algebraic system was formulated and solved to determine the unknown coefficients of the approximate solution. The optimized point was then incorporated into the interpolation and collocation framework, leading to the construction of a continuous numerical scheme. Evaluating the continuous formulation at selected grid points produced the final discrete OCM, which serves as an efficient numerical algorithm for solving Quadratic Riccati Differential Equations. The inclusion of the optimized point enhances the approximation capability of the method and contributes to its improved numerical performance.

The fundamental properties of the derived OCM establish its suitability for solving nonlinear differential equations. Analysis of the associated linear operator showed that the method possesses a uniform order of seven, indicating a high level of accuracy and a correspondingly small truncation error. Since the order of the method exceeds one, an OCM

satisfies the consistency requirement, ensuring that the numerical solution approaches the exact solution as the step size decreases. The zero-stability analysis further confirmed that the roots of the characteristic polynomial satisfy the necessary stability conditions, thereby guaranteeing bounded propagation of numerical errors. Consequently, by the Dahlquist convergence theorem, the method is convergent. In addition, the region of absolute stability obtained through the boundary locus technique revealed that an OCM is A_α -stable, implying that it maintains stable numerical solutions over a wide range of problems. Collectively, these properties demonstrate that the OCM is a reliable, accurate, stable, and convergent computational method for the numerical solution of Quadratic Riccati Differential Equations.

The numerical results presented in Tables 4.1 to 4.3 demonstrate the effectiveness of an Optimized Computational Method (OCM) in solving Quadratic Riccati Differential Equations. For all the examples considered, the computed solutions obtained using the OCM show excellent agreement with the corresponding exact solutions at every point of evaluation. The close correspondence between the numerical and analytical solutions indicates that the proposed method accurately captures the behavior of the underlying differential equations and produces highly reliable approximations throughout the integration interval. This consistency confirms the capability of the OCM to provide stable and accurate solutions for nonlinear Riccati problems.

For Example 4.1, the numerical approximations generated by the OCM closely follow the exact solution across the entire domain. The computed values remain virtually identical to the analytical values, demonstrating the ability of the method to preserve the solution characteristics without any noticeable deviation. When compared with the existing methods of Sunday, (2017) and Sunday and Philip, (2018), the OCM exhibits superior agreement with the exact solution, reflecting its improved computational formulation and enhanced convergence properties. The results clearly indicate that the optimization strategy incorporated into the method contributes significantly to its high level of accuracy and numerical efficiency.

The performance of the OCM in Example 4.2 further validates its robustness and reliability. The numerical solutions obtained remain in excellent agreement with the exact solution at all evaluation points, confirming the consistency of the method. In comparison with the approaches proposed by Kashkari and Syam, (2019) and Kamoh et al., (2021), the OCM provides a more precise approximation of the analytical solution. The method maintains its accuracy as the independent variable increases, demonstrating strong numerical stability and indicating that it is capable of handling Riccati differential equations effectively without deterioration in solution quality. These observations highlight the suitability of an OCM for solving a wide range of nonlinear initial value problems.

Similarly, the results for Example 4.3 reveal the exceptional capability of an OCM in reproducing the exact solution throughout the computational interval. The computed solutions coincide perfectly with the analytical values, illustrating the remarkable precision of the method. Compared with the existing approach reported by Isah et al. (2020),

an OCM provides a more accurate representation of the exact solution and demonstrates superior computational performance. Taken together, the numerical results from all three examples confirm that an OCM is a highly accurate, stable and efficient technique for solving Quadratic Riccati Differential Equations. Its consistent performance across different test problems establishes it as a reliable numerical tool for applications involving nonlinear differential equations.

6 Conclusion

This study developed an Optimized Computational Method (OCM) for the numerical solution of Quadratic Riccati Differential Equations (QRDEs). The method was derived using a power series polynomial through interpolation and collocation techniques with the incorporation of an optimized mesh point within a one-step interval. The exact location of the optimized point was obtained through a systematic optimization procedure, leading to the formulation of a continuous scheme and its corresponding discrete computational method. Theoretical analysis of the proposed method established that it possesses a uniform order of seven, thereby ensuring high accuracy. Further investigations confirmed that the method is consistent, zero-stable, convergent, and A_α -stable, making it suitable for solving a wide class of nonlinear differential equations. Numerical experiments were conducted on three benchmark Quadratic Riccati Differential Equations, and the results were compared with existing methods from the literature. The computed solutions obtained by the OCM showed excellent agreement with the exact solutions and demonstrated superior numerical performance across all test problems considered.

The results obtained in this study have demonstrated that an Optimized Computational Method is an efficient and reliable numerical technique for solving Quadratic Riccati Differential Equations. The incorporation of an optimized mesh point significantly enhanced the accuracy and stability of the method, while the theoretical properties confirmed its robustness and convergence. The numerical simulations revealed that the OCM consistently produced highly accurate approximations and outperformed several existing methods used for similar problems. Its ability to maintain excellent agreement with exact solutions across different examples highlights its effectiveness in handling nonlinear differential equations. Therefore, an OCM can be regarded as a valuable contribution to the field of numerical analysis and is recommended for the solution of QRDEs and other related nonlinear initial value problems encountered in science and engineering.

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